Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 32: 5-4-2022: https://youtu.be/C3E4J2Q1OW4

slides:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture32.pdf

Plan for the day: Lecture 32: May 2, 2022:

Topics: Differential Equation

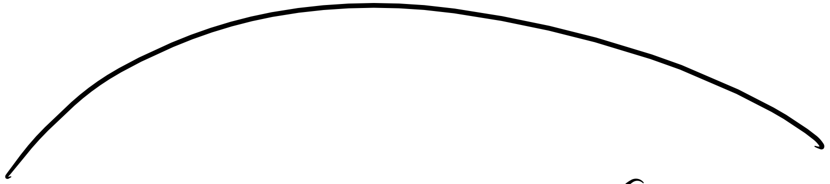
- **Solving differential equations in one variable if separable.**
- **Solving systems of differential equations.**
- **Application: Battle of Trafalgar.**

$F = ma$ $\qquad dv/dt = a(t)$ $\qquad dx/dt = v(t)$ $\qquad$ so $d^2x/dt^2 = a(t)$

$\downarrow F = mg$

$g \approx 9.8 \frac{m}{sec^2}$

$F = \frac{G m_1 m_2}{R^2}$ $\qquad g = \frac{G}{R^2}$



$\frac{dv}{dt} = g \Rightarrow \int dv = \int g\, dt \Rightarrow v = gt + C$

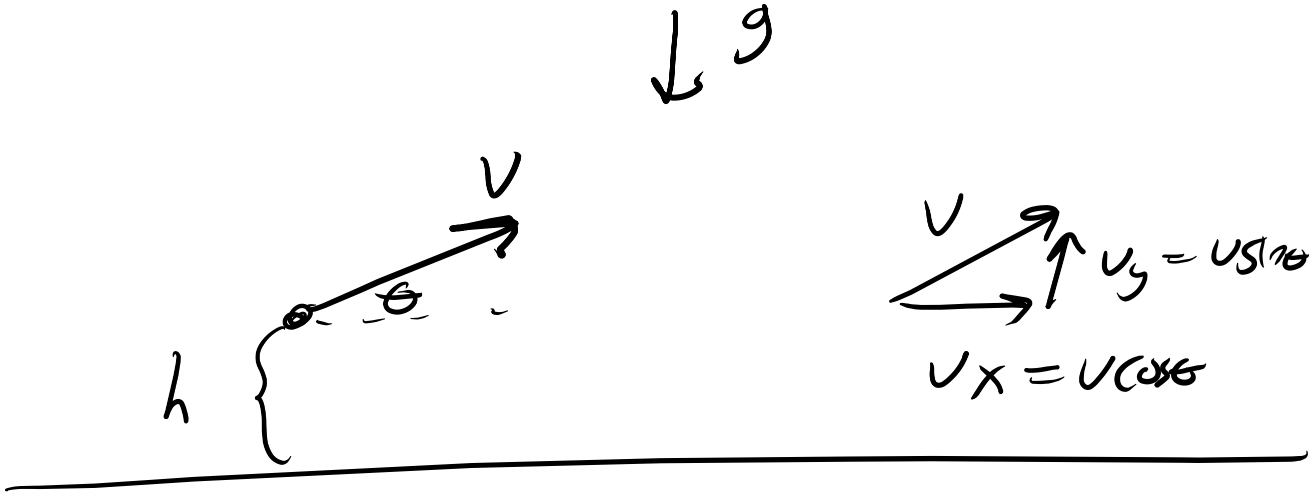
if speed is v_0 at time $t=0$, get $v(t) = gt + v_0$

$\frac{dy}{dt} = v = gt + v_0 \Rightarrow \int dy = \int (gt + v_0)\, dt$

$y = \frac{1}{2} g t^2 + v_0 t + C$

if at y_0 at time $t=0$ get

$y(t) = \frac{1}{2} g t^2 + v_0 t + y_0$



$$\frac{d^2y}{dt^2} = -9.8 \qquad y_0,\ v_y$$

$$\frac{d^2x}{dt^2} = 0 \qquad x_0,\ v_x$$

Imagine: $\frac{dy}{dt} = f(t)$ and we know $F' = f$

$$\int dy = \int f(t)\,dt$$

$$y = F(t) + C$$

$dy/dt = e^{t^2} \Longrightarrow \int dy = \int e^{t^2} dt$

$e^u = \sum_{n=0}^{\infty} u^n/n!$ so $\int e^{t^2} dt = \int \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} dt$ Fubini's it!

