Math 150: Multivariable Calculus: MWF 9-9:50am: Spring 2023: Williams College

Professor Steven Miller (sjm1 AT williams.edu), Wachenheim 339

My Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/

Course Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp23/

Slides:
https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp23/Math150Sp23LectureNotes.pdf

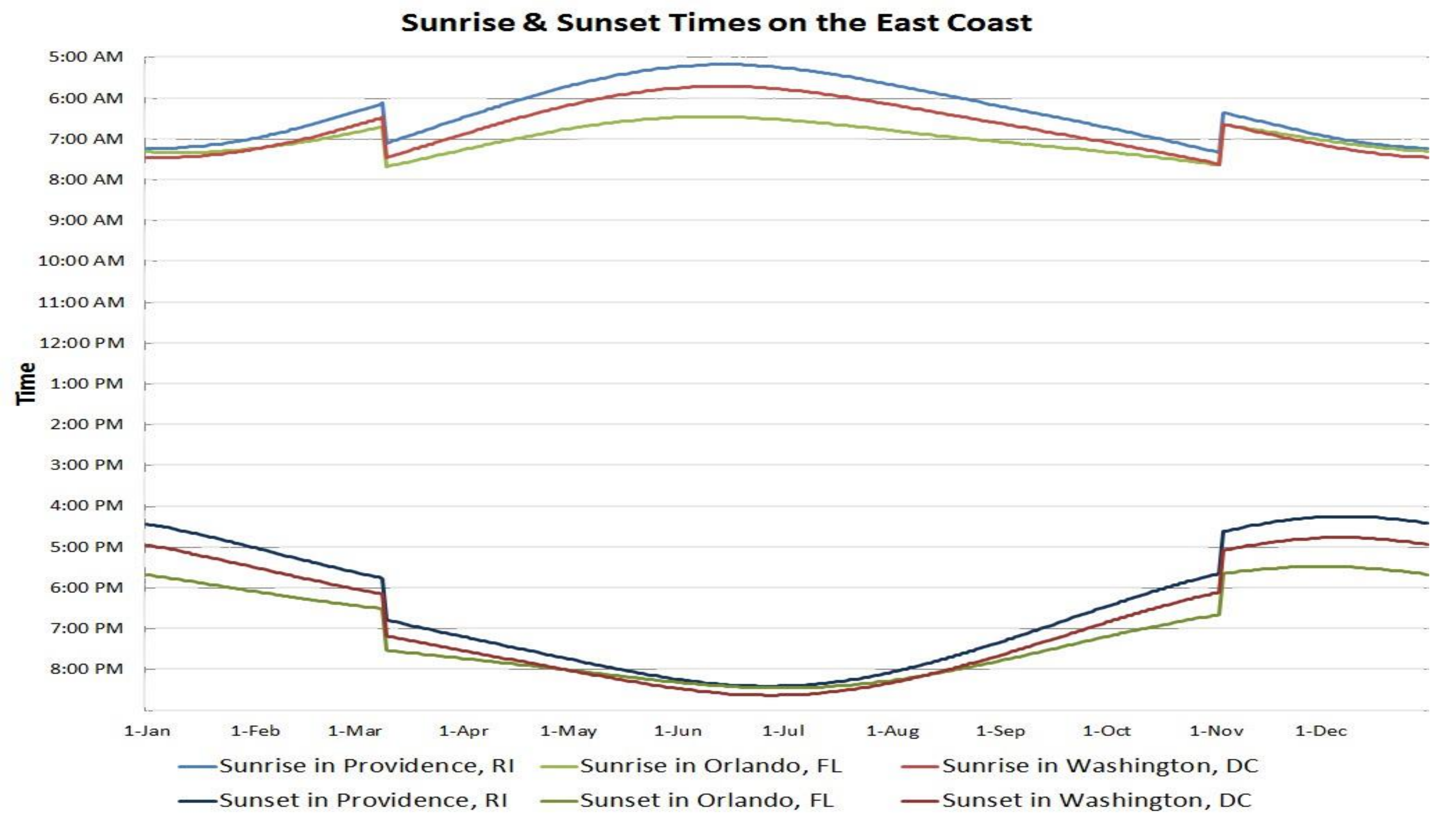
Other: Advice from Jeff Miller

- Party less than the person next to you.
- Take advantage of office hours / mentoring.
- Learn to manage your time: no one else wants to.

Happy to do practice interviews, adjust deadlines....

Lecture 1 Video: https://youtu.be/tEkxqje0_A8

Who America is rooting for in the Super Bowl:

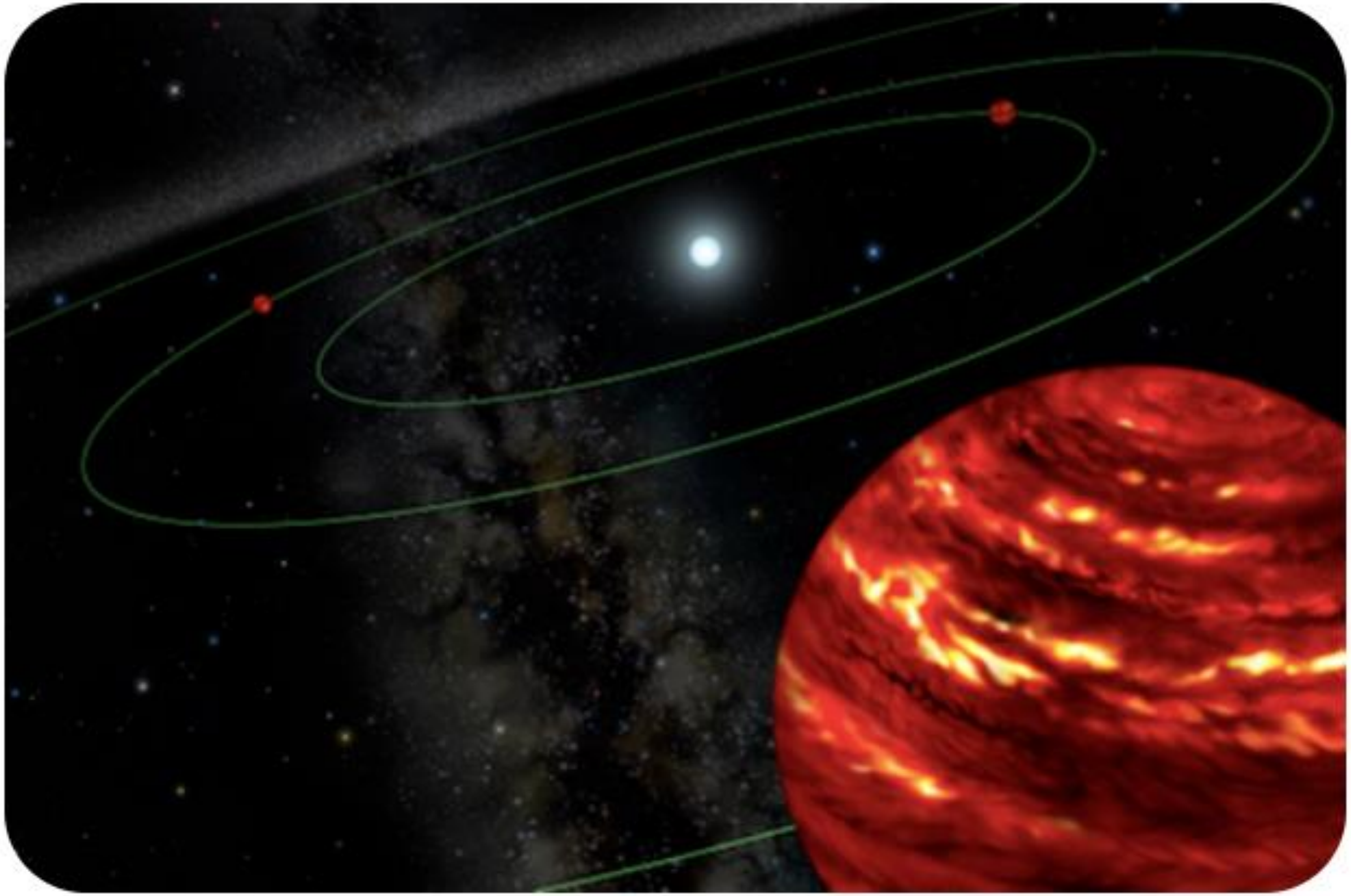

Sunrise & Sunset Times on the East Coast
5:00 AM
6:00 AM
7:00 AM
8:00 AM
9:00 AM
10:00 AM
11:00 AM
12:00 PM
1:00 PM
2:00 PM
3:00 PM
4:00 PM
5:00 PM
6:00 PM
7:00 PM
8:00 PM
Time
1-Jan
1-Feb
1-Mar
1-Apr
1-May
1-Jun
1-Jul
1-Aug
1-Sep
1-Oct
1-Nov
1-Dec
Sunrise in Providence, RI
Sunrise in Orlando, FL
Sunrise in Washington, DC
Sunset in Providence, RI
Sunset in Orlando, FL
Sunset in Washington, DC

Plan for the day: Lecture 1: February 3, 2023:

- Discuss motivation of calculus
- Motivate integration: passing to the limit of a sum
- Dangers of extrapolating and what happens when you ass|u|me.

$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$ $\frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}$ $\frac{4\cancel{9}}{\cancel{9}8} = \frac{1}{2}$ $\frac{1\cancel{2}}{\cancel{2}4} = \frac{1}{4}$

- Review Calc I and II.



