Introduction to Logarithms

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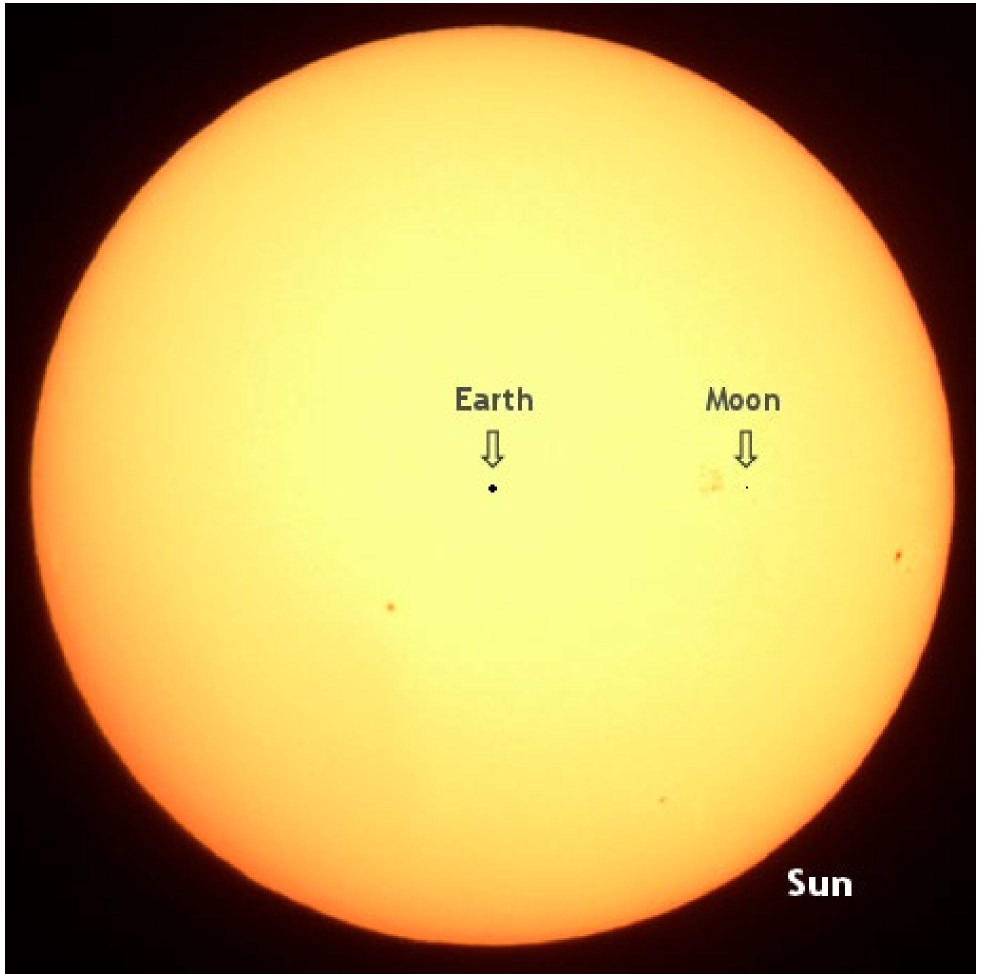
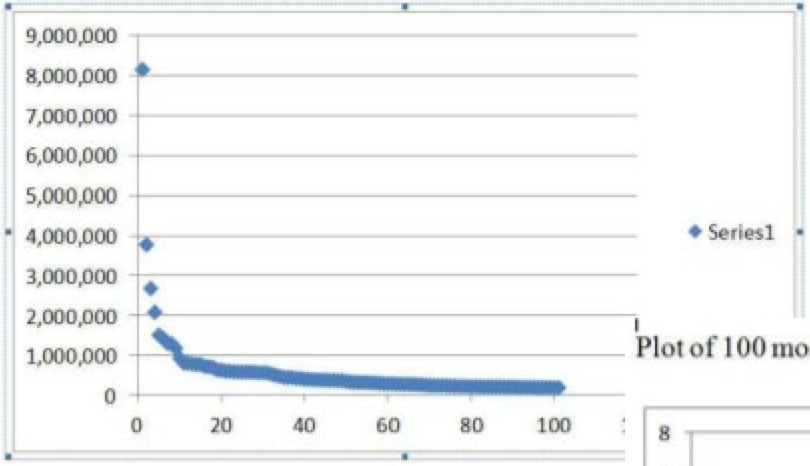
Why do we care about Logarithms

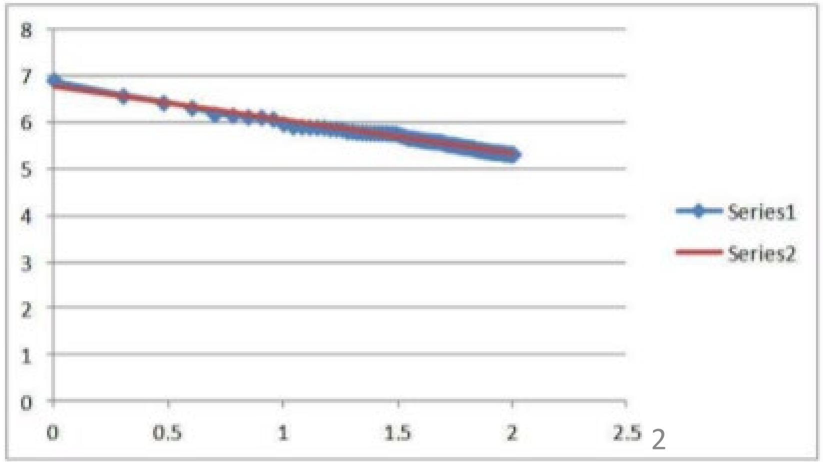
- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).