$= \sum_{n=0}^{\infty} \frac{1}{n!} \int t^{2n} dt = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)n!}$

$\frac{dy}{dt} = f(y,t) = f_1(y) f_2(t)$ Separable

$dy = f_1(y) f_2(t) \, dt$

$\int \frac{dy}{f_1(y)} = \int f_2(t) \, dt$

Define $\int$ by $\int f$ means $\int_0^x f(t)\,dt = F(x) - F(0)$ where $F' = f$

Solve $\int f = f - 1$ or $\int_0^x f(t)\,dt = f(x) - f(0)$. Answer: $f(x) = e^x$

Same as $f - \int f = 1$

$$\left(1 - \int\right) f = 1$$

$$\left(1-\int\right)^{-1}\left(1-\int\right) f = \left(1-\int\right)^{-1} 1$$

$$f = \left(1-\int\right)^{-1} 1$$

$$f = \left(1 + \int + \int\int + \int\int\int + \cdots\right) 1$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Recall: $1 + r + r^2 + \cdots = (1-r)^{-1}$, $|r| < 1$

Take $r = \int$

$$\int 1 = \int_0^x 1\,dt = x = \frac{x}{1!}$$

$$\int\int 1 = \int_0^x t\,dt = \frac{x^2}{2!}$$

$$\int\int\int 1 = \int_0^x \frac{t^2}{2!}\,dt = \frac{x^3}{3 \cdot 2!} = \frac{x^3}{3!}$$

https://en.wikipedia.org/wiki/Battle_of_Trafalgar

Battle of Trafalgar

From Wikipedia, the free encyclopedia

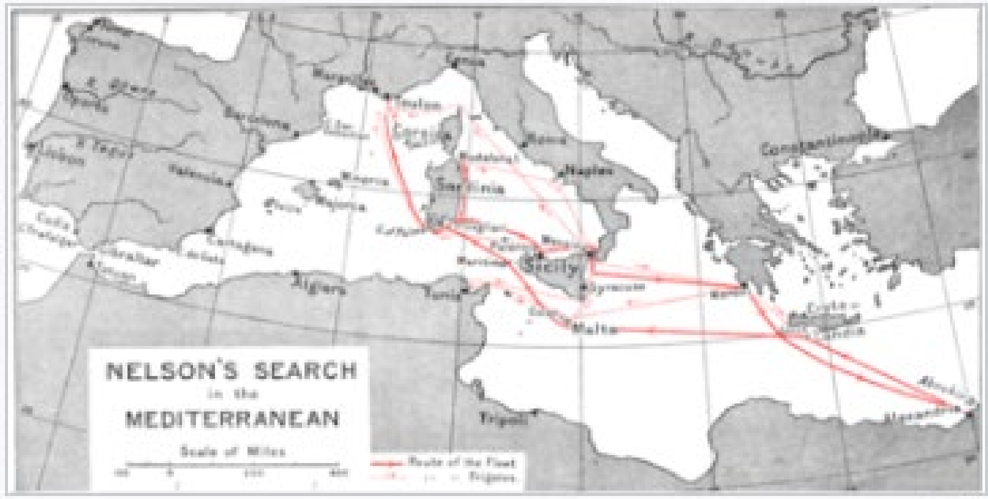
For the painting, see The Battle of Trafalgar (painting).

