Math 150: Calculus III: Multivariable Calculus

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Lecture 6: 2-16-2022:

https://youtu.be/f8JkOmTuuyY (slides here)

Class 06: http://youtu.be/aIZmP640Sn4 (February 12, 2014: Dot Products, Cross Products, Determinants, Areas)

Plan for the day: Lecture 6: February 16, 2022:

Topics:

Dot Product

Cross Product

Equations of Planes

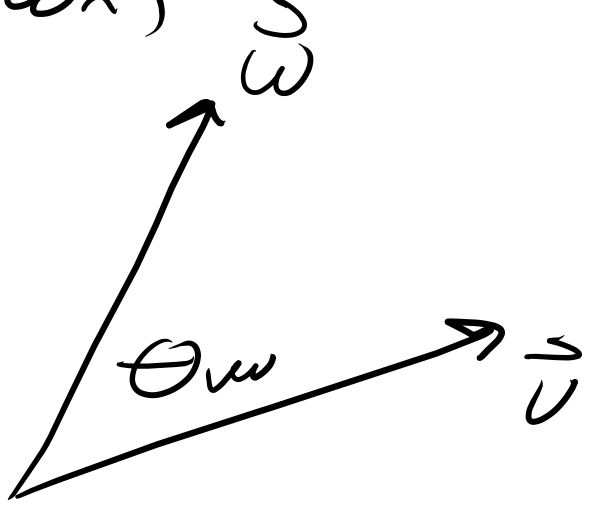
Determinants

Areas

(Matrices as time permits)

Dot product:

$\vec{v} = (v_1, \dots, v_n)$ $\quad$ $\vec{w} = (w_1, \dots, w_n)$



$\vec{v} \cdot \vec{w} = v_1 w_1 + \dots + v_n w_n$

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta_{vw}$

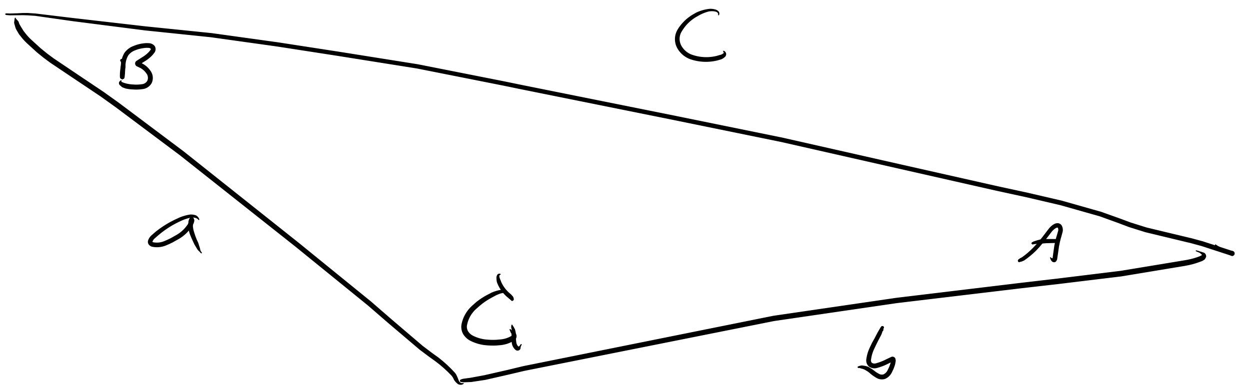
$= \vec{w} \cdot \vec{v}$

See if replace $\vec{v}$ with $a\vec{v}$ then $(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w})$

so wlog assume $\|\vec{v}\| = 1$

Special case: $\vec{v} = (1, 0, 0, \dots, 0)$

Input for dot product formula is the Law of Cosines

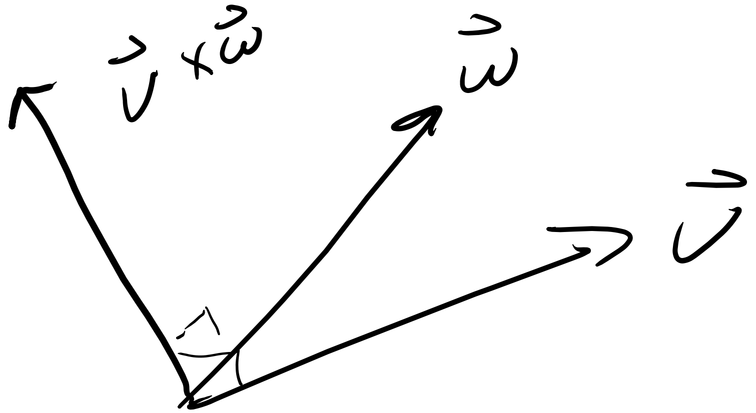


If $C = 90$,
$\cos(C) = 0$
get Pythagoras!