In this illustration, you can see three young planets tracing orbits around a star called HR 8799 that lies about 130 light-years from Earth. Image credit: Gemini Observatory Artwork by Lynette Cook https://spaceplace.nasa.gov/other-solar-systems/en/

Newton's Law of Gravity

$$F = G\frac{m_1 m_2}{r^2}$$

F = force

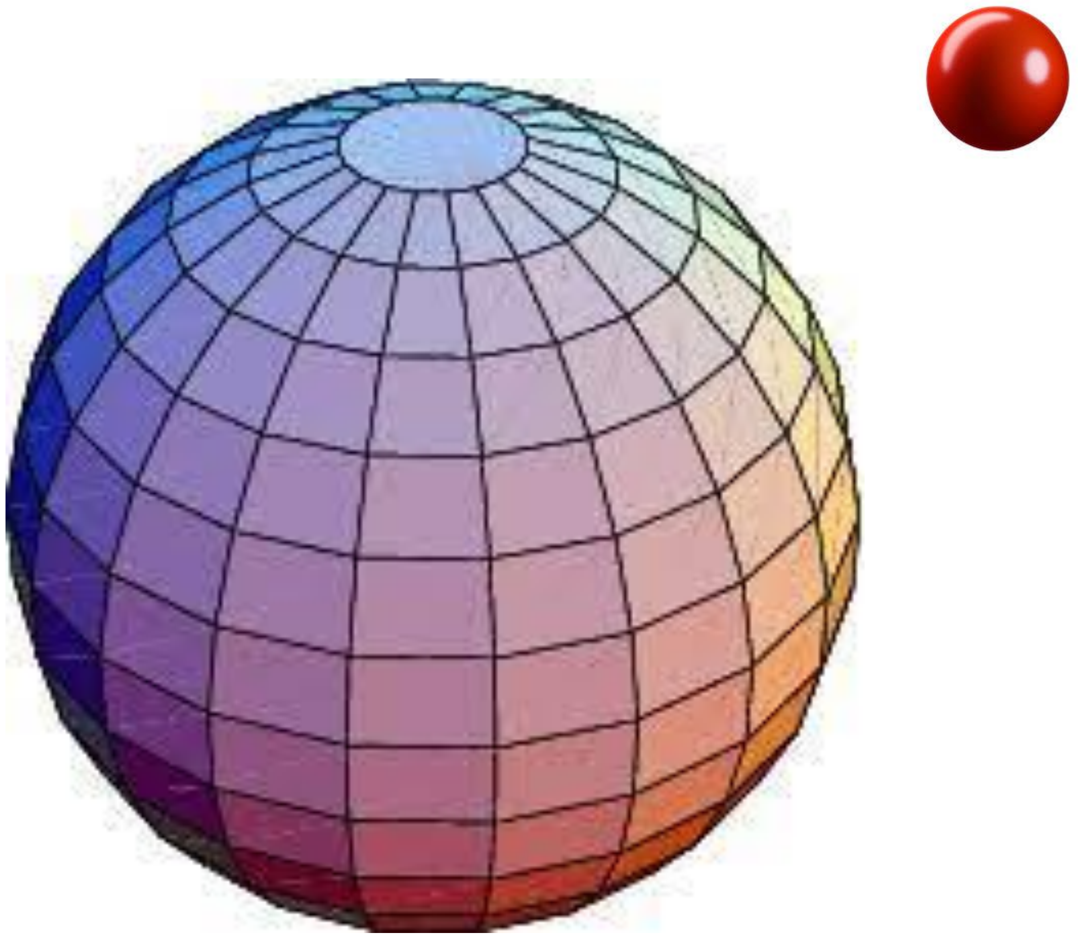
G = gravitational constant

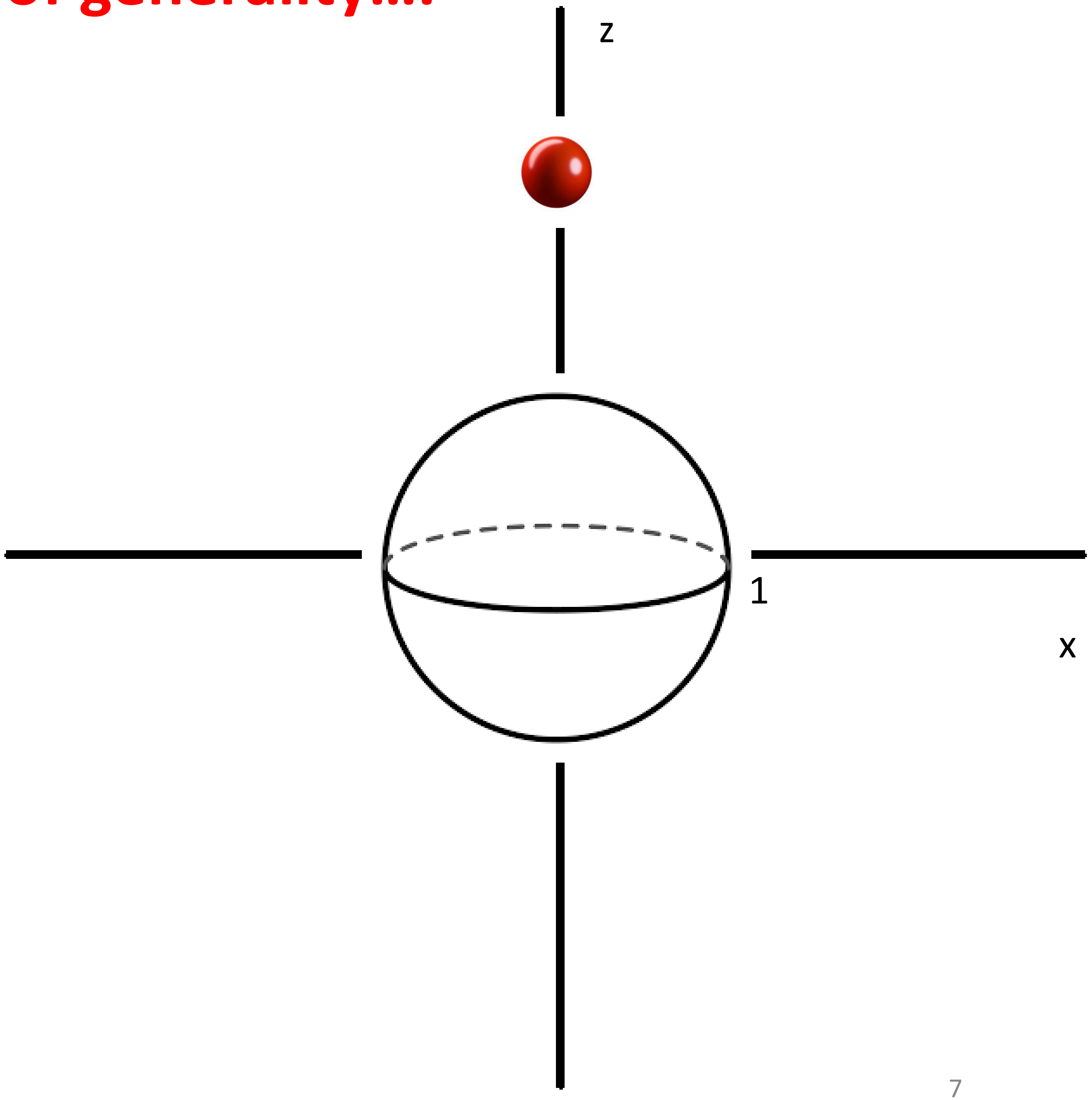
m_1 = mass of object 1

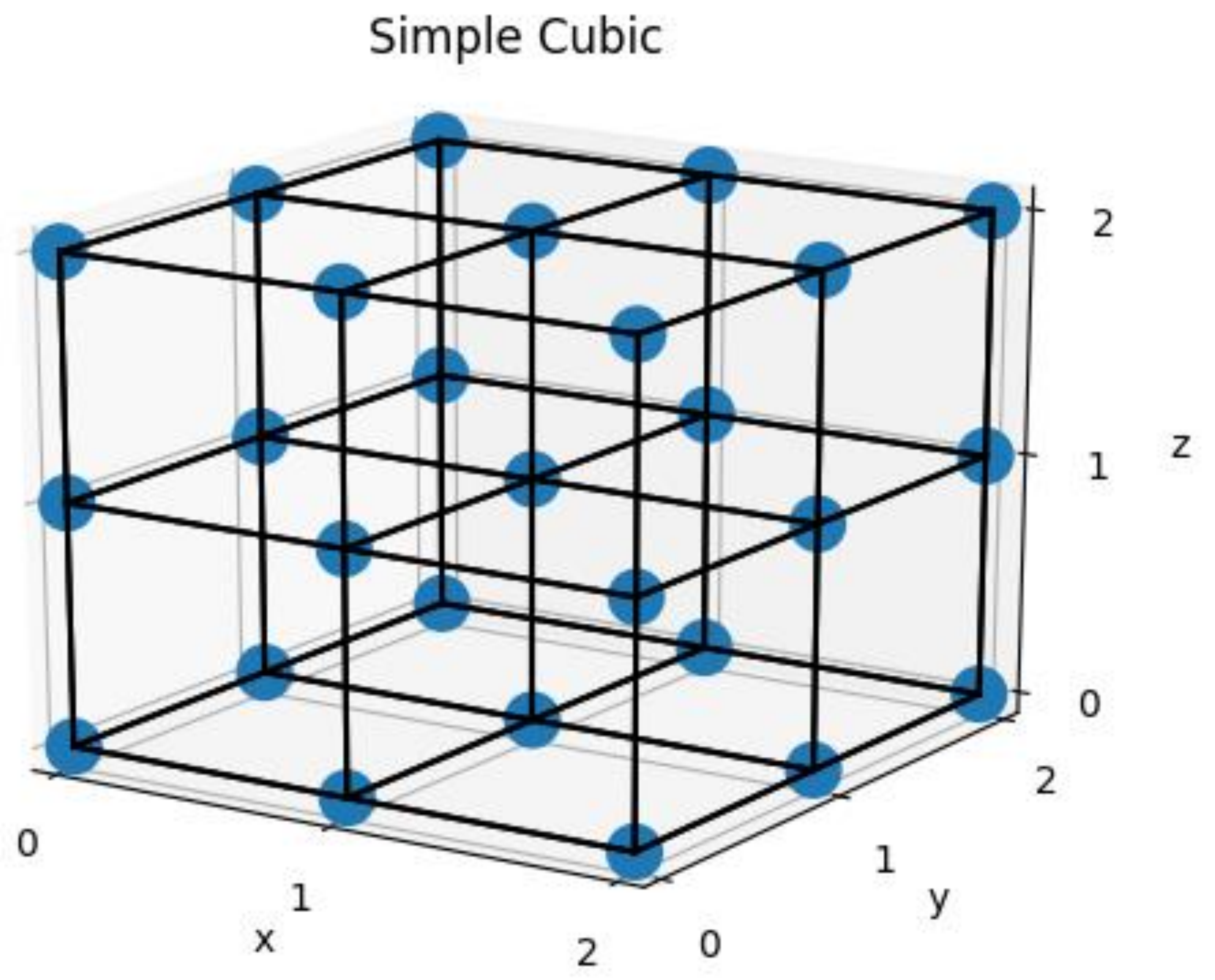
m_2 = mass of object 2

r = distance between centers of the masses

Symmetry Arguments: Without loss of generality….







https://www.juliabloggers.com/computationally-visualizing-crystals/

```
symmgravityapprox[range_, height_, printme_] := Module[{},
  (* we assume the object is "height" units above the center of a sphere of radius 1 *)
  force = 0;
  numpoints = 0;
  For[x = 0, x ≤ range, x++,
   {
    If[printme == 1, If[Mod[x, range / 10] == 0, Print["Have done ", x, " of ", range, "."]]];
    For[y = 0, y ≤ range, y++,
     For[z = -range, z ≤ range, z++,
       {
        (* only find contribution if point in sphere *)
        If[x^2 + y^2 + z^2 ≤ range^2,
          {
           distsquared = (x / range)^2 + (y / range)^2 + (z / range - height)^2;
           (* two vectors: (0,0,-height) and (x/range,y/range,z/range-height) *)
           (* dot product is product of lengths times cos(angle) *)
           (* we take the force and multiply by cos(angle) *)
           contribution = (z / range - height) * (-height) / (distsquared * height * Sqrt[distsquared]);
           multiplier = (Sign[x]^2 + 1) * (Sign[y]^2 + 1);
           force = force + contribution * multiplier;
           numpoints = numpoints + multiplier;
          }]; (* end of if loop *)
       }]; (* end of z *)
    ]; (* end of y *)
   }]; (* end of x *)
  If[printme == 1,
   {
    Print["Discrete approx: ", 1.0 force / numpoints];
    Print["Theory: ", 1.0 / height^2];
    Print["Numpoints inside = ", numpoints];
    Print["Predicted numpoints inside = ", 1.0 ((4 Pi / 3) / 8) * (2 range + 1)^3];
   }];
  Return[{1.0 force / numpoints, 1.0 / height^2}];
 ]
```

Notice ranges:

x, y from 0 to range
z from –range to range.