Plot of 100 most populous cities



Plot of 100 most populous cities: log-log plot



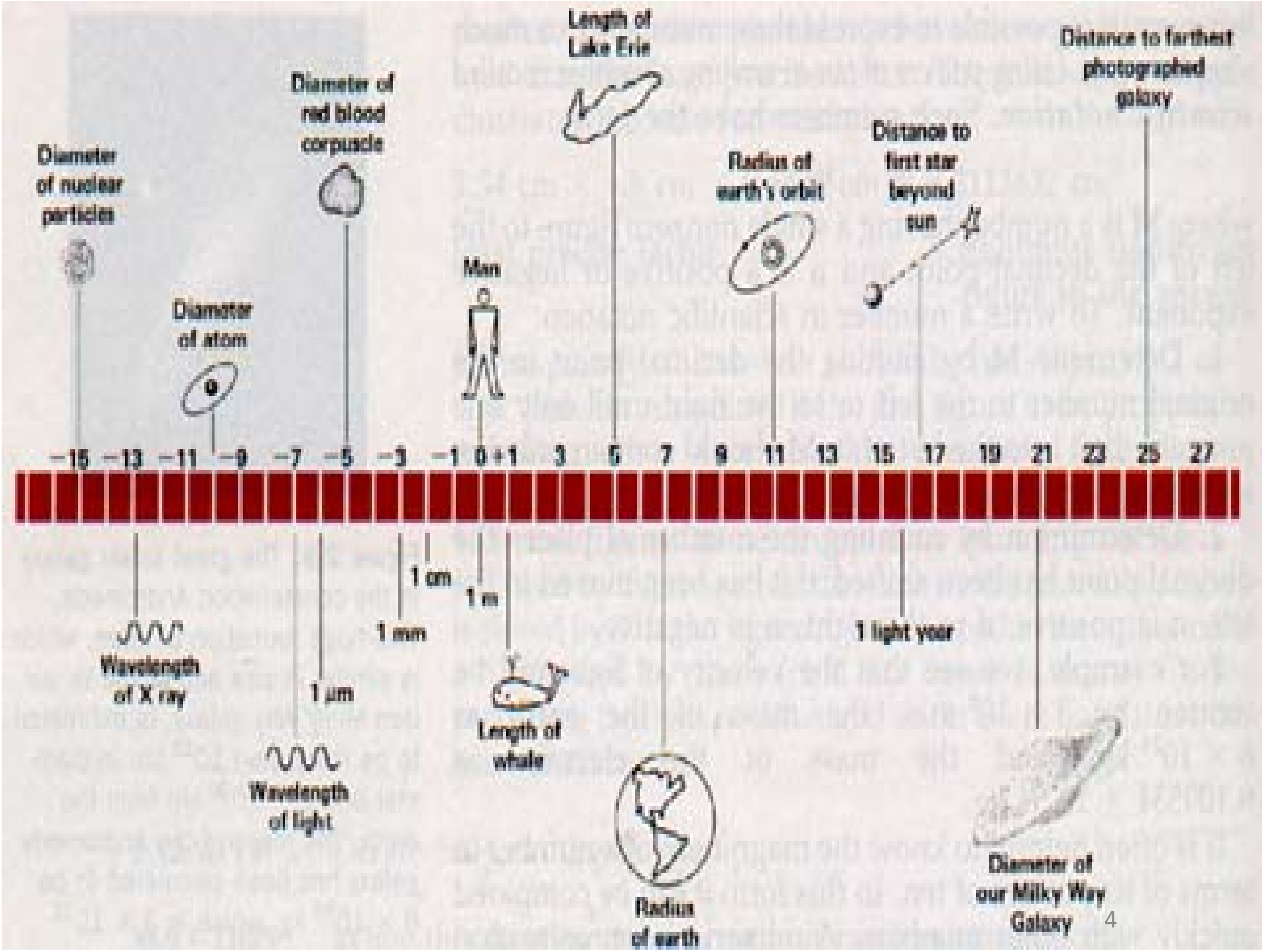
Definition of Logarithms

- **If $\boldsymbol{x = b^y}$ then $\boldsymbol{\log_b x = y}$.**
- Read as the logarithm of x base b is y.
- Often use base 10, and some authors suppress the subscript 10.
- Other popular bases are 2 for computers, and e for calculus; many sources write $\ln x$ for the natural logarithm of x, which is its logarithm base e (e is approximately 2.71828).

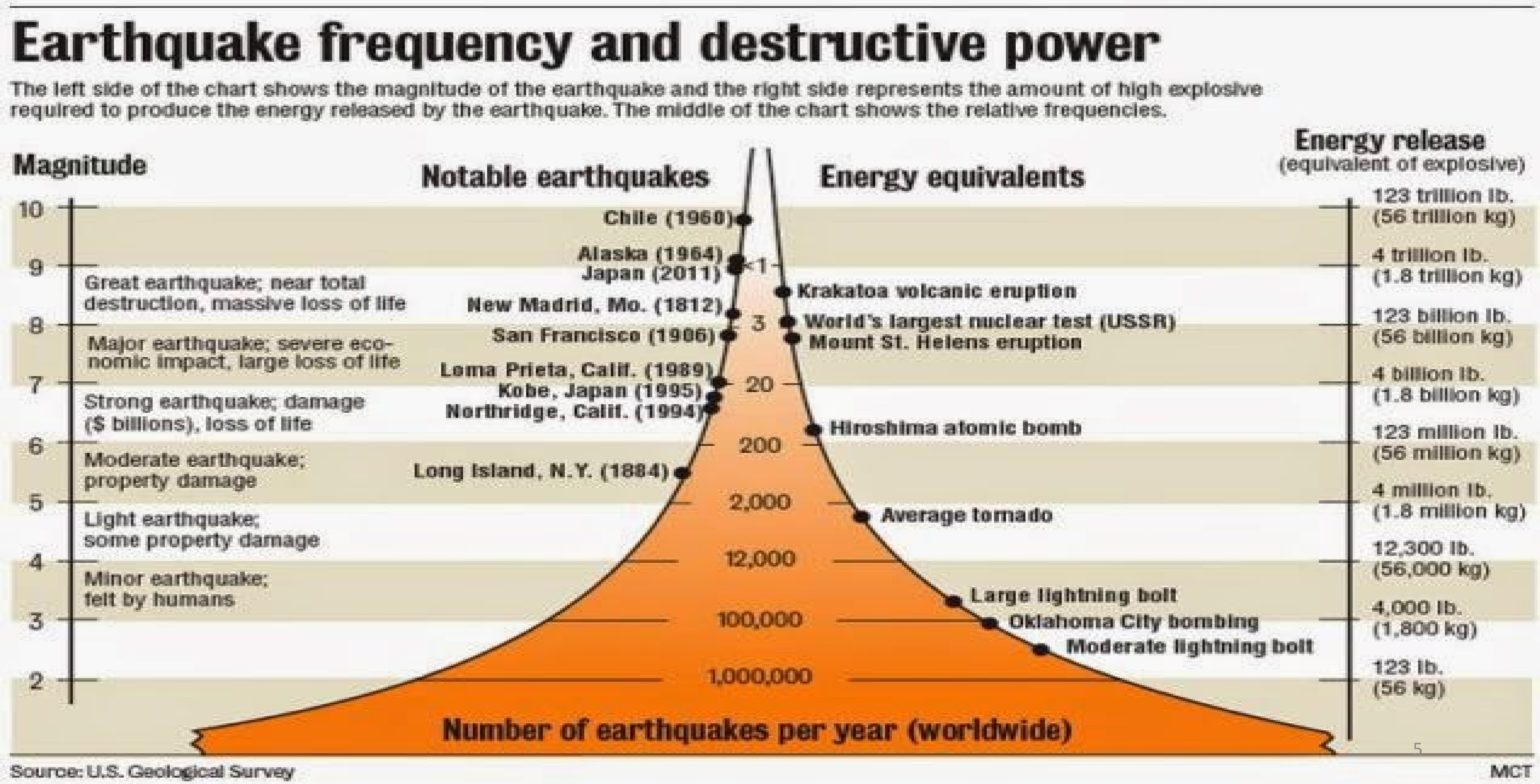
- **Examples: $\boldsymbol{\log_b x = y}$ means we need $\boldsymbol{y}$ powers of $\boldsymbol{b}$ to get $\boldsymbol{x}$.**
  - $100 = 10^2$ becomes $\log_{10} 100 = 2.$ In base e it is about 4.6.
  - $1 = 10^0$ becomes $\log_{10} 1 = 0.$ In base e it is still 0.
  - $.001 = 10^{-3}$ becomes $\log_{10} .001 = -3.$ In base e it is about -6.9.