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

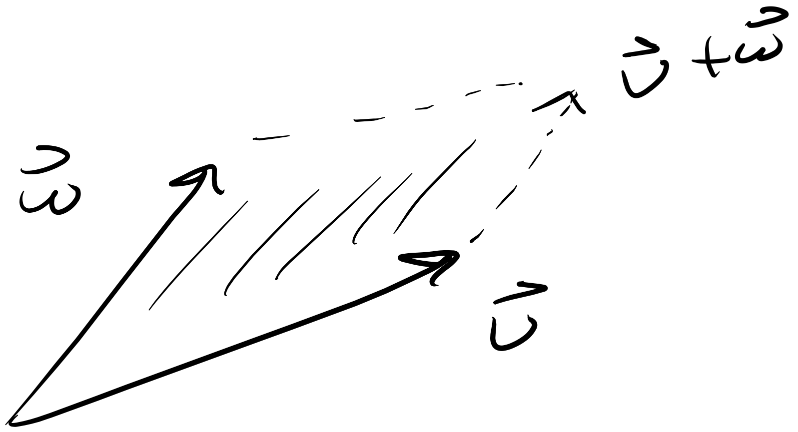
Prove by drawing lines and making lots of right triangles and using Pythagoras.

Cross Product



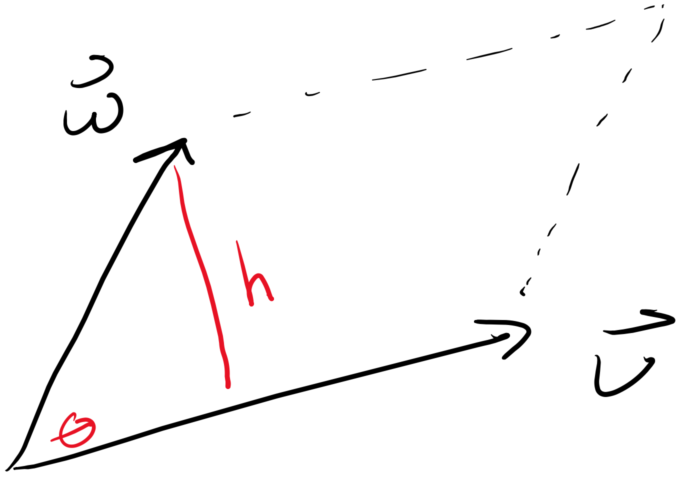
$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

$\|\vec{v} \times \vec{w}\|$ = area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$



Build intuition: $\vec{v} = (1, 0, 0, \ldots, 0)$

Area of the parallelogram



$$h = \|\vec{w}\| \sin\theta$$

$$\text{Area} = \text{base} * \text{height}$$

$$= \|\vec{v}\| \, \|\vec{w}\| \sin\theta$$

$$\vec{v}\cdot\vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta \, , \quad \sin\theta = \sqrt{1-\cos^2\theta}$$

$$\underset{\text{(area)}}{\sqrt{\|\vec{v}\|^2\|\vec{w}\|^2 - (\vec{v}\cdot\vec{w})^2}} = \sqrt{(v_1^2+v_2^2)(w_1^2+w_2^2) - (v_1w_1+v_2w_2)^2}$$

$$\underset{(\text{area})}{\sqrt{\|\vec{v}\|^2\|\vec{w}\|^2-(\vec{v}\cdot\vec{w})^2}} = \sqrt{(v_1^2+v_2^2)(w_1^2+w_2^2)-(v_1w_1+v_2w_2)^2}$$

$$= \sqrt{v_1^2w_2^2+v_2^2w_1^2-2v_1w_1v_2w_2}$$

$$= \sqrt{(v_1w_2)^2-2(v_1w_2)(v_2w_1)+(v_2w_1)^2}$$

$$= \sqrt{(v_1w_2-v_2w_1)^2}$$

$$= |v_1w_2-v_2w_1|$$

area in the plane

Determinant

$$\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = v_1w_2-v_2w_1$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

Connection in plane b/w determinant and area

$\vec{v}$ and $\vec{w}$

$$\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = v_1 w_2 - v_2 w_1$$ is the signed area