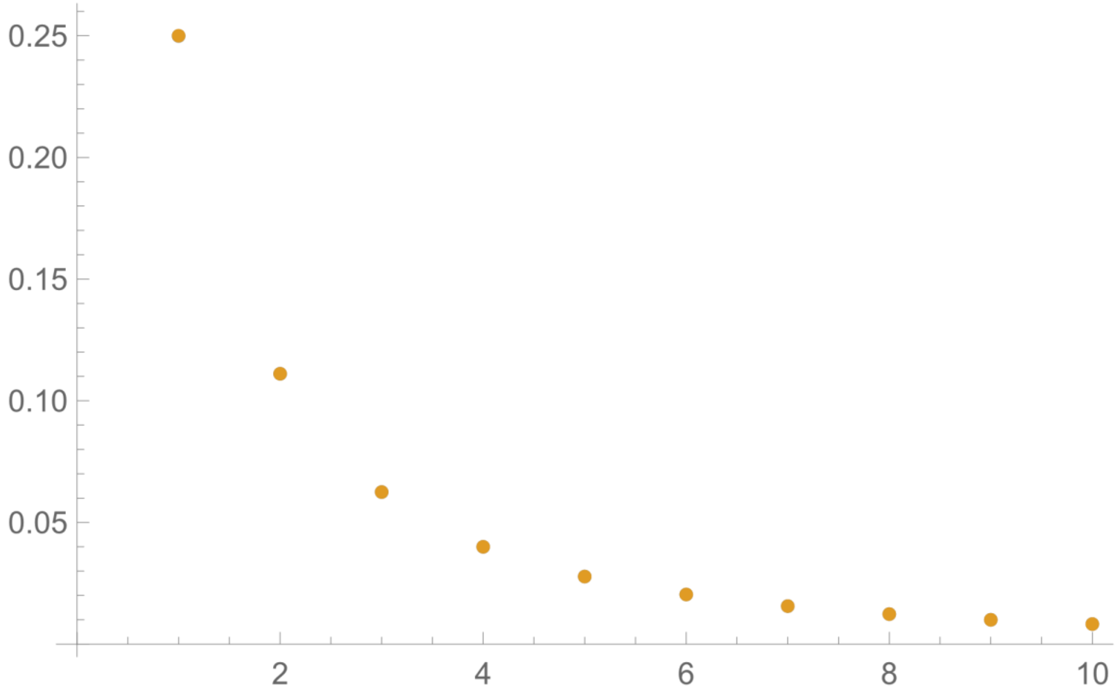
Reason is the force is down.

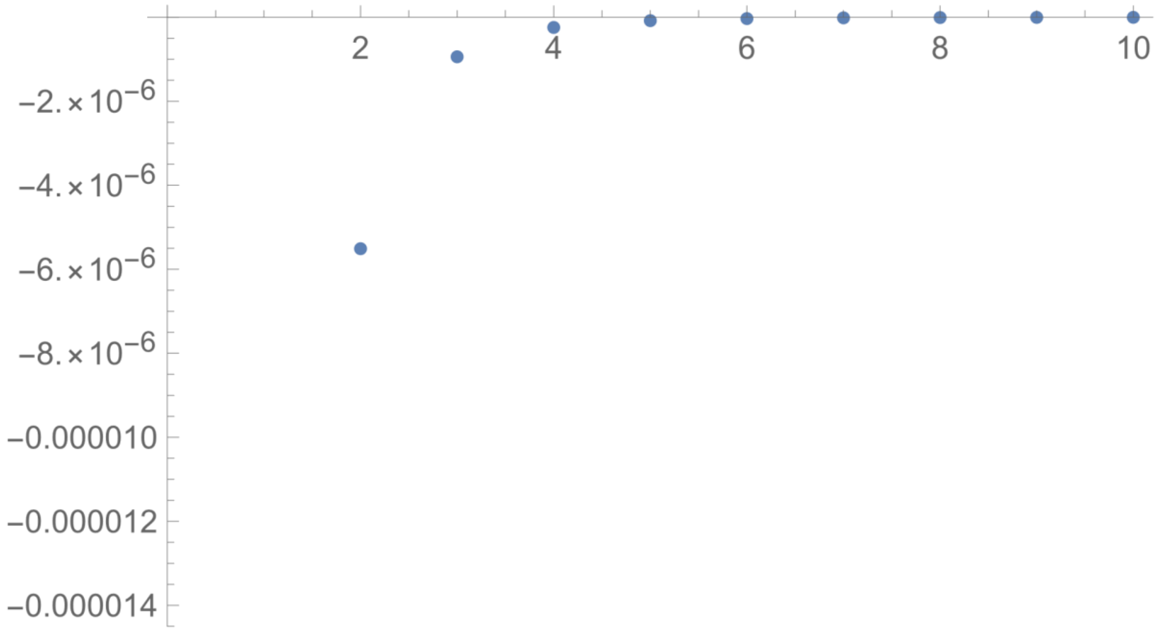
The following four points have the same contribution:

- (x, y, z)
- (-x, y, z)
- (x, -y, z)
- (-x, -y, z)

Thus saves a factor of four if compute contribution of one of these and multiply by 4.

Note if x or y is zero would multiply by 2 (if both are zero multiply by 1).





Comparing the gravitational force on an object at height h above the north pole of a unit sphere two ways:

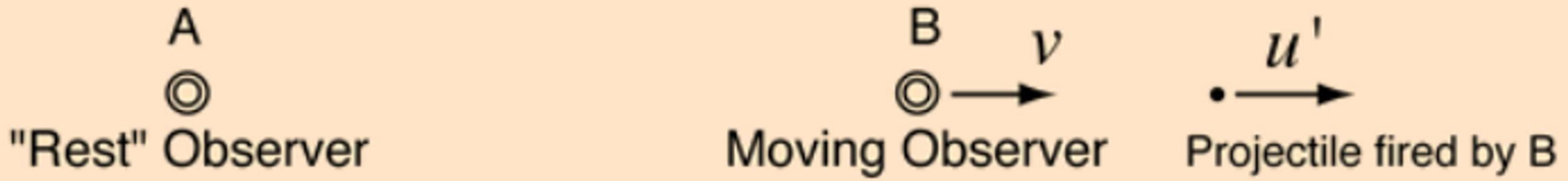
(1) all the mass is at the center,

(2) compute the force from points at (x/N, y/N, z/N) for x, y and z integers.

The left is the plot of both, the right is the difference between the two.

Einstein Velocity Addition

The relative velocity of any two objects never exceeds the velocity of light. Applying the Lorentz transformation to the velocities, expressions are obtained for the relative velocities as seen by the different observers. They are called the Einstein velocity addition relationships.



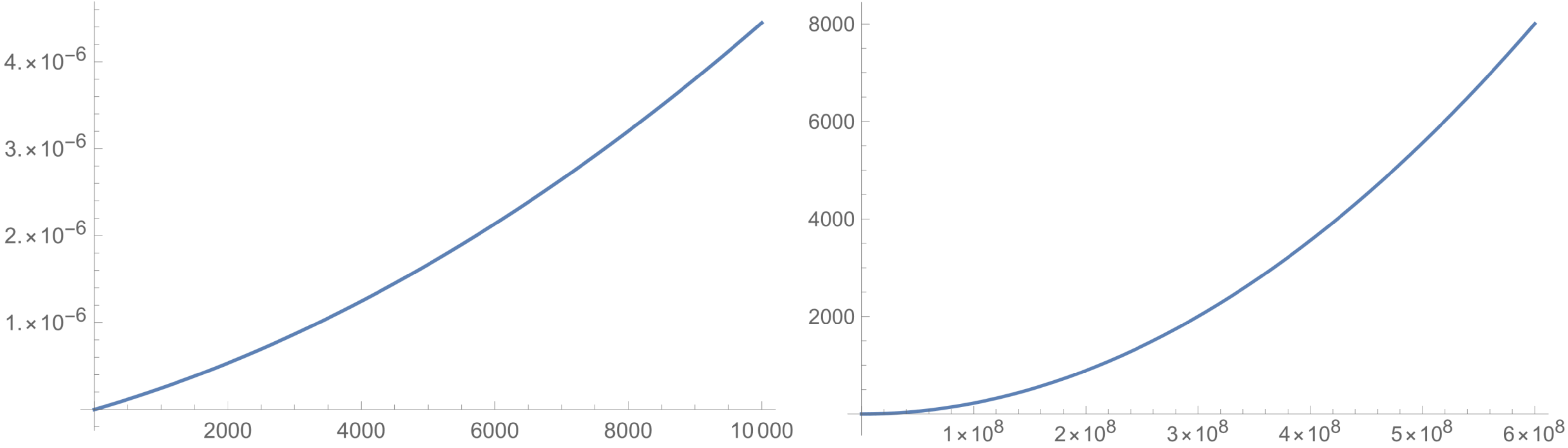
$$u = \frac{v + u'}{1 + \frac{vu'}{c^2}}$$