The **Battle of Trafalgar** (21 October 1805) was a naval engagement between the British Royal Navy and the combined fleets of the French and Spanish Navies during the War of the Third Coalition (August–December 1805) of the Napoleonic Wars (1803–1815).[4]

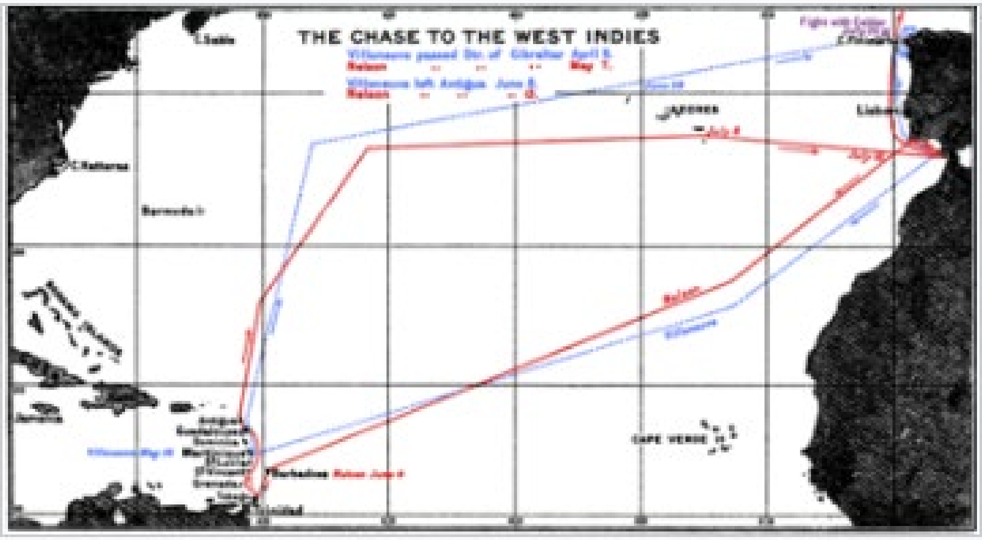
As part of Napoleon's plans to invade England, the French and Spanish fleets combined to take control of the English Channel and provide the Grande Armée safe passage. The allied fleet, under the command of French Admiral Villeneuve, sailed from the port of Cádiz in the south of Spain on 18 October 1805. They encountered the British fleet under Lord Nelson, recently assembled to meet this threat, in the Atlantic Ocean along the southwest coast of Spain, off Cape Trafalgar.

Nelson was outnumbered, with 27 British ships of the line to 33 allied ships including the largest warship in either fleet, the Spanish *Santisima Trinidad*. To address this imbalance, Nelson sailed his fleet directly at the allied battle line's flank, hoping to break it into pieces. Villeneuve had worried that Nelson might attempt this tactic but, for various reasons, had made no plans in case this occurred. The plan worked almost perfectly; Nelson's columns split the Franco-Spanish fleet in three, isolating the rear half from Villeneuve's flag aboard *Bucentaure*. The allied vanguard sailed off while it attempted to turn around, giving the British temporary superiority over the remainder of their fleet. The ensuing fierce battle resulted in 22 allied ships being lost, while the British lost none.