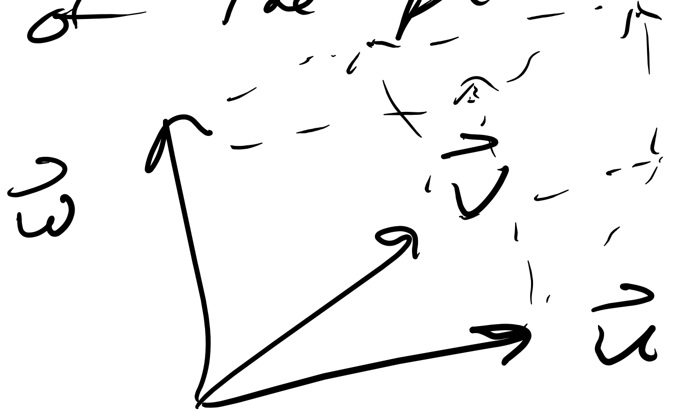
$$\begin{vmatrix} w_1 & w_2 \\ v_1 & v_2 \end{vmatrix} = w_1 v_2 - w_2 v_1$$

$$= -(v_1 w_2 - v_2 w_1)$$

Det of $\vec{u}, \vec{v}, \vec{w}$,

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Vol of the parallelpiped



Trick only works in 3-dim

$$
\begin{array}{ccc|cc}
 & & - & - & - \\
u_1 & u_2 & u_3 & u_1 & u_2 \\
v_1 & v_2 & v_3 & v_1 & v_2 \\
w_1 & w_2 & w_3 & w_1 & w_2 \\
 & & + & + & +
\end{array}
$$

$\hat{i} = (1,0,0) = \vec{e}_1$

$\hat{j} = (0,1,0) = \vec{e}_2$

$\hat{k} = (0,0,1) = \vec{e}_3$

$$u_1 v_2 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2$$

$$- u_3 v_2 w_1 - u_1 v_3 w_2 - u_2 v_1 w_3$$

Cross Product

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ v_1 & v_2 \\ w_1 & w_2 \end{matrix}$$

$$= \hat{i}(v_2 w_3) + \hat{j}(v_3 w_1) + \hat{k}(v_1 w_2) - \hat{k}(v_2 w_1) - \hat{i}(v_3 w_2) - \hat{j}(v_1 w_3)$$

$$= (v_2 w_3 - v_3 w_2)\hat{i} + (v_3 w_1 - v_1 w_3)\hat{j} + (v_1 w_2 - v_2 w_1)\hat{k}$$

$$= (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1)$$

Thm:

$\vec{v} \times \vec{w}$ is perpendicular ($\perp$) to $\vec{v}$ and $\vec{w}$

Show $(\vec{v} \times \vec{w}) \cdot \vec{v} = 0 \implies \cos\theta = 0$

$\implies \theta = \pi/2$ or $3\pi/2$

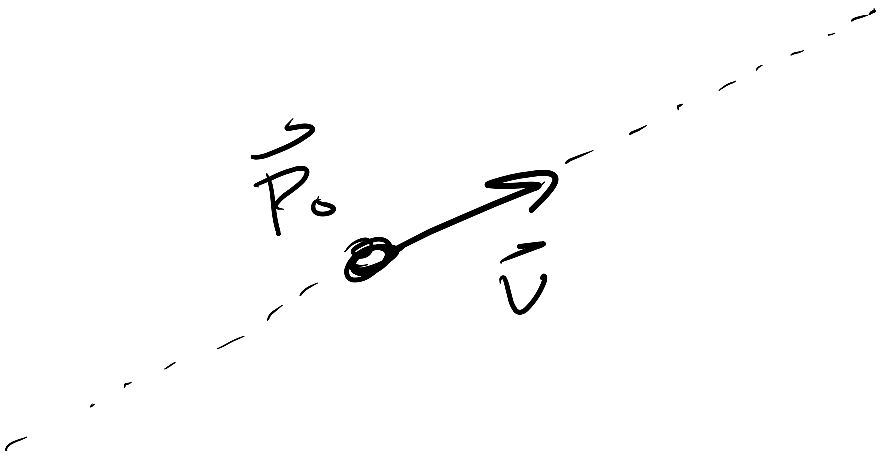
Study $(\vec{u} \times \vec{v}) \times \vec{w}$ YES!