Examples of Logarithms

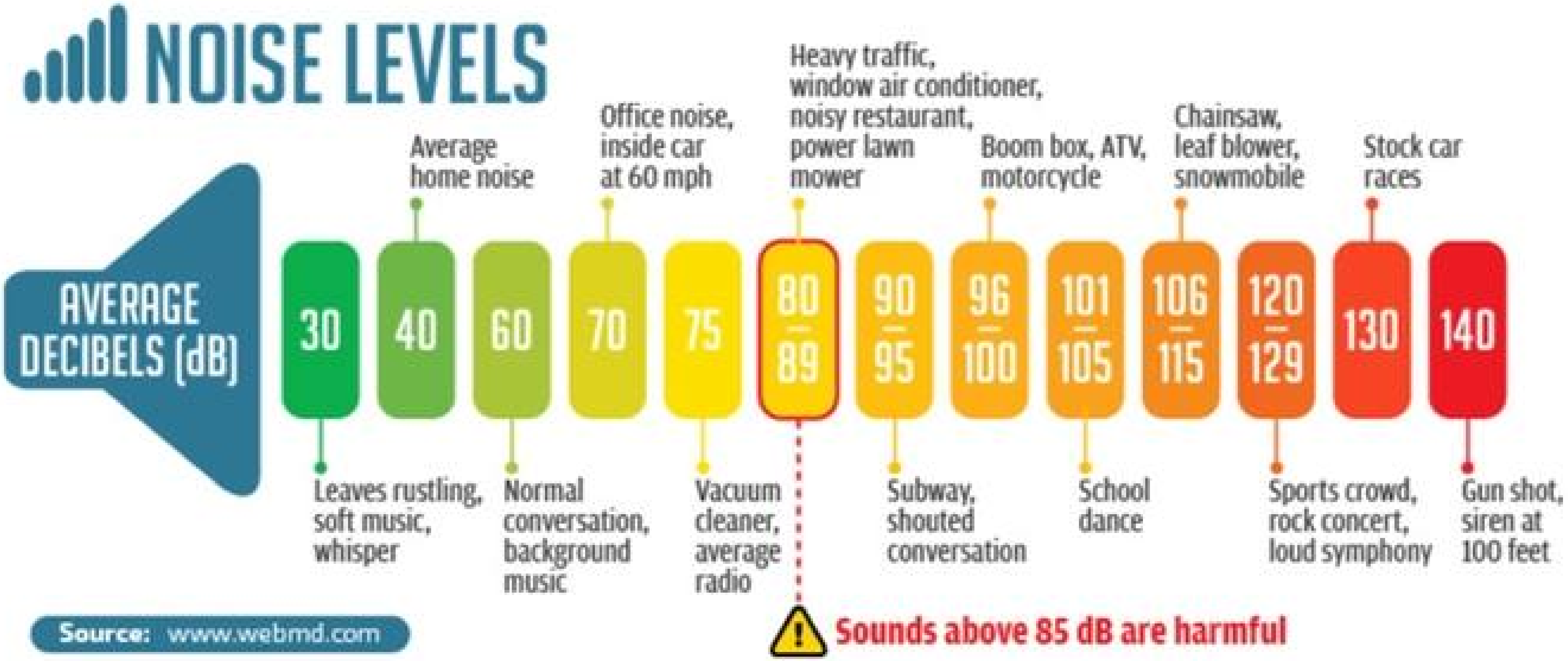
Order of Magnitude of some Lengths	
LENGTH	**meters**
radius of proton	10^{-15}
radius of atom	10^{-10}
radius of virus	10^{-7}
radius of amoeba	10^{-4}
height of human being	10^{0}
radius of earth	10^{7}
radius of sun	10^{9}
earth-sun distance	10^{11}
radius of solar system	10^{13}
distance of sun to nearest star	10^{16}
radius of milky way galaxy	10^{21}
radius of visible Universe	10^{26}



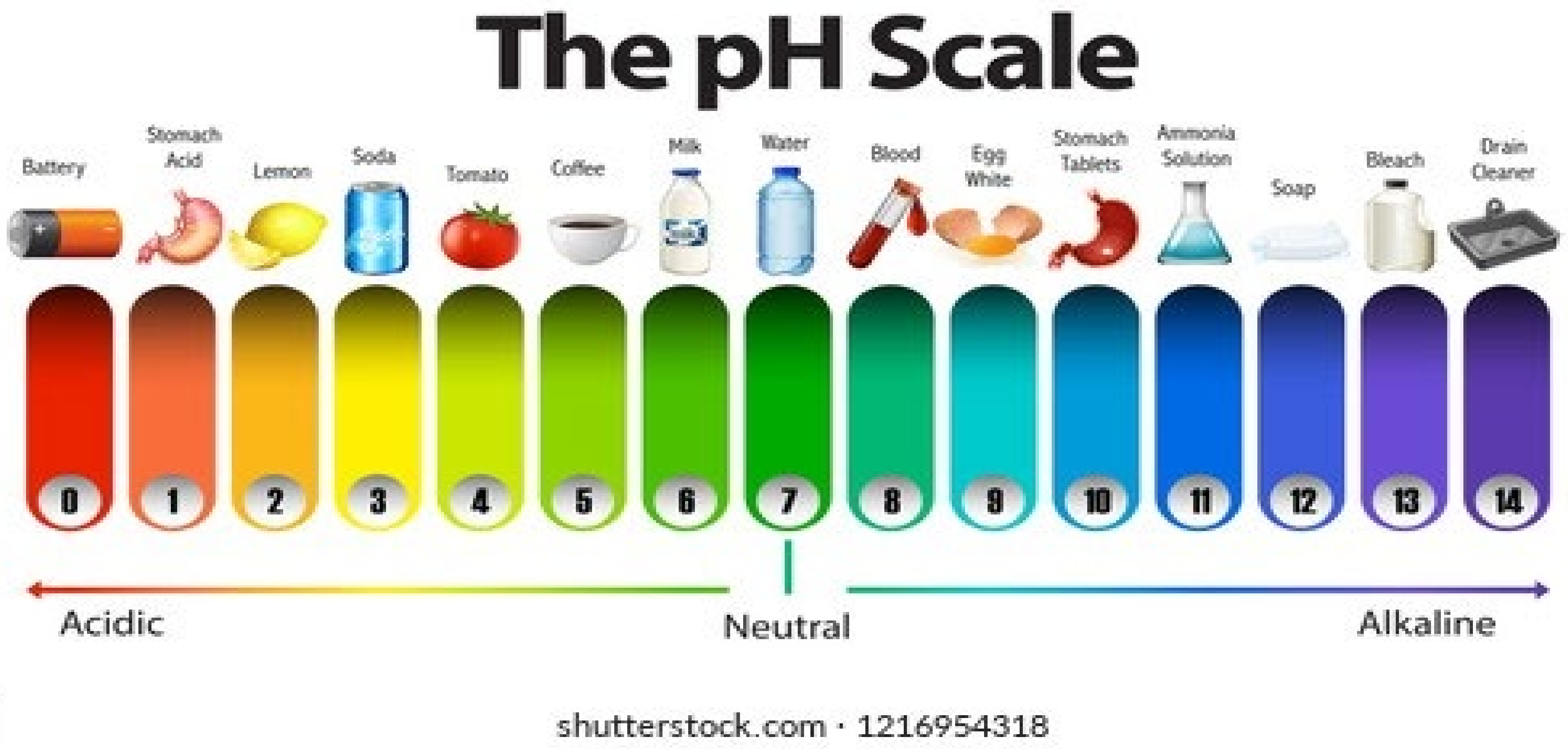
Examples of Logarithms



Examples of Logarithms



Examples of Logarithms



Recall: Definition of Logarithms

- **If $x = b^y$ then $\log_b x = y$.**
- Read as the logarithm of x base b is y.
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- Other popular bases are 2 for computers, and e for calculus; many sources write $\ln x$ for the natural logarithm of x, which is its logarithm base e (e is approximately 2.71828).