u' = velocity of projectile as seen by B

u = velocity of projectile as seen by A

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

http://hyperphysics.phy-astr.gsu.edu/hbase/Relativ/einvel.html



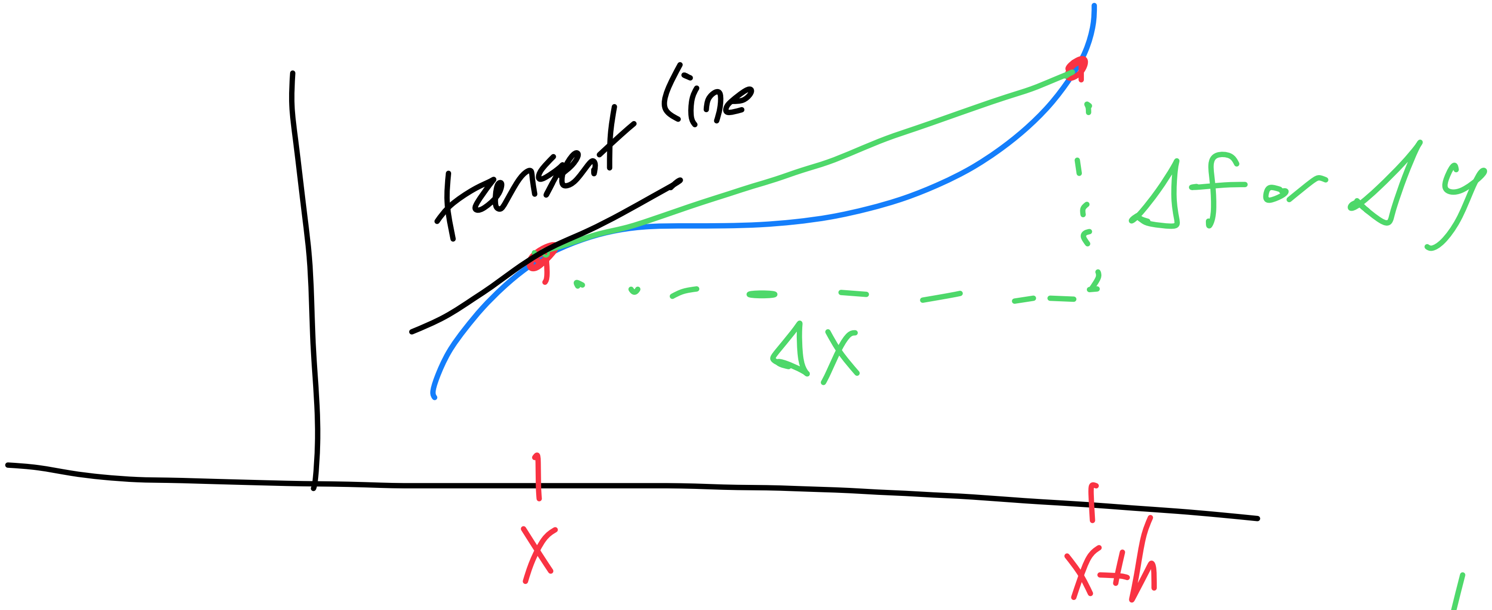
Plotting the difference between the Einstein correction and the classical prediction for adding two speeds.

We throw a projectile forward on a train (or rocket ship) traveling in the same direction at 10,000 mph.

The x-axis is the speed of the thrown object, the y-axis is the difference between the relativistic correction and the classical prediction. Note the order of the error for speeds up to 10,000 mph is on the same order as our integration approximation!

Note the Apollo 11's fastest speed was about 25,000 miles per hour! Lightspeed is about 6.7 x 10^8 mph.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{df}{dx}$$



average rate of change from x to $x+h$ is

$$\frac{f(x+h) - f(x)}{x+h-x}$$

h

Math 150: Multivariable Calculus: Spring 2023: Lecture 02: Review of Calc I https://youtu.be/C73M7A-KN54

Plan for the day.

- Discuss how information is presented (theme of the class!).

- Discuss how one does calculations (another theme of the term!).

- Review Calculus I and II (if time permits).

Images from the National World War II Museum – New Orleans: https://www.nationalww2museum.org/

WAR DEPARTMENT
AAF FORM NO. 5
APPROVED DEC. 7, 194[illegible]

[IN]DIVIDUAL FLIGHT REC[OR]D

(1) SERIAL NO. 0-361713 (2) NAME TIBBETS, PAUL W. JR. (3) RANK Colonel (4) AGE 1917
LAST MIDDLE
(5) PERS. CLASS 01 (6) BRANCH Air Corps (7) STATION APO 336,
(8) ORGANIZATION ASSIGNED 20th 313th 509th
AIR FORCE COMMAND WING GROUP SQUADRON ATTACHED FOR FLYING
(9) ORGANIZATION ATTACHED
AIR FORCE COMMAND WING GROUP SQUADRON DETACHMENT
(10) PRESENT RATING & DATE Sr. Pilot 6-2-43 (11) ORIGINAL RATING & DATE Pilot 2-16-38
(12) TRANSFERRED FROM
(13) FLIGHT RESTRICTIONS None
(15) TRANSFERRED TO
(14) TRANSFER DATE
(16)
DO NOT WRITE IN THIS SPACE

PERS CLASS	RANK	RTG.	A. F.	COMMAND	WING	GROUP NO.	GROUP TYPE	SQUADRON NO.	SQUADRON TYPE	STATION	MO.	YR.
:			:	: :	: :	: :	: :	: : :	: :	: : :		

(17) MONTH August 1945

DAY 18	AIRCRAFT TYPE, MODEL & SERIES 19	NO. LANDINGS 20	FLYING INST. (INCL. IN 1ST PIL. TIME) S 21	~~XXXXX~~ CA 22	CO-PILOT CP 23	QUALIFIED PILOT DUAL QD 24	FIRST PILOT DAY P 25	FIRST PILOT NIGHT P N OR NI 26	RATED PERS. NON-PILOT 27	28	29	NON-RATED OTHER ARMS & SERVICES 30	NON-RATED OTHER CREW & PASS'GR 31	SPECIAL INFORMATION INSTRUMENT I 32	NIGHT N 33	INSTRUMENT TRAINER 34	PILOT NON-MIL. AIRCRAFT OVER 400 H.P. 35	PILOT NON-MIL. AIRCRAFT UNDER 400 H.P. 36
6	B-29	1		12:15														
24	C-54	1					7:25								3:00			
26	C-54	2					7:05								6:05			
COLUMN TOTALS		4		12:15			14:30								9:05			

CERTIFIED CORRECT:
George W Marquardt
GEORGE W. MARQUARDT,
Captain, Air Corps,
Asst. Operations Officer

	(42) TOTAL STUDENT PILOT TIME	(43) TOTAL FIRST PILOT TIME		(44) TOTAL PILOT TIME		
(37) THIS MONTH		14:30	0:00	26:45		
(38) PREVIOUS MONTHS THIS F. Y.		97:00	0:00	105:40		
(39) THIS FISCAL YEAR		111:30	0:00	132:25		
(40) PREVIOUS FISCAL YEARS	296:45	3726:55	46:20	3768:25		3:50
(41) TO DATE	296:45	3838:25	46:20	3900:50	3577:45	3:50

AIRCRAFT 19	NL 20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
B-29	1		12														
C-54	3					15											

CARD NO. 1 — CARD NO. 2 — CARD NO. 3

C-54 14:30
B-29 12:15

WAR DEPARTMENT
AAF FORM NO. 5
APPROVED DEC. 7, 1941

INDIVIDUAL FLIGHT RECORD

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(5) PERS. CLASS 01 (6) BRANCH Air Corps

(8) ORGANIZATION ASSIGNED 20th (AIR FORCE) (COMMAND) 313th (WING) 509th (GROUP) (SQUADRON) (7) STATION APO 336, (ATTACHED FOR FLYING)

(9) ORGANIZATION ATTACHED (AIR FORCE) (COMMAND) (WING) (GROUP) (SQUADRON) (DETACHMENT)

(10) PRESENT RATING & DATE Sr. Pilot 6-2-43

(12) TRANSFERRED FROM (11) ORIGINAL RATING & DATE Pilot 2-16-38

(15) TRANSFERRED TO (13) FLIGHT RESTRICTIONS None

(16) (14) TRANSFER DATE

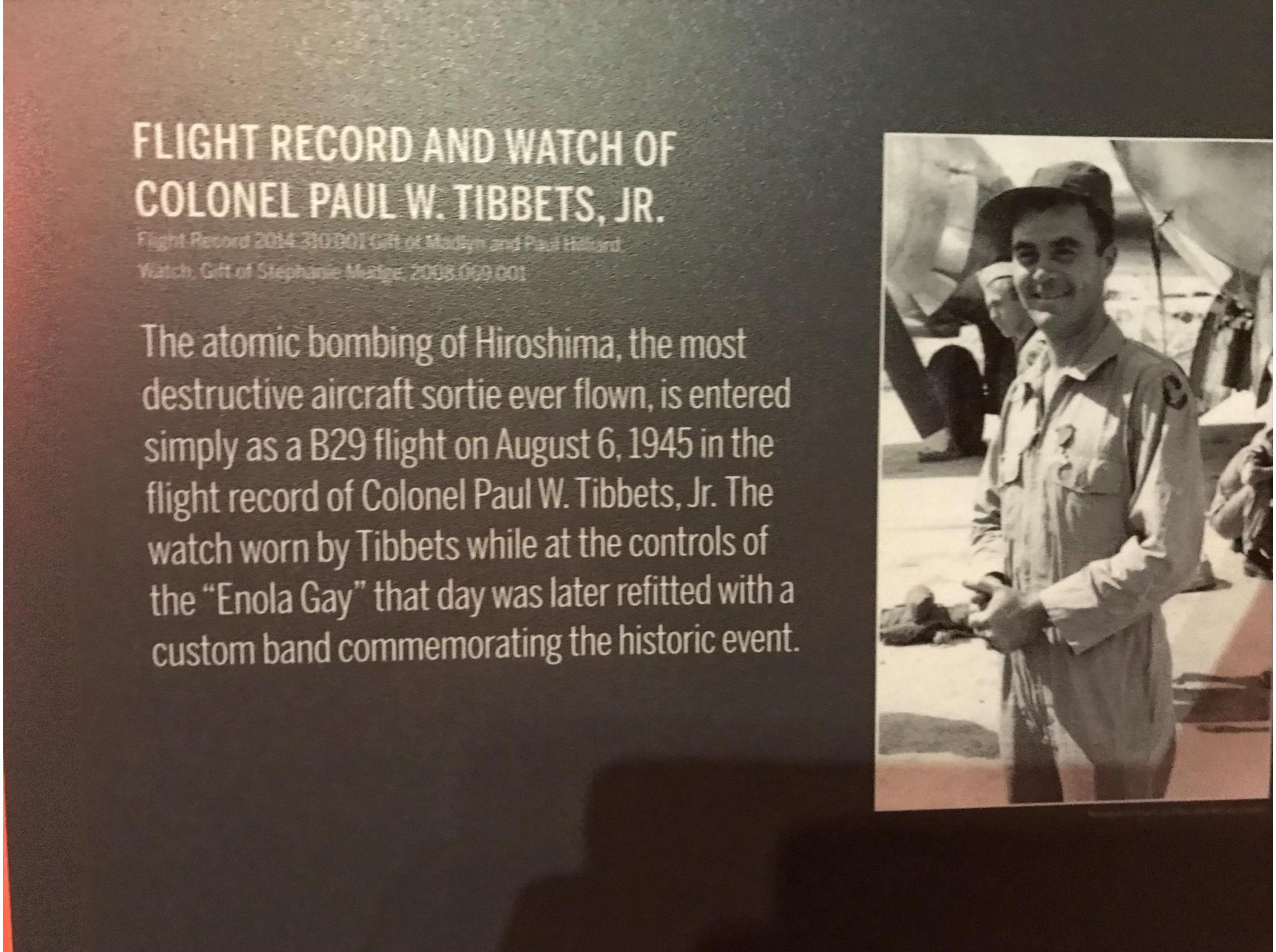
DO NOT WRITE IN THIS SPACE

PERS CLASS	RANK	RTG.	A. F.	COMMAND	WING	GROUP NO.	GROUP TYPE	SQUADRON NO.	SQUADRON TYPE	STATION	MO.	YR.