Study $(\vec{u} \times \vec{v}) \cdot \vec{w}$ YES

Lines and Planes

$$\vec{P}(t) = \vec{P}_0 + t\vec{v}$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 + tv_1 \\ y_0 + tv_2 \\ z_0 + tv_3 \end{pmatrix}$$



Symmetric version

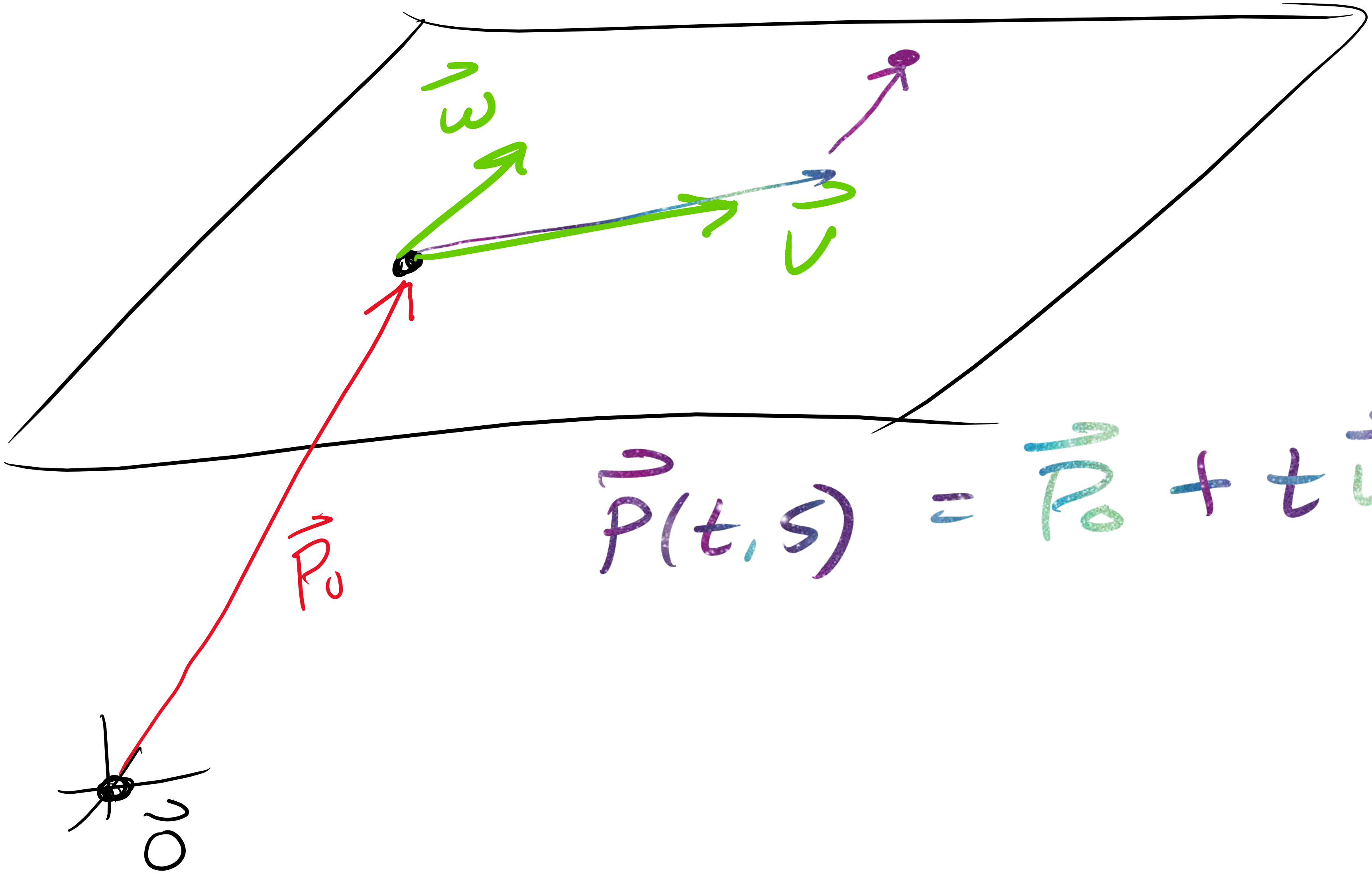
$$t = \frac{x - x_0}{v_1}$$

$$t = \frac{y - y_0}{v_2}$$

$$t = \frac{z - z_0}{v_3}$$

if $v_1, v_2, v_3 \neq 0$

Plane



$$\vec{P}(t,s) = \vec{P}_0 + t\vec{v} + s\vec{w}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + s \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$x = x_0 + t v_1 + s w_1$$
$$y = y_0 + t v_2 + s w_2$$
$$z = z_0 + t v_3 + s w_3$$