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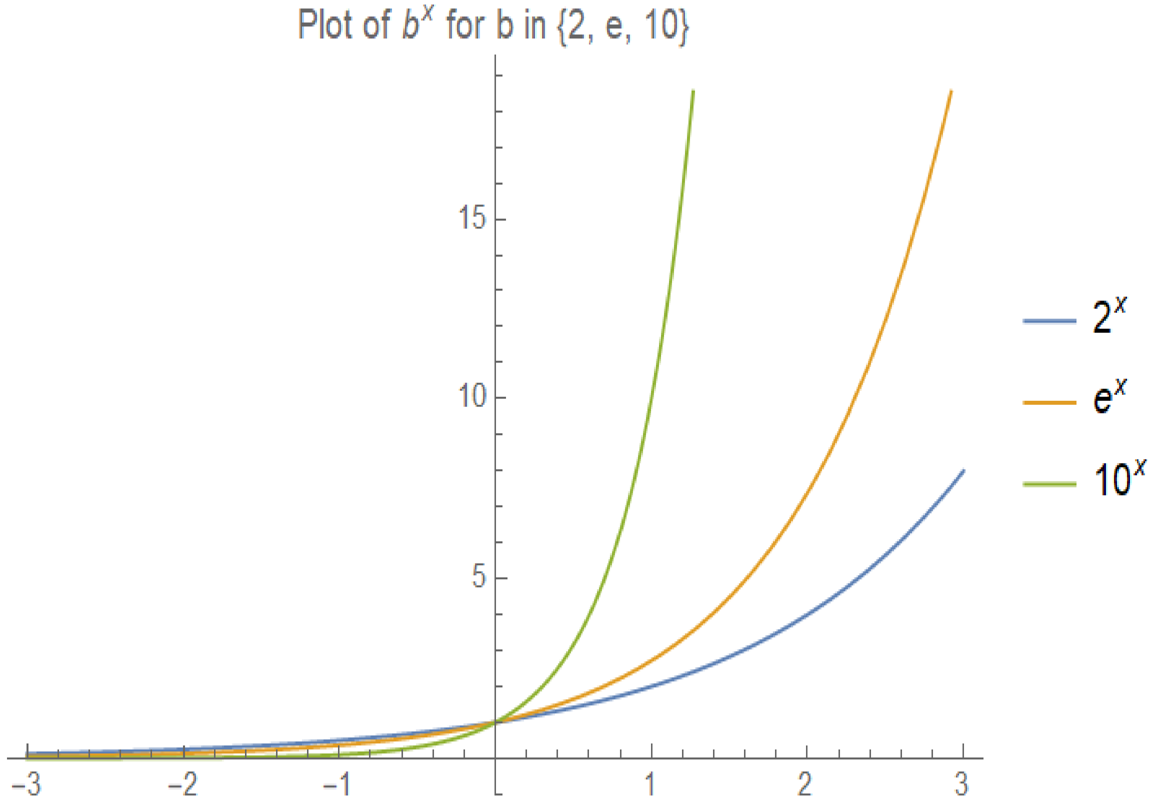
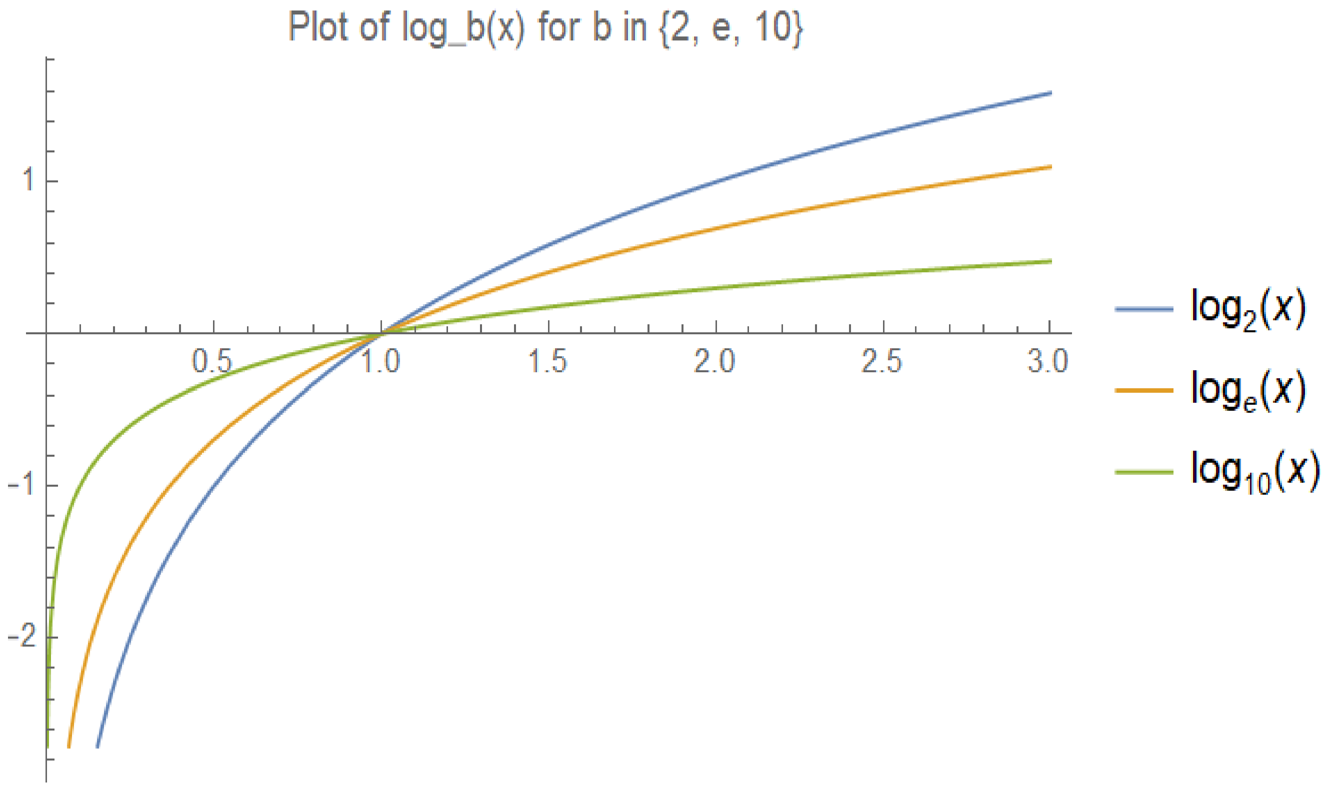
Plots of Exponentiation and Logarithms

- If $x = b^y$ then $\log_b x = y$.
- Read as the logarithm of x base b is y.