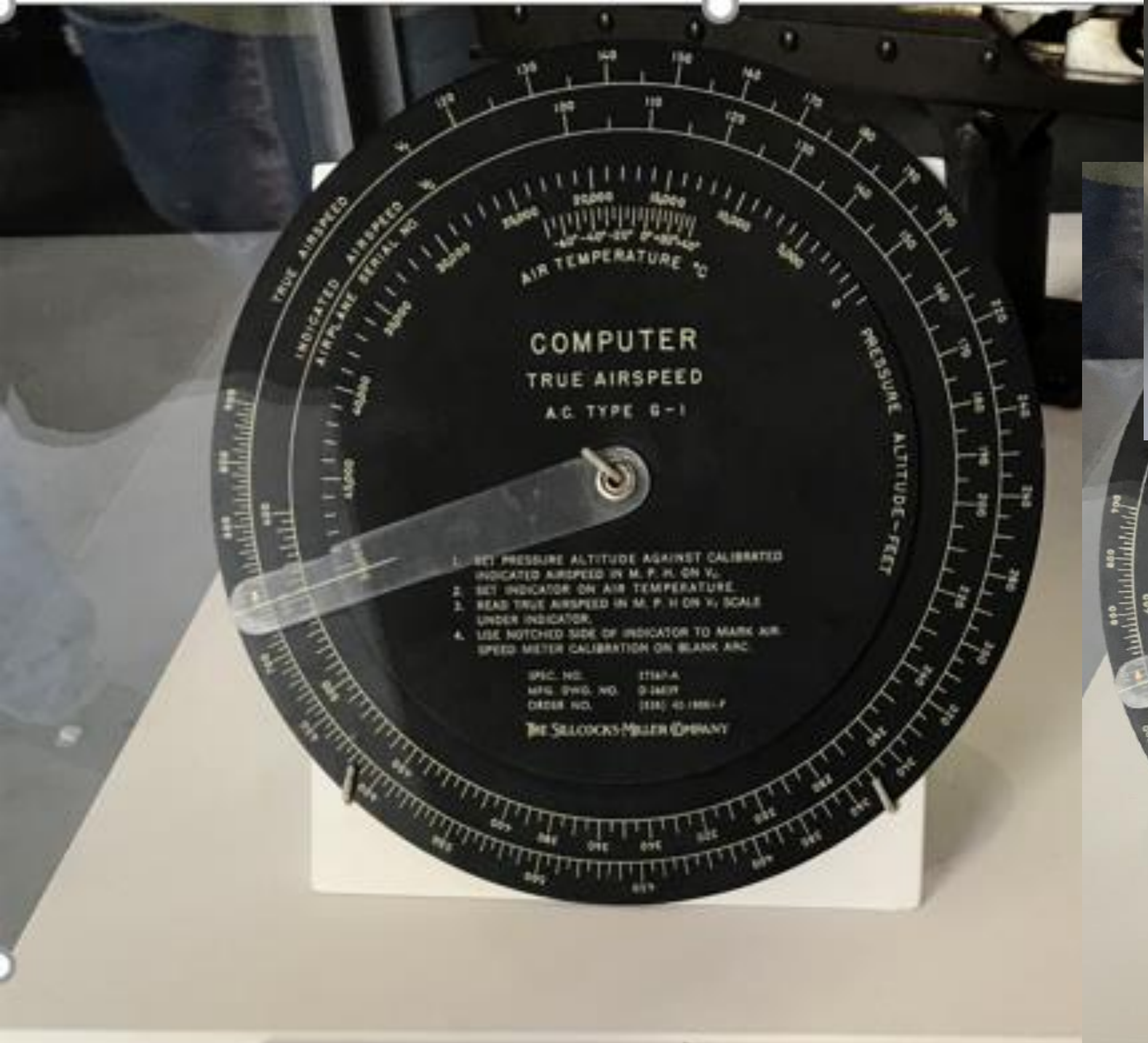
(17) MONTH August 1945

DAY	AIRCRAFT TYPE, MODEL & SERIES	NO. LANDINGS	FLYING INST. (INCL. IN 1ST PIL. TIME) S	CA	CO-PILOT CP	QUALIFIED PILOT DUAL QD	FIRST PILOT DAY P	FIRST PILOT NIGHT P N OR NI	RATED PERS. NON-PILOT			NON-RATED OTHER ARMS & SERVICES	NON-RATED OTHER CREW & PASS'GR	SPECIAL INFORMATION INSTRUMENT I	NIGHT N	INSTRUMENT TRAINER	PILOT NON-MIL. AIRCRAFT OVER 400 H.P.	PILOT NON-MIL. AIRCRAFT UNDER 400 H.P.
18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
6	B-29	1		12:15														
24	C-54	1					7:25								3:00			
26	C-54	2					7:05								6:05			

Six degrees of separation….



Oracle of Bacon: https://www.oracleofbacon.org/

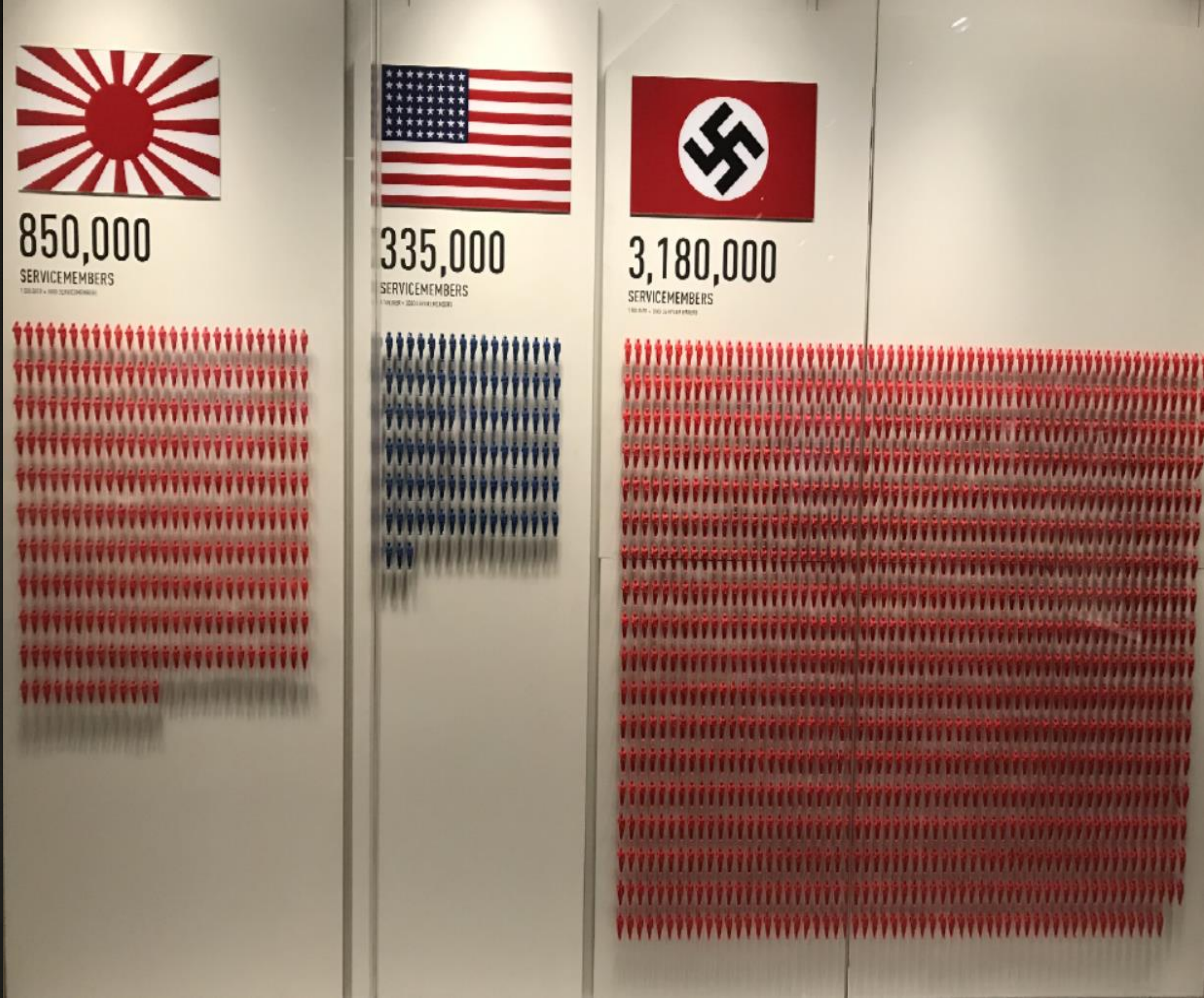


TRUE AIRSPEED COMPUTER

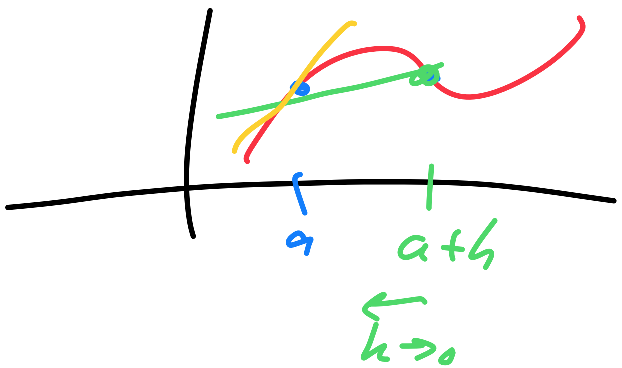
True airspeed is the aircraft's speed in relation to the ground. Due to the influence of air pressure at various altitudes, the indicated airspeed the pilot sees on his instruments is different than true airspeed. For the sake of accuracy, bombardiers had to enter the aircraft's speed in relation to the ground into the bombsight. This circular slide rule computer would quickly calculate true airspeed and is still used by pilots today.

MILITARY STRENGTH

When World War II broke out in 1939, the United States was not a great military power. The number of US service personnel was just 335,000, and the US Army was comparable in size to much smaller states like Bulgaria, Portugal, and Romania. Equipment was so scarce that only a tiny fraction of US troops had ever trained with modern weapons. By contrast, Germany had been rapidly rebuilding its military strength since 1933, and had more than three million men under arms. Japan, fighting an all-out war of conquest in China since 1937, had 850,000 men in the field. The world had become a dangerous place, and the US was dangerously unready.



Definition of the derivative: Standard, and what will generalize well….