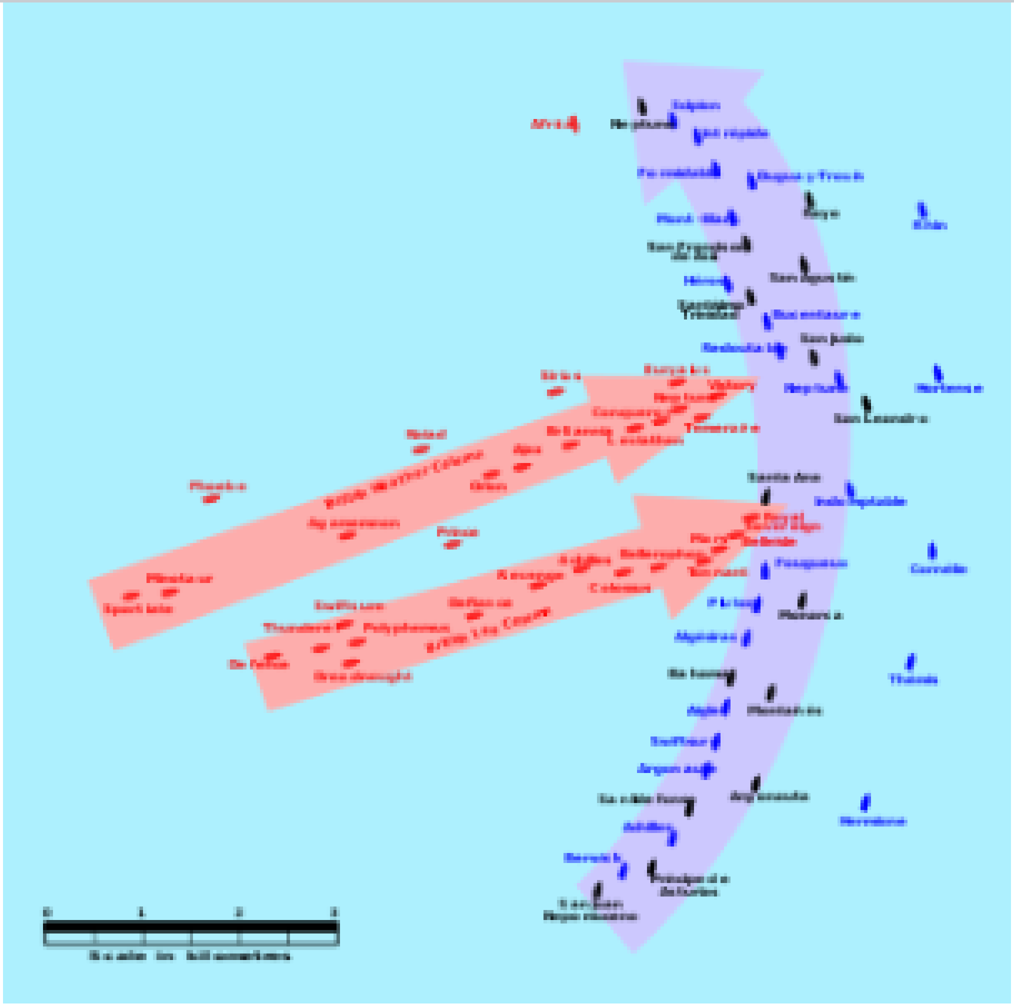
Date	21 October 1805
Location	Off Cape Trafalgar, Atlantic Ocean 36.293°N 6.255°W[1]
Result	British victory
Belligerents	
France Spain	United Kingdom
Commanders and leaders	
Pierre Villeneuve (POW) Federico Gravina (DOW)	Horatio Nelson † Cuthbert Collingwood
Strength	
33 ships of the line 5 frigates 2 brigs 35,000 men	27 ships of the line 4 frigates 1 schooner 1 cutter 20,000 men
Casualties and losses	
4,395 killed 2,541 wounded 7,000–8,000 captured 21 ships of the line captured 1 ship of the line destroyed.[2]	458 killed 1,208 wounded.[3]



Nelson's Search in the Mediterranean



The Chase to the West Indies



Artist's conception of the situation at noon as *Royal Sovereign* was breaking into the Franco-Spanish line

	British	Franco-Spanish
First rates	3	4
Second rates	4	0
Third rates	20	29
Total ships of the line	27	33
Other ships	6	7

Battle of Trafalgar by William Lionel Wyllie, Juno Tower, CFB Halifax, Nova Scotia, Canada

1 Mathematics in Warfare

By FREDERICK WILLIAM LANCHESTER

THE PRINCIPLE OF CONCENTRATION.
THE "N-SQUARE" LAW.