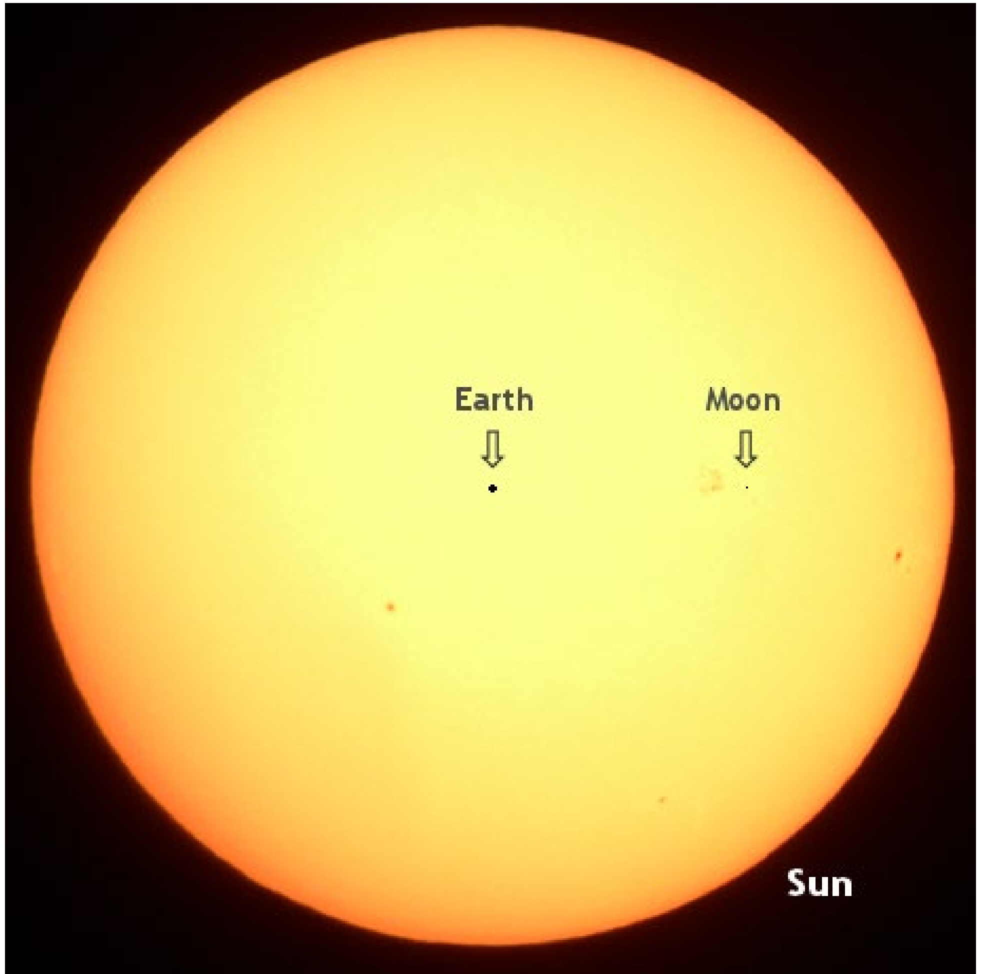




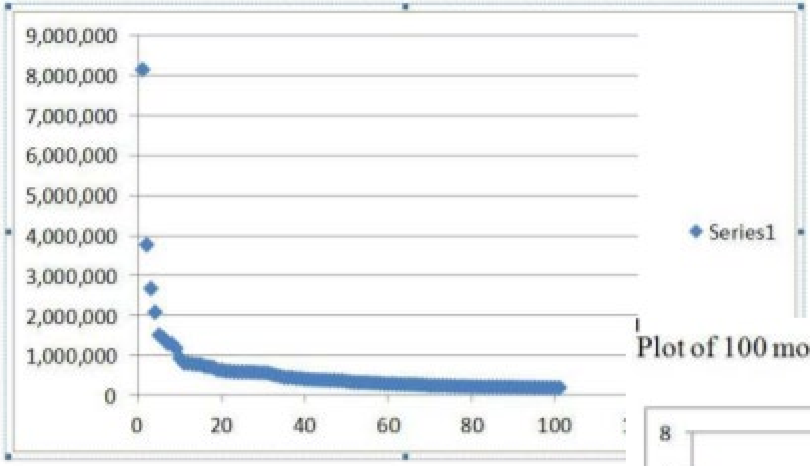
$$10^n \leq x \leq 10^{n+1} \quad \text{implies} \quad n \leq \log_{10} x \leq n + 1.$$

Why do we care about Logarithms

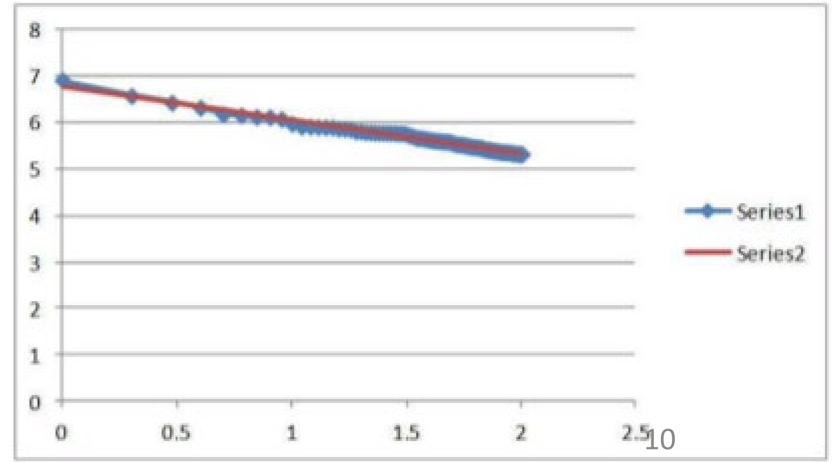
- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).



Plot of 100 most populous cities



Plot of 100 most populous cities: log-log plot



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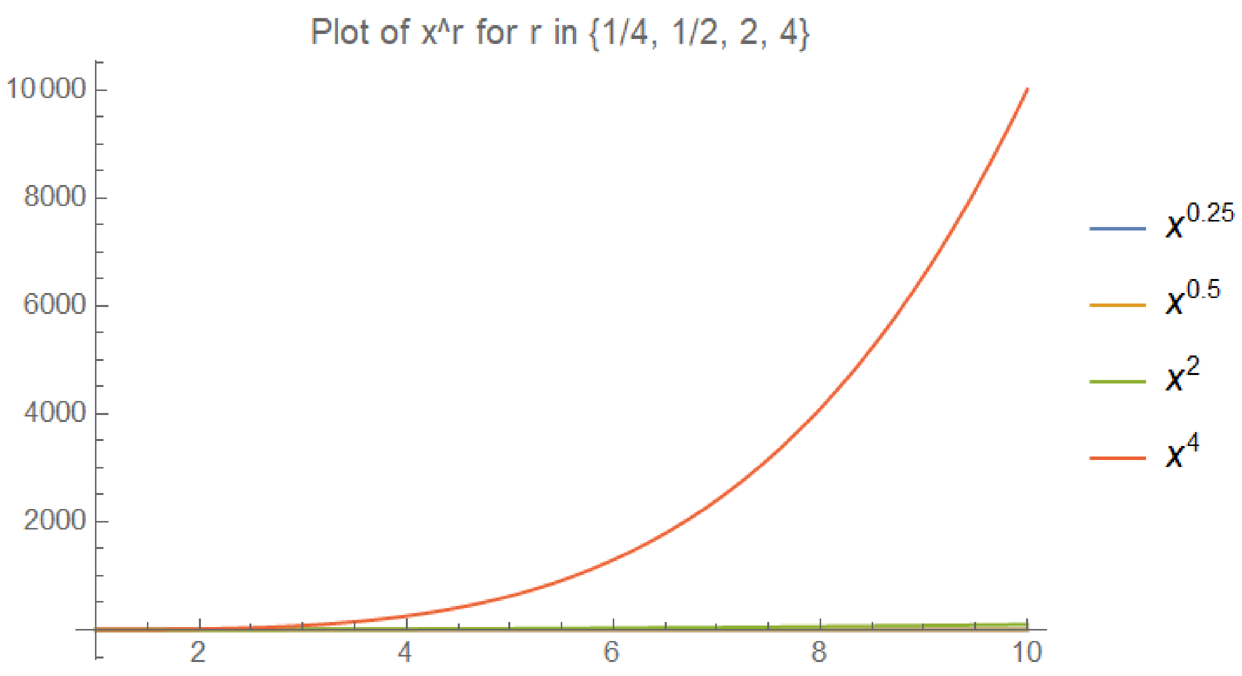