$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

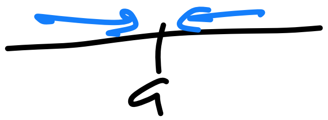
let $x = a+h$

as $h\to 0$, $x\to a$ and vice-versa

Same as $\displaystyle\lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{x-a} = 0$

multiplied by 1, ok as $x\to a$ but is never a!

$$\lim_{x\to a}\frac{f(x) - \text{tangent line approx}}{x-a}$$ goes to zero



error is small relative to elapsed time $x-a$

$f(x) = x^{1/2}$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{1/2} - x^{1/2}}{h} * \frac{(x+h)^{1/2} + x^{1/2}}{(x+h)^{1/2} + x^{1/2}}$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h\left((x+h)^{1/2} + x^{1/2}\right)} = \lim_{h\to 0} \frac{1}{(x+h)^{\frac{1}{2}} + x^{\frac{1}{2}}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2} x^{-\frac{1}{2}}$$

$g(x) = f(x)^q$ if $f(x) = x^{p/q}$ q pos integer so $g(x) = x^p$

$g'(x) = q f(x)^{q-1} f'(x)$ so $f'(x) = \dfrac{g'(x)}{q f(x)^{q-1}} = \dfrac{p x^{p-1}}{q x^{p(q-1)/q}}$

now do alg, see get $f'(x) = p/q * x^{p/q - 1}$

$g(x) = x^{\sqrt{2}} = e^{f(x)} \longrightarrow \ln(x^{\sqrt{2}}) = \ln(e^{f(x)})$

so $\sqrt{2} \ln x = f(x)$ so $g(x) = e^{\sqrt{2}\ln x}$

thus $g'(x) = e^{\sqrt{2}\ln x} \cdot \sqrt{2}\frac{1}{x} = \sqrt{2} x^{\sqrt{2}-1}$

$f(x)$	$f'(x)$
x^3	$3x^2$
$x^{3/2}$	$\frac{3}{2}x^{1/2}$
$x^{\sqrt{2}}$	$\sqrt{2}\,x^{\sqrt{2}-1}$
x^r	$r x^{r-1}$

PROOFS

$$\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

Pascal / Binomial Thm

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$(x+h)^3 = x^3 + 3x^2h + (\text{crap with at least } h^2)$$

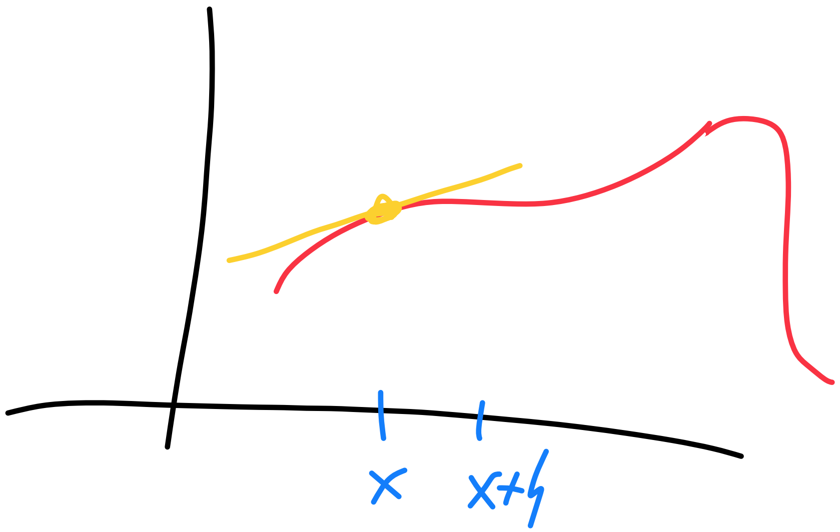
$$\underline{-x^3} \qquad\qquad \underline{-x^3}$$

$$\frac{\text{num}}{h} = \frac{3x^2h}{h} + \frac{(\text{crap } h^2 \text{ and higher})}{h}$$

has h or higher

Meaning of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



If $f'(x) > 0$ Then f is ↑ to right
and ↓ to left

If $f'(x) < 0$ Then f is ↓ to right and ↑ to left

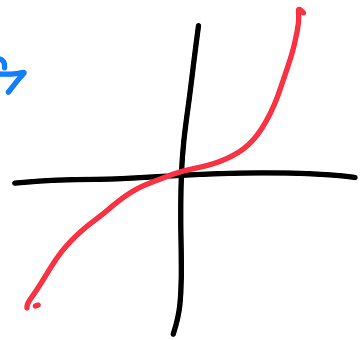
$f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

If $f'(x) = 0$ say x is a critical point

This is a CANDIDATE for a
local max/min



A.2.1 Intermediate Value Theorem

Theorem A.2.1 (Intermediate Value Theorem (IVT))**.** *Let f be a continuous function on $[a, b]$. For all C between $f(a)$ and $f(b)$ there exists a $c \in [a, b]$ such that $f(c) = C$. In other words, all intermediate values of a continuous function are obtained.*

Sketch of the proof. We proceed by **Divide and Conquer**. Without loss of generality, assume $f(a) < C < f(b)$. Let x_1 be the midpoint of $[a, b]$. If $f(x_1) = C$ we are done. If $f(x_1) < C$, we look at the interval $[x_1, b]$. If $f(x_1) > C$ we look at the interval $[a, x_1]$.

In either case, we have a new interval, call it $[a_1, b_1]$, such that $f(a_1) < C < f(b_1)$ and the interval has half the size of $[a, b]$. We continue in this manner, repeatedly taking the midpoint and looking at the appropriate half-interval.

If any of the midpoints satisfy $f(x_n) = C$, we are done. If no midpoint works, we divide infinitely often and obtain a sequence of points x_n in intervals $[a_n, b_n]$. This is where rigorous mathematical analysis is required (see §A.3 for a brief review, and [Rud] for complete details) to show x_n converges to an $x \in (a, b)$.

For each n we have $f(a_n) < C < f(b_n)$, and $\lim_{n\to\infty} |b_n - a_n| = 0$. As f is continuous, this implies $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(x) = C$. □

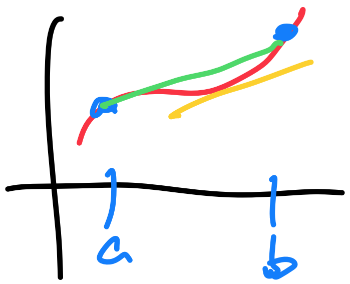
Theorem A.2.2 (Mean Value Theorem (MVT)). *Let $f(x)$ be differentiable on $[a, b]$. Then there exists a $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c) \cdot (b - a). \tag{A.14}$$

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$

or at point c, instantaneous speed equals average speed

We give an interpretation of the Mean Value Theorem. Let $f(x)$ represent the distance from the starting point at time x. The average speed from a to b is the distance traveled, $f(b) - f(a)$, divided by the elapsed time, $b - a$. As $f'(x)$ represents the speed at time x, the Mean Value Theorem says that there is some intermediate time at which we are traveling at the average speed.



To prove the Mean Value Theorem, it suffices to consider the special case when $f(a) = f(b) = 0$; this case is known as Rolle's Theorem:

Theorem A.2.3 (Rolle's Theorem). *Let f be differentiable on $[a, b]$, and assume $f(a) = f(b) = 0$. Then there exists a $c \in (a, b)$ such that $f'(c) = 0$.*

Exercise A.2.4. *Show the Mean Value Theorem follows from Rolle's Theorem.* Hint: *Consider*

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a). \tag{A.15}$$

Note $h(a) = f(a) - f(a) = 0$ and $h(b) = f(b) - (f(b) - f(a)) - f(a) = 0$. The conditions of Rolle's Theorem are satisfied for $h(x)$, and

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}. \tag{A.16}$$

Proof of Rolle's Theorem. Without loss of generality, assume $f'(a)$ and $f'(b)$ are non-zero. If either were zero we would be done. Multiplying $f(x)$ by -1 if needed, we may assume $f'(a) > 0$. *For convenience, we assume $f'(x)$ is continuous.* This assumption simplifies the proof, but is not necessary. In all applications in this book this assumption will be met.

Case 1: $f'(b) < 0$: As $f'(a) > 0$ and $f'(b) < 0$, the Intermediate Value Theorem applied to $f'(x)$ asserts that all intermediate values are attained. As $f'(b) < 0 < f'(a)$, this implies the existence of a $c \in (a, b)$ such that $f'(c) = 0$.

Case 2: $f'(b) > 0$: $f(a) = f(b) = 0$, and the function f is increasing at a and b. If x is real close to a then $f(x) > 0$ if $x > a$. This follows from the fact that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{A.17}$$

As $f'(a) > 0$, the limit is positive. As the denominator is positive for $x > a$, the numerator must be positive. Thus $f(x)$ must be greater than $f(a)$ for such x. Similarly $f'(b) > 0$ implies $f(x) < f(b) = 0$ for x slightly less than b.

Therefore the function $f(x)$ is positive for x slightly greater than a and negative for x slightly less than b. If the first derivative were always positive then $f(x)$ could never be negative as it starts at 0 at a. This can be seen by again using the limit definition of the first derivative to show that if $f'(x) > 0$ then the function is increasing near x. Thus the first derivative cannot always be positive. Either there must be some point $y \in (a, b)$ such that $f'(y) = 0$ (and we are then done) or $f'(y) < 0$. By the Intermediate Value Theorem, as 0 is between $f'(a)$ (which is positive) and $f'(y)$ (which is negative), there is some $c \in (a, y) \subset [a, b]$ such that $f'(c) = 0$. $\square$

Math 150: Multivariable Calculus: Spring 2023: Lecture 03: Review of Calc I and II: https://youtu.be/ICj4EdLh4Ak

Plan for the day.

- Review Calculus I and II.

- Start discussing Calculus III (if time permits).

Derivatives of Standard Functions

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
(x^n)' &= nx^{n-1} \\
(\sin x)' &= \cos x \\
(\cos x)' &= -\sin x \\
(e^x)' &= e^x \\
(b^x)' &= (\log_e b) b^x \\
(\log_e x)' &= \frac{1}{x} \\
(\log_b x)' &= \frac{1}{\log_e b} \frac{1}{x}
\end{aligned}
$$

radians

(minus sign)

$b^x = e^{x \log b} = e^{x \ln(b)}$

$\log b = \ln(b)$

Useful Rules

Sum Rule:	$h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$
Constant Rule:	$h(x) = af(x)$	$h'(x) = af'(x)$
Product Rule:	$h(x) = f(x)g(x)$	$h'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient Rule:	$h(x) = \frac{f(x)}{g(x)}$	$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
Chain Rule:	$h(x) = g(f(x))$	$h'(x) = g'(f(x)) \cdot f'(x)$
	$h(x) = (f(x))^n$	$h'(x) = n(f(x))^{n-1} \cdot f'(x)$
Multiple Rule:	$h(x) = f(ax)$	$h'(x) = af'(ax)$
Reciprocal Rule:	$h(x) = f(x)^{-1}$	$h'(x) = -f'(x)f(x)^{-2}$

$= \frac{1}{f(x)}$

$A(x) = f(x)\,g(x)$ $\qquad A'(x) \overset{?}{=} f, g, f', g'$

$f(x) = x^n$ $\qquad g(x) = x^m$ $\qquad A(x) = x^{n+m}$

$f'(x) = n x^{n-1}$ $\qquad g'(x) = m x^{m-1}$ $\qquad A'(x) = (n+m)\,x^{n+m-1}$

$$f'(x)g(x) + f(x)g'(x) = A'(x)$$

$f(x) = \sin x$ $\qquad g(x) = \cos x$ $\qquad A(x) = \sin x \cos x = \frac{1}{2}\sin(2x)$