Know x and y, z is forced

Know x and y, solve for t and s

Knowing t and s, know z

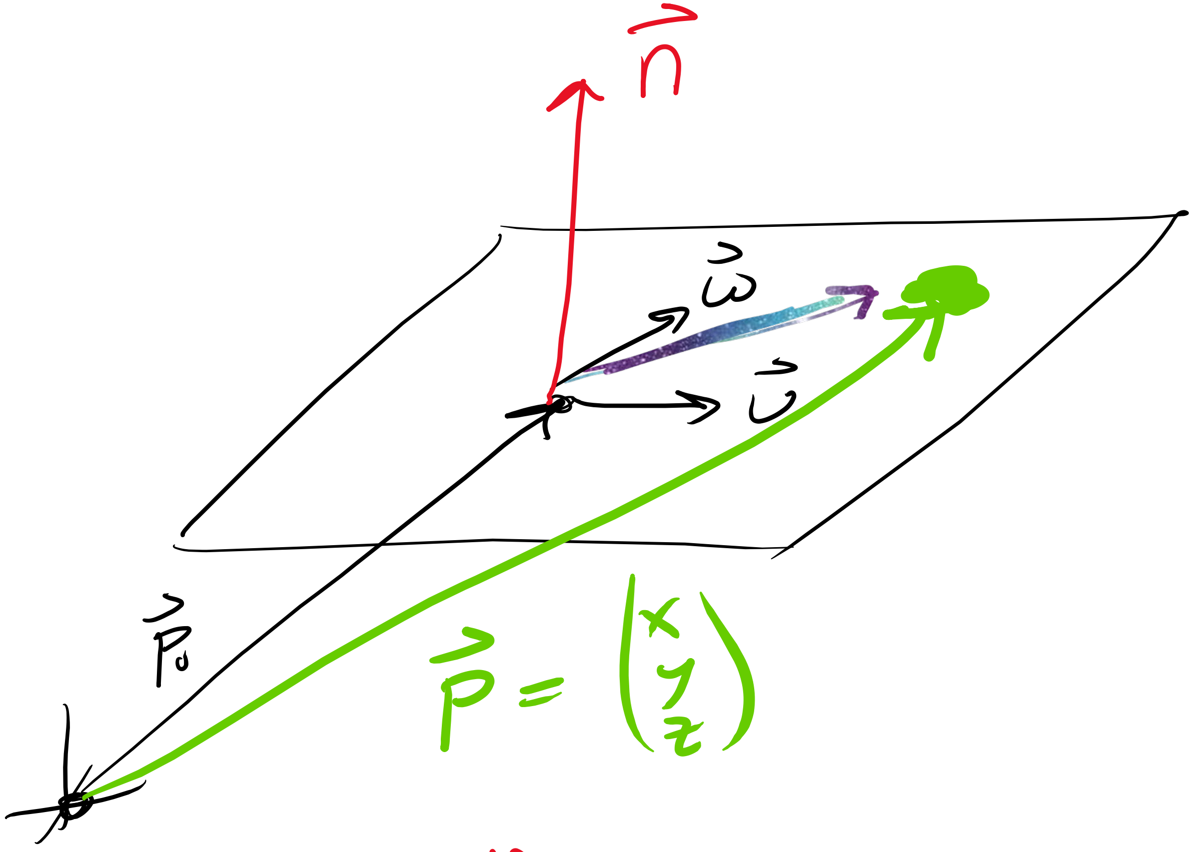
$$\Rightarrow z = z_0 + \underline{t(x,y)} v_3 + \underline{s(x,y)} w_3$$

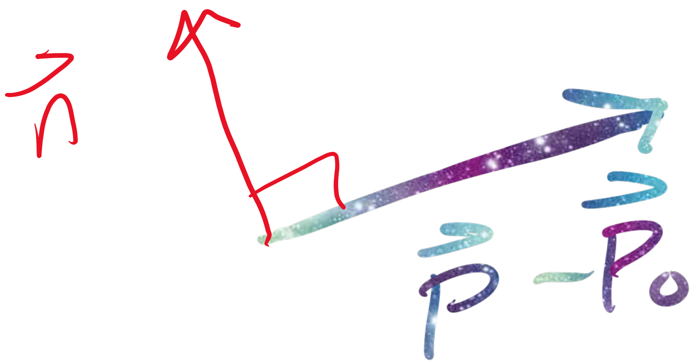
Linear in x and y

algebra

$$ax + by + cz = d$$

eq of a plane





$\vec{n}$ normal, it is $\perp$ to the plane

Could use $\vec{n} = \vec{v} \times \vec{w}$

unit normal use $\dfrac{\vec{v} \times \vec{w}}{\|\vec{v} \times \vec{w}\|}$

$$\vec{n} \cdot (\vec{P} - \vec{P}_0) = 0$$

$$\vec{n} \cdot \vec{P} - \vec{n} \cdot \vec{P}_0 = 0$$

$$\vec{n} \cdot \vec{P} = \underbrace{\vec{n} \cdot \vec{P}_0}_{d} \qquad \vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$ax + by + cz = d$$