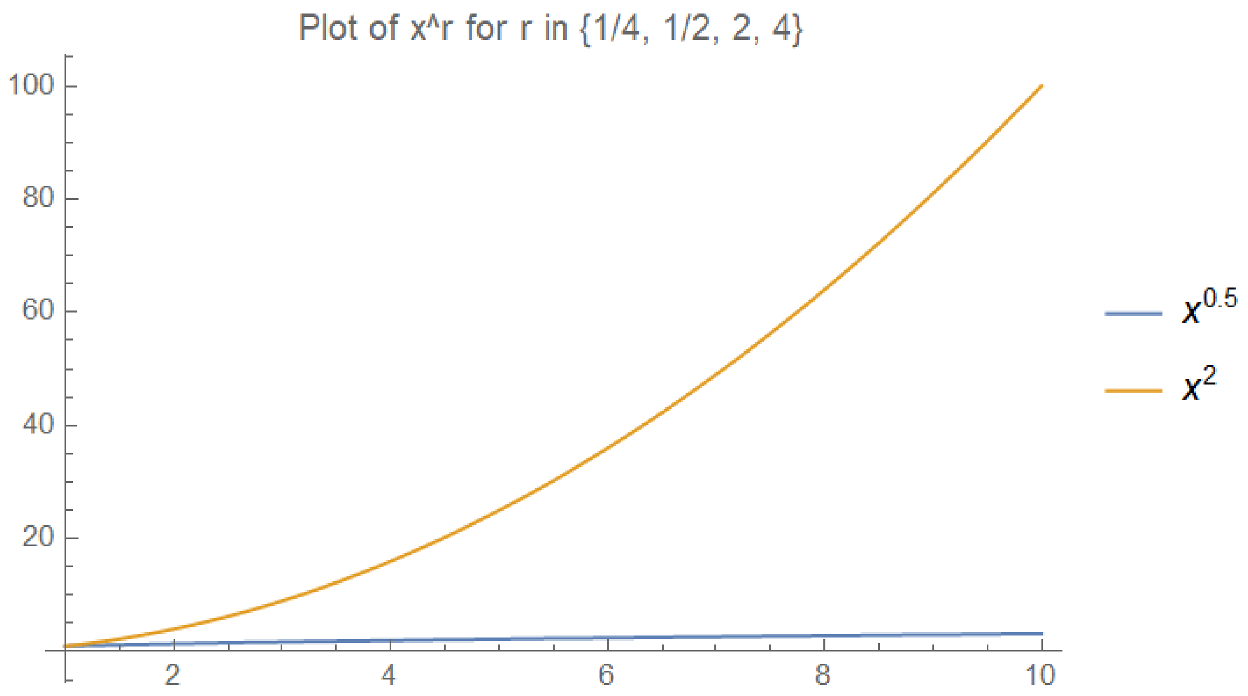

Notice that even on a small range, from 1 to 10, the polynomial of highest degree drowns out the others and can barely see.

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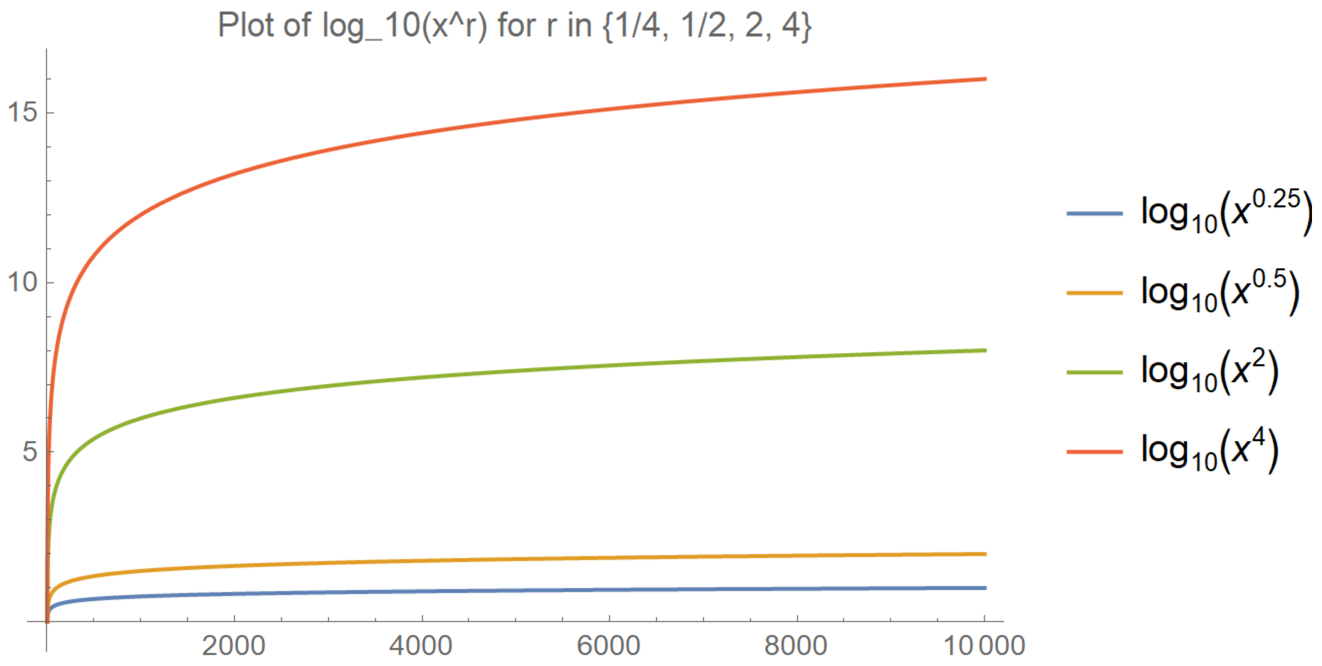
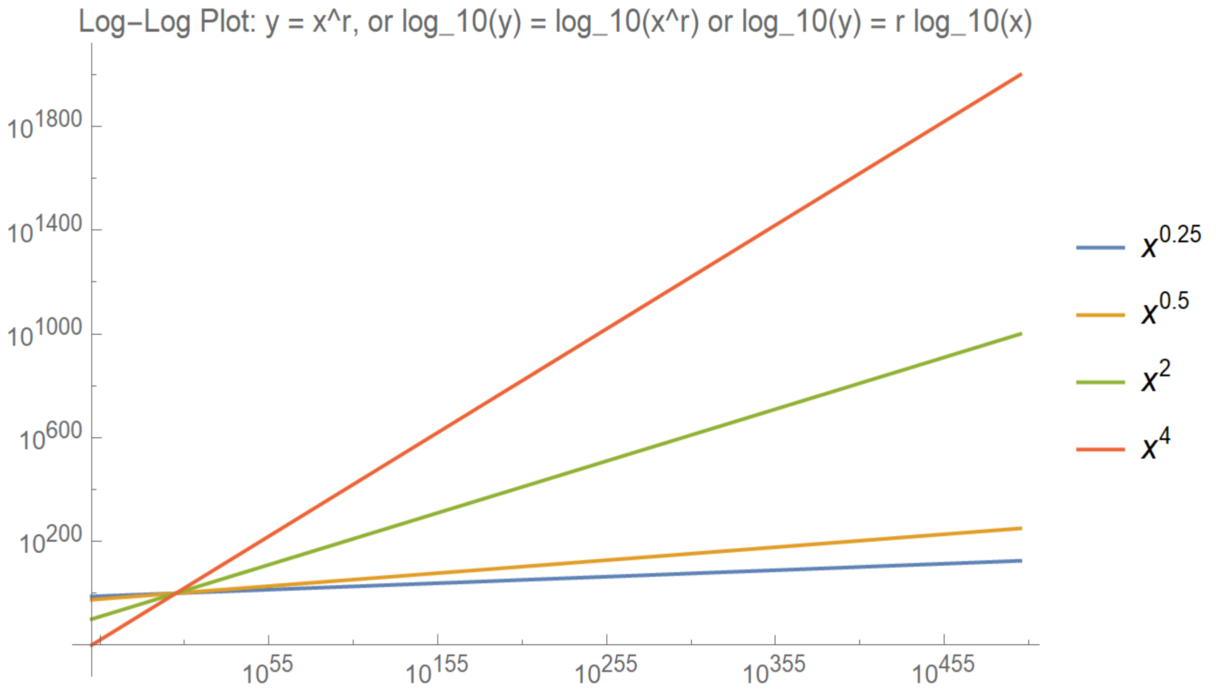