$f'(x) = \cos x$ $\qquad g'(x) = -\sin x$ $\qquad A'(x) = \frac{1}{2}\cos(2x)\cdot 2$

$$f'(x)g(x) + f(x)g'(x) = \cos^2 x - \sin^2 x$$
$$= \cos(2x)$$

Proof of Product Rule: $A(x) = f(x)g(x)$

$$A'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right]$$

$$= \underbrace{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}_{f'(x)} \underbrace{\lim_{h\to 0} g(x+h)}_{g(x)} + \underbrace{\lim_{h\to 0} f(x)}_{f(x)} \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)}$$

$$f'(x)\,g(x) + f(x)\,g'(x)$$

$\blacksquare$

Limit Caveats

$$\lim_{x\to\infty} (x^2 - x) = \infty \quad \text{vs} \quad \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty$$

$$\lim_{x\to\infty} (x^2 - x^2) = 0$$

$$\lim_{x\to 0} \frac{x}{x^3} = \infty$$

$$\lim_{x\to 0} \frac{x^3}{x} = 0$$

$$\lim_{x\to 0} \frac{x}{x} = 1$$

No list

- $\infty - \infty$
- ∞ / ∞
- $0/0$
- $\infty \cdot 0$

$$f(x) = \left(\left(3x^2 + \sqrt{\cos x + 4x^3}\right)^{17} * x^3\right)^2$$

$$f(x) = A(x)^2$$

$$f'(x) = 2A(x)A'(x)$$

Know $A(x) = \left(3x^2 + \sqrt{\cos x + 4x^3}\right)^{17} * x^3$

$= B(x)C(x)$ and $A'(x) = B'(x)C(x) + B(x)C'(x)$

$B(x) =$ ___ $\qquad C(x) = x^3$

$B'(x) =$ ___ $\qquad C'(x) = 3x^2$

OF $\longleftrightarrow$ Chain Rule TO $\longleftrightarrow$ Power rule

$$\int (f(x)g(x))'\,dx = f(x)g(x)$$

$$= \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

So $\int f(x)g'(x)dx = \underbrace{\int (f(x)g(x))'\,dx}_{f(x)g(x)} - \int f'(x)g(x)dx$

$u = f(x)$ $\qquad$ $dv = g'(x)dx$ $\qquad$ $\frac{dv}{dx} = g'(x)$

$du = f'(x)$ $\qquad$ $\frac{du}{dx} = f'(x)$ $\qquad$ $v = g(x)$

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^{\pi/2} x\cos x\,dx$$

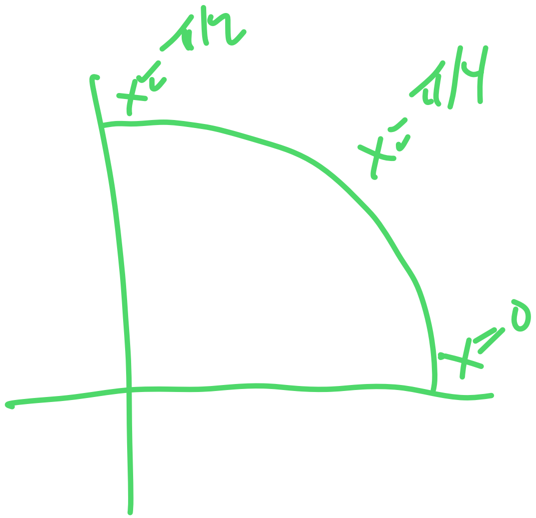
$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$$\int_0^{\pi/2} x\cos x\,dx = x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx$$

$$= \left[\frac{\pi}{2} - 0\right] + \cos x\Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} - 1$$



Reasonable?

$0 \le x\cos x \le \frac{\pi}{2}$

$0\cdot\frac{\pi}{2} \le \text{Integral} \le \frac{\pi}{2}\cdot\frac{\pi}{2}$

Max value of $x\cos x$ for $0 \le x \le \pi/2$

$x=0$ get 0 $\qquad x = \pi/2$ get 0

Critical point: $(x\cos x)' = 0$

So $1 \cdot \cos x + x(-\sin x) = 0$

$\cos x = x \sin x$

$1 = x\tan(x)$

x_c satisfies $1 = x\tan(x)$

Max value is $x_c \cos x_c$

U-Substitution

$A(x) = f(g(x))$ $\quad A'(x) = f'(g(x))\, g'(x)$

So $\int f'(g(x))\, g'(x)\, dx = f(g(x))$

Ex: $\frac{1}{2} \int 2x\, e^{x^2}\, dx$

$g(x) = x^2$

$g'(x) = 2x \qquad g'(x)dx = 2x\,dx$

$f(x) = e^x \qquad f'(x) = e^x$

$f(g(x)) = e^{x^2} \qquad f'(g(x)) = e^{x^2}$

$= \frac{1}{2} \int e^{x^2} \cdot 2x\, dx$

$= \frac{1}{2} e^{x^2}$

$$\int_0^{20} x e^{x^2}\,dx \qquad \text{u-substitution: } u = g(x)$$

$$u = x^2 \qquad du = 2x\,dx \qquad \text{so } x\,dx = \frac{1}{2}\,du$$

$$x: 0 \to 20 \qquad u: 0 \to 400$$

$$= \frac{1}{2}\int_{u=0}^{400} e^u\,du = \frac{1}{2} e^u \Big|_0^{400} = \frac{1}{2} e^{400} - \frac{1}{2} e^0$$

Math 150: Multivariable Calculus: Spring 2023: Lecture 04: Introduction to Sequences and Series: https://youtu.be/-FGp2M9tMO4

Plan for the day.

- Understanding finite and infinite sums.
- Conjecturing limiting values.
- Famous sequences.

Fun Sequences

$1 =$ 1

$1 + 3 =$ 4

$1 + 3 + 5 =$ 9

$1 + 3 + 5 + 7 =$ 16

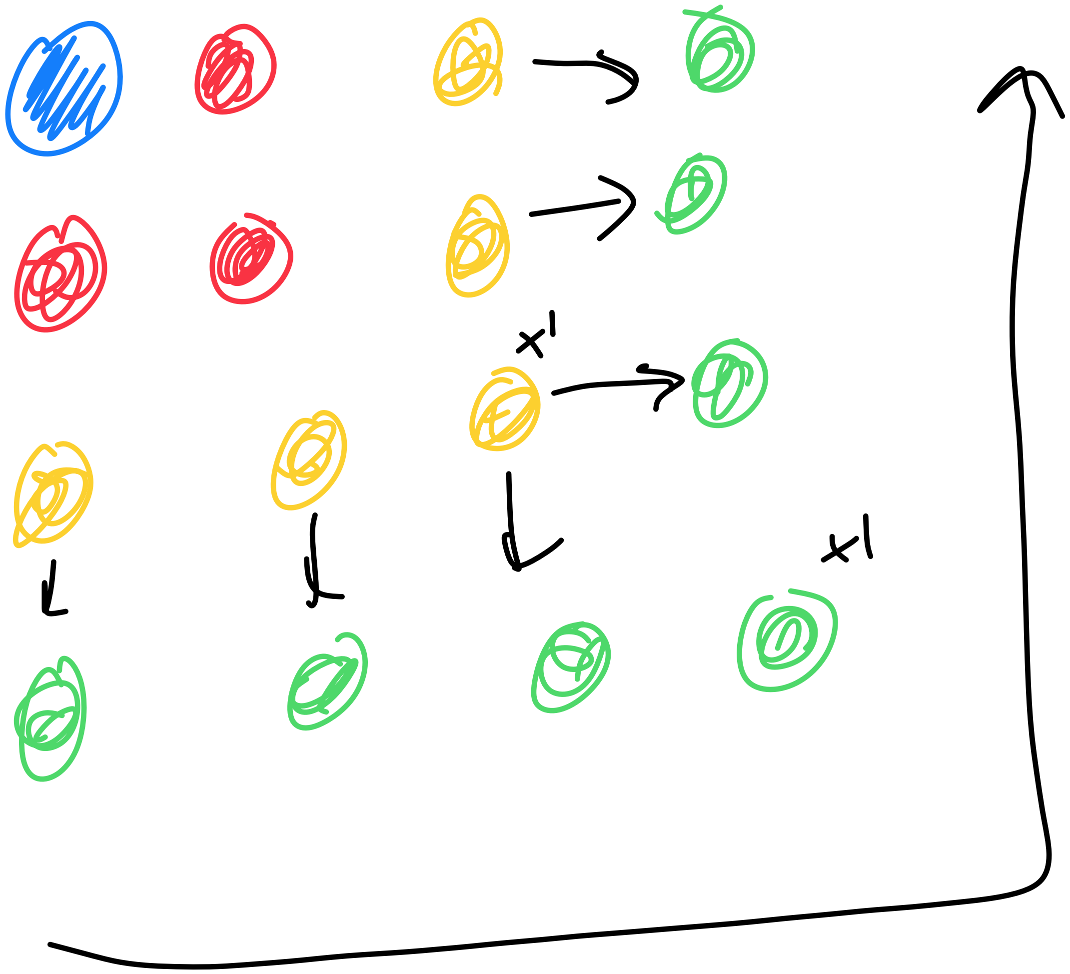
$1 + 3 + 5 + 7 + 9 =$ 25

$1 + 3 + 5 + 7 + 9 + 11 =$ 36

1, 4, 9, 16, 25, 36, ...

Conj: sum of first n odds 's n^2

Do you notice a pattern? Can you make a conjecture?