Taking, first, the ancient conditions where man is opposed to man, then, assuming the combatants to be of equal fighting value, and other conditions equal, clearly, on an average, as many of the "duels" that go to make up the whole fight will go one way as the other, and there will be about equal numbers killed of the forces engaged; so that if 1,000 men meet 1,000 men, it is of little or no importance whether a "Blue" force of 1,000 men meet a "Red" force of 1,000 men in a single pitched battle, or whether the whole "Blue" force concentrates on 500 of the "Red" force, and, having annihilated them, turns its attention to the other half; there will, presuming the "Reds" stand their ground to the last, be half the "Blue" force wiped out in the annihilation of the "Red" force [1] in the first battle, and the second battle will start on terms of equality—*i.e.*, 500 "Blue" against 500 "Red."

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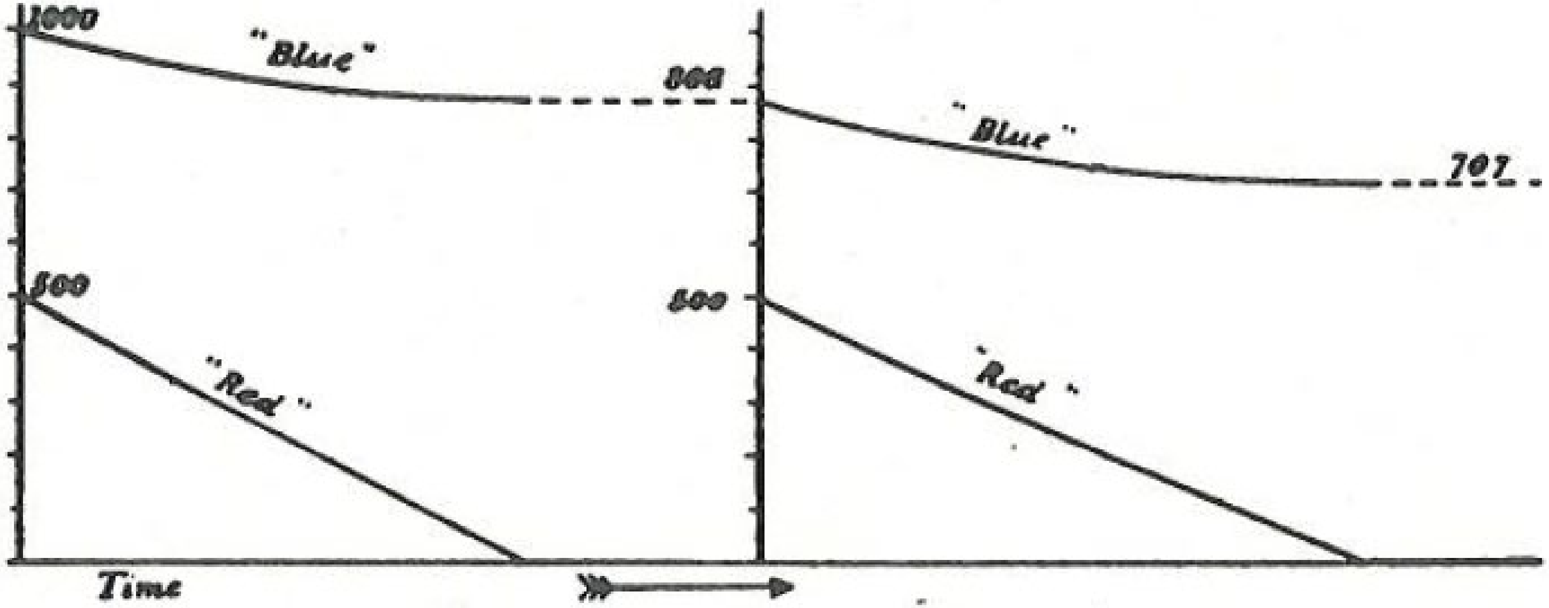
Modern Conditions Investigated. Now let us take the modern conditions. If, again, we assume equal individual fighting value, and the combatants otherwise (as to "cover," etc.) on terms of equality, each man will in a given time score, on an average, a certain number of hits that are effective; consequently, the number of men knocked out per unit time will be directly proportional to the numerical strength of the opposing force. Putting this in mathematical language, and employing symbol b to represent the numerical strength of the "Blue" force, and r for the "Red," we have:—

$$\frac{db}{dt} = -r \times c \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{dr}{dt} = -b \times k \quad . \quad . \quad . \quad . \quad (2)$$



(a) (b)

The reduction of strength of the two forces may be represented by two conjugate curves following the above equations. In Figure 1 (*a*) graphs are given representing the case of the "Blue" force 1,000 strong encountering a section of the "Red" force 500 strong, and it will be seen that the "Red" force is wiped out of existence with a loss of only about 134 men of the "Blue" force, leaving 866 to meet the remaining 500 of the "Red" force with an easy and decisive victory; this is shown in Figure 1 (*b*), the victorious "Blues" having annihilated the whole "Red" force of equal total strength with a loss of only 293 men.

The n – square law in warfare

Modified from Lanchester's article

The differential equations are db/dt = −r and dr/dt = −b.

The solutions to the system is easily determined as the equations are `separable',

namely we find d^2 b / dt^2 = b and d^2 r / dr^2 = r.

Following Lancaster,

we call the forces blue and red (and not English and French, or ...). We normalize so that there is initially one red unit,

and there are more blue units than red units.

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(* the lines below solve the equations and output the critical time where the red
 force is annihilated. *)
DSolve[{b'[t] == - r[t], r'[t] == - b[t], b[0] == B, r[0] == R}, {b, r}, t]
```

$$\left\{\left\{b \to \text{Function}\left[\{t\}, \frac{1}{2} e^{-t} \left(B + B e^{2t} + R - e^{2t} R\right)\right], r \to \text{Function}\left[\{t\}, -\frac{1}{2} e^{-t} \left(-B + B e^{2t} - R - e^{2t} R\right)\right]\right\}\right\}$$

```
Solve[-B + B e^(2 t) - R - e^(2 t) R == 0, t]
```

$$\left\{\left\{t \to \frac{1}{2}\left(2 i \pi c_1 + \text{Log}\left[\frac{B+R}{B-R}\right]\right) \text{ if } c_1 \in \mathbb{Z}\right\}\right\}$$

The blue function is $\frac{1}{2}(B+R)\, e^{-t} + \frac{1}{2}(B-R)\, e^{t}$.

The red function is $\frac{1}{2}(B+R)\, e^{-t} - \frac{1}{2}(B-R)\, e^{t}$.

$$\frac{db}{dt} = -r \times c \qquad \frac{dr}{dt} = -b \times k$$

$$\frac{db}{dt} = -c\, r(t) \qquad \frac{dr}{dt} = -k\, b(t)$$ take deriv wrt t

$$\frac{d^2b}{dt^2} = -c\frac{dr}{dt} \quad \text{and} \quad \frac{d^2r}{dt^2} = -k\frac{db}{dt}$$

$$\Longrightarrow \frac{d^2b}{dt^2} = (-c)(-k\, b(t)) = ck\, b(t)$$

: Solve $\frac{d^2b}{dt^2} = b''(t) = ck\, b(t)$

Try (Divine Inspiration): $b(t) = e^{at}$ Then $b'(t) = a e^{at}$ so $b''(t) = a^2 e^{at} = a^2 b(t)$

So $a^2 = ck \Rightarrow a = \sqrt{ck}$ or $-\sqrt{ck}$

General Soln: $b(t) = A_1 e^{\sqrt{ck}\, t} + A_2 e^{-\sqrt{ck}\, t}$

Maybe A_1 is zero as otherwise #ships goes to ∞ as time passes.... BUT battle ends after finite time