Left: Semi-log plot: $y = \log x^r$. Right: log-log plot: $\log y = \log x^r$.
Note that we can now see the four functions on one plot, and the log-log plot now has linear relations.

Review: Exponent Laws

Laws

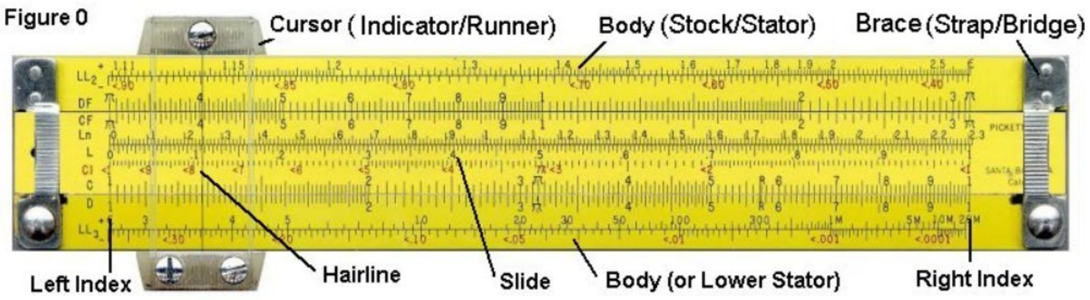
- $b^m\ b^n = b^{m+n}$
- $b^m\ /\ b^n = b^{m-n}$
- $(b^m)^n = b^{mn}$

Examples

- $10^3 10^2 = (10 * 10 * 10) * (10 * 10) = 10^5$
- $10^3/10^2 = (10 * 10 * 10)/(10 * 10) = 10^1$
- $(10^3)^2 = 10^3 * 10^3 = (10 * 10 * 10) * (10 * 10 * 10) = 10^6$

Logarithm Laws

Parts of a Slide Rule



Remember if $x = b^y$ then $\log_b x = y$.

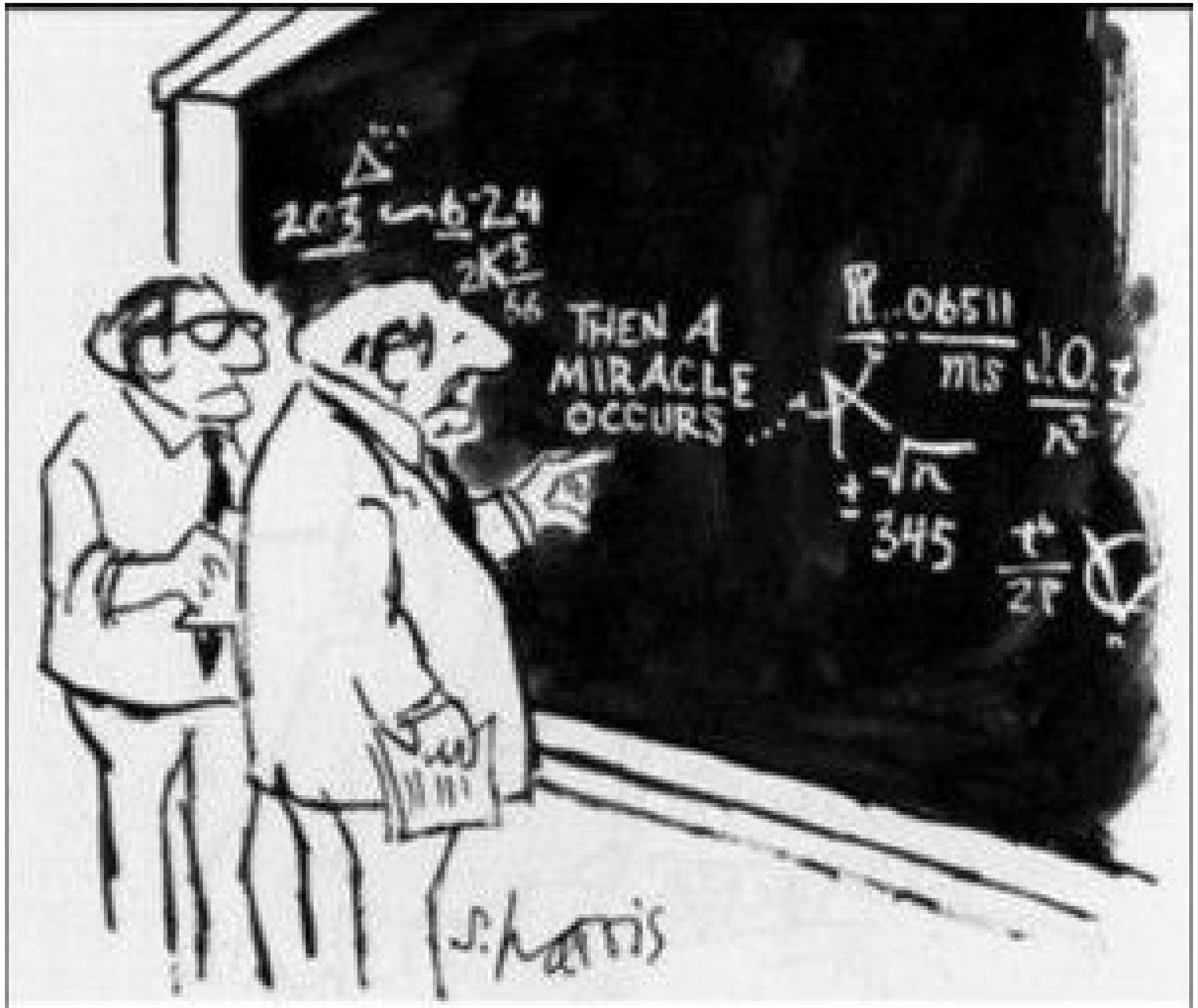
Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

These allow us to simplify computations with logarithms.

THEOREM

- $\log_b(x^n) = \mathrm{n} \log_b x$. Log of a power is that power times the log.
- $\log_b(x_1\ x_2) = \log_b(x_1) + \log_b(x_2)$. Log of a product is the sum of the logs.
- $\log_b(x_1/x_2) = \log_b(x_1) - \log_b(x_2)$. Log of a quotient is the difference of the logs.
- $\log_b x = \log_c x / \log_c b$. If know logs in one base, know in all.

OPTIONAL – PROOFS OF THE LOG LAWS



"I think you should be more explicit here in step two."