Fun Sequences II

$1 =$ 1

$1 + 2 =$ 3

$1 + 2 + 3 =$ 6

$1 + 2 + 3 + 4 =$ 10

$1 + 2 + 3 + 4 + 5 =$ 15

$1 + 2 + 3 + 4 + 5 + 6 =$ 21

$n(n+1)/2$

$1 + 2 + 3 + \cdots + 100$

$100 + 99 + 98 + \cdots + 1$

$101 + 101 + \cdots + 101$

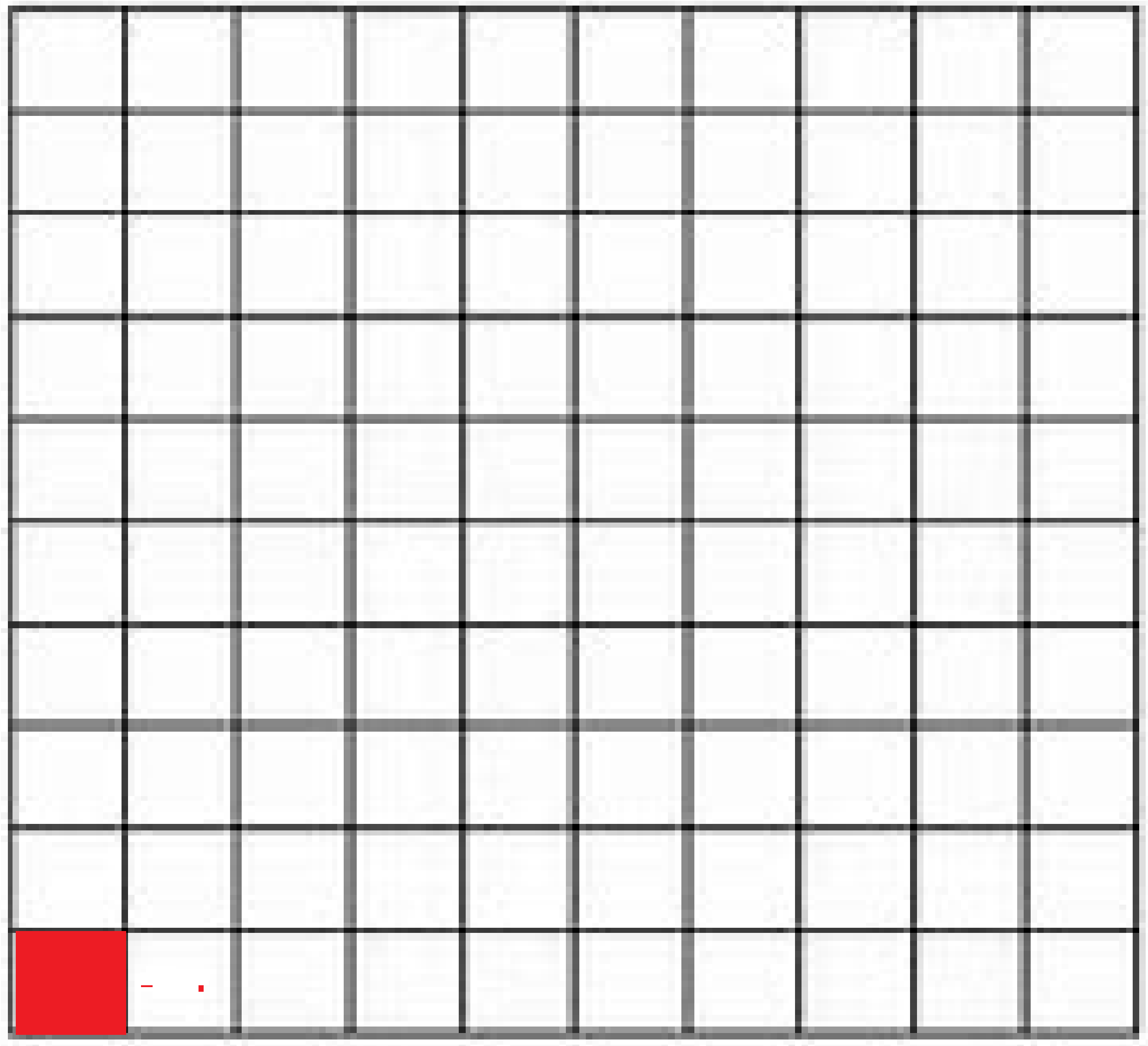
$= (101)100/2$

Do you notice a pattern? Can you make a conjecture?

Our goal is to explore **tilings**.
What is a tiling?

We have a collection of objects and we want to place them down to cover a space.

For example, imagine you want to cover the floor and the floor is a giant square, say 10 feet by 10 feet. What would be a good shape to use to cover it? We want the shape to be smaller that the floor, and we want all the pieces to fit together with no gaps. Answer: 1



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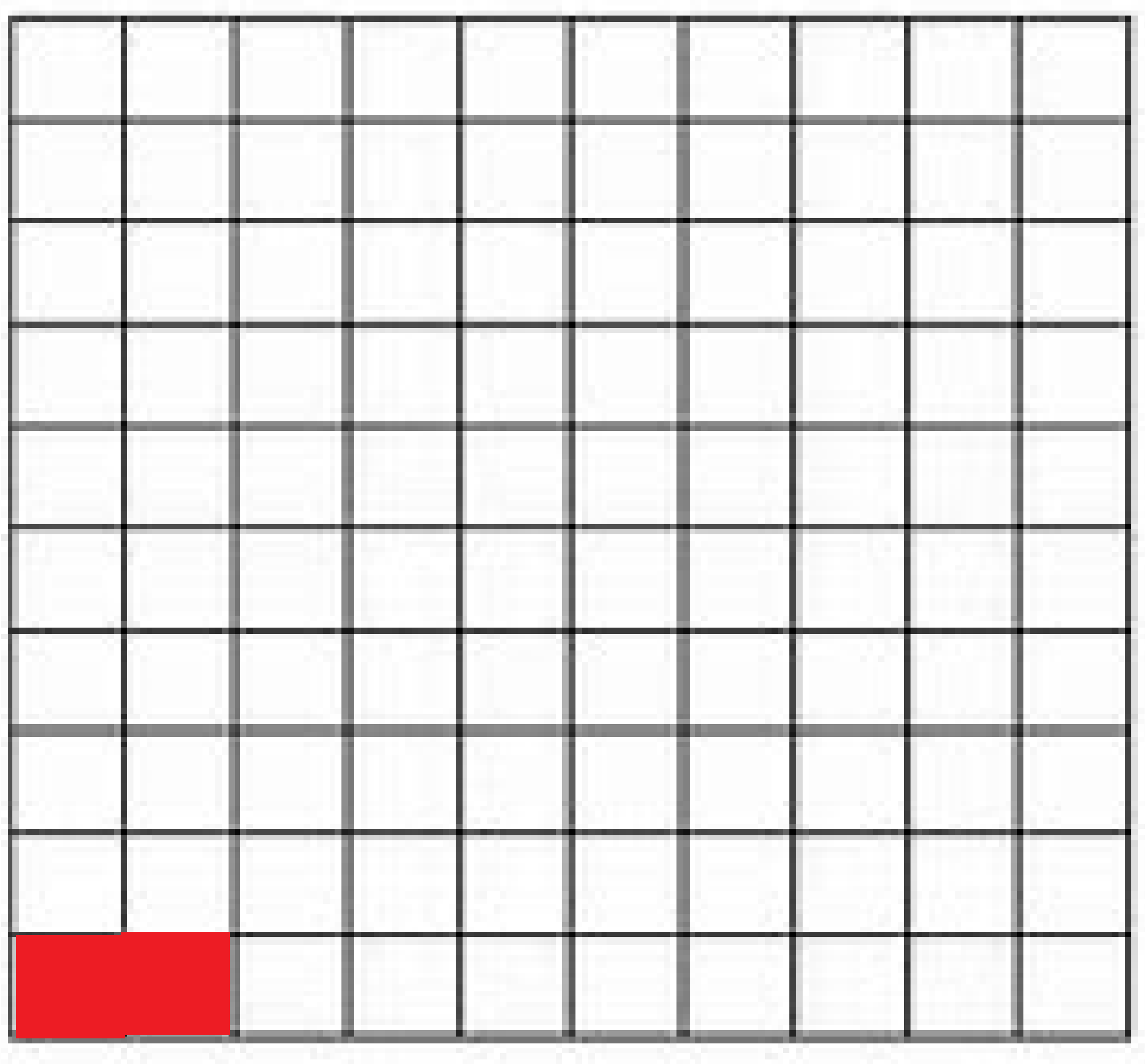

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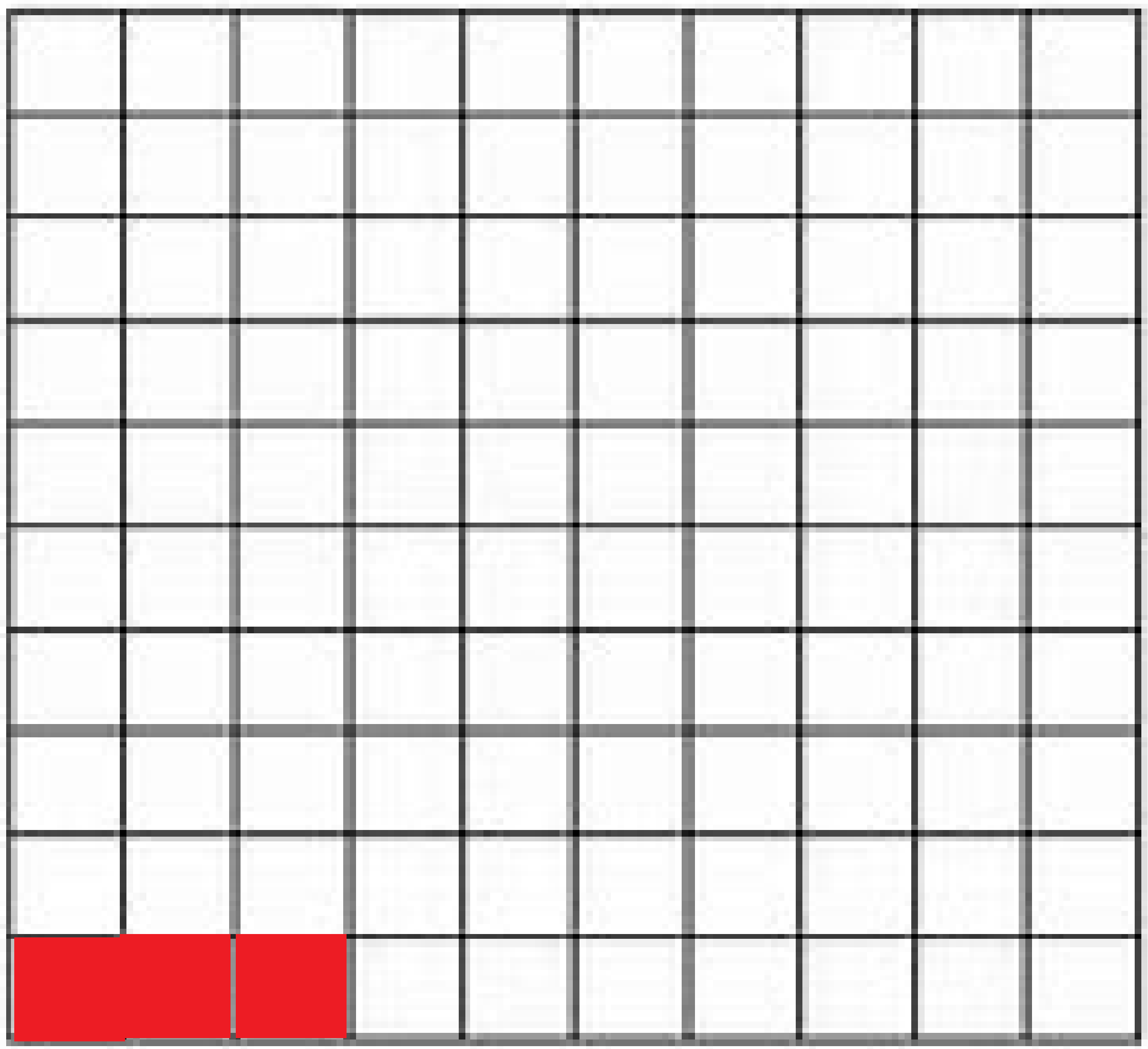