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

- $\boldsymbol{\log_b(x^n) = n \log_b x}$. **Log of a power is that power times the log.**

Proof:

- $\log_b x = y$ means $x = b^y$.

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

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- $\boldsymbol{\log_b(x^n) = n \log_b x}$. **Log of a power is that power times the log.**

Proof:

- $\log_b x = y$ means $x = b^y$.
- Thus $x^n = (b^y)^n = b^{ny}$.

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

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Proof:

- $\log_b x = y$ means $x = b^y$.
- Thus $x^n = (b^y)^n = b^{ny}$.
- Taking logarithms: $\log_b(xn) = ny = n \log_b x$. ■

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

- $\boldsymbol{\log_b(x_1\ x_2) = \log_b(x_1) + \log_b(x_2)}$. **Log of a product is the sum of the logs.**

Proof:

- As $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$, we have $x_1 = b^{y_1}$ and $x_2 = b^{y_2}$.

Logarithm Laws: Proofs

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Proof:

- As $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$, we have $x_1 = b^{y_1}$ and $x_2 = b^{y_2}$.
- Thus $x_1\ x_2 = b^{y_1} b^{y_2} = b^{y_1+y_2}$.

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

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Proof:

- As $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$, we have $x_1 = b^{y_1}$ and $x_2 = b^{y_2}$.
- Thus $x_1\ x_2 = b^{y_1} b^{y_2} = b^{y_1 + y_2}$.
- Therefore $\log_b(x_1\ x_2) = y_1 + y_2 = \log_b x_1 + \log_b x_2$. ∎

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_c x = u$ (so $x = c^u$) and $\log_c b = v$ (so $b = c^v$).

- $\boldsymbol{\log_b x = \log_c x / \log_c b}$. **Know logs in one base, know in all.**

Proof:

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.

Logarithm Laws: Proofs

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Proof:

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.
- Thus $x = b^y = (c^v)^y = c^{vy}$.

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

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- As also have $x = c^u$ we have $u = vy$ or $y = u/v$.

Logarithm Laws: Proofs

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Proof:

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.
- Thus $x = b^y = (c^v)^y = c^{vy}$.
- As also have $x = c^u$ we have $u = vy$ or $y = u/v$.
- Substituting gives $\log_b x = \log_c x / \log_c b$. $\blacksquare$

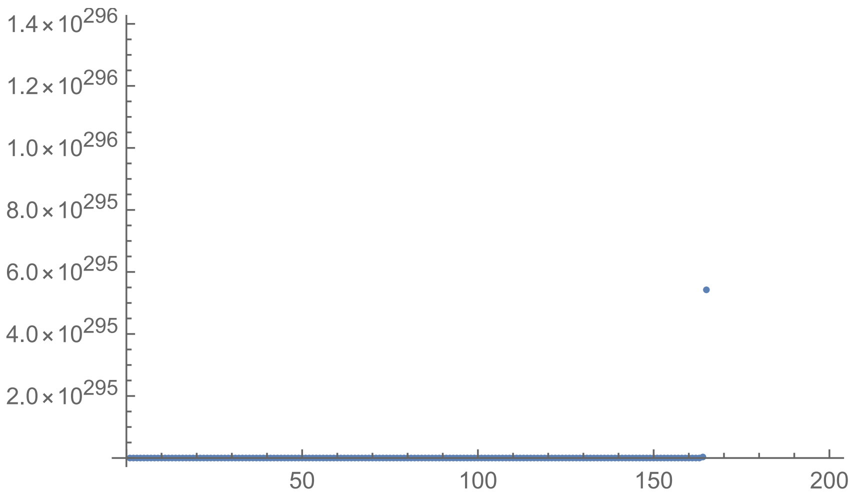
Example: Factorial Function:
Number ways to order n *objects when order matters:*

$$n! = n * (n-1) * \bullet\bullet\bullet * 3 * 2 * 1.$$

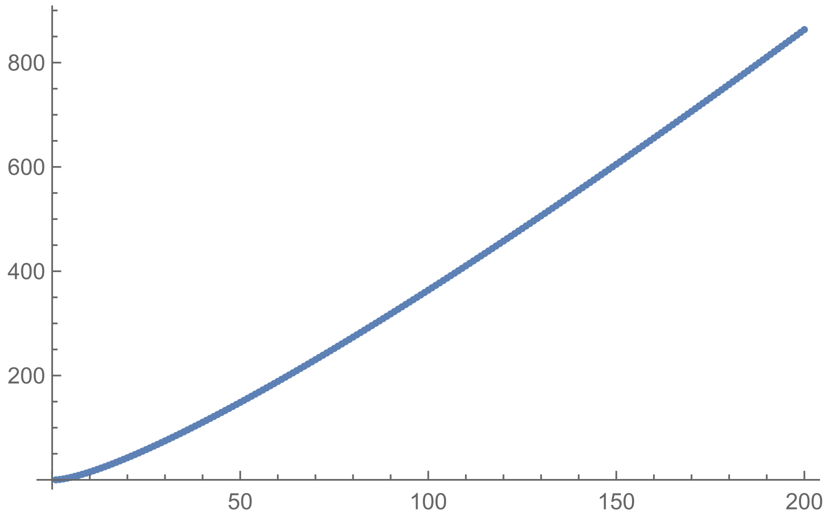
```
list = {}; semiloglist = {}; logloglist = {};
For[n = 1, n <= 200, n++,
  {
   list = AppendTo[list, {n, n!}];
   semiloglist = AppendTo[semiloglist, {n, Log[n!]}];
   logloglist = AppendTo[logloglist, {Log[n], Log[n!]}];
   }];
Print[ListPlot[list]]; Print[ListPlot[semiloglist]]; Print[ListPlot[logloglist]];
```

Example: Factorial Function:
Number ways to order *n objects when order matters:*

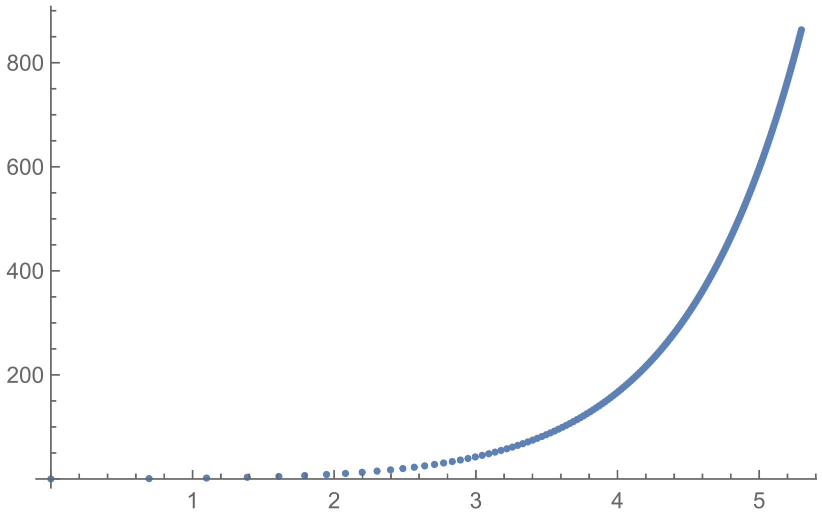
$$n! = n \ * \ (n-1) \ * \bullet\bullet\bullet * \ 3 \ * \ 2 \ * \ 1.$$



Normal Plot



Semi-log Plot



Log-Log Plot

For large n, have $n! \approx n^n e^{-n} \sqrt{2\pi n}$, so $\log n \approx \frac{n}{e} \log n$ (plus a much smaller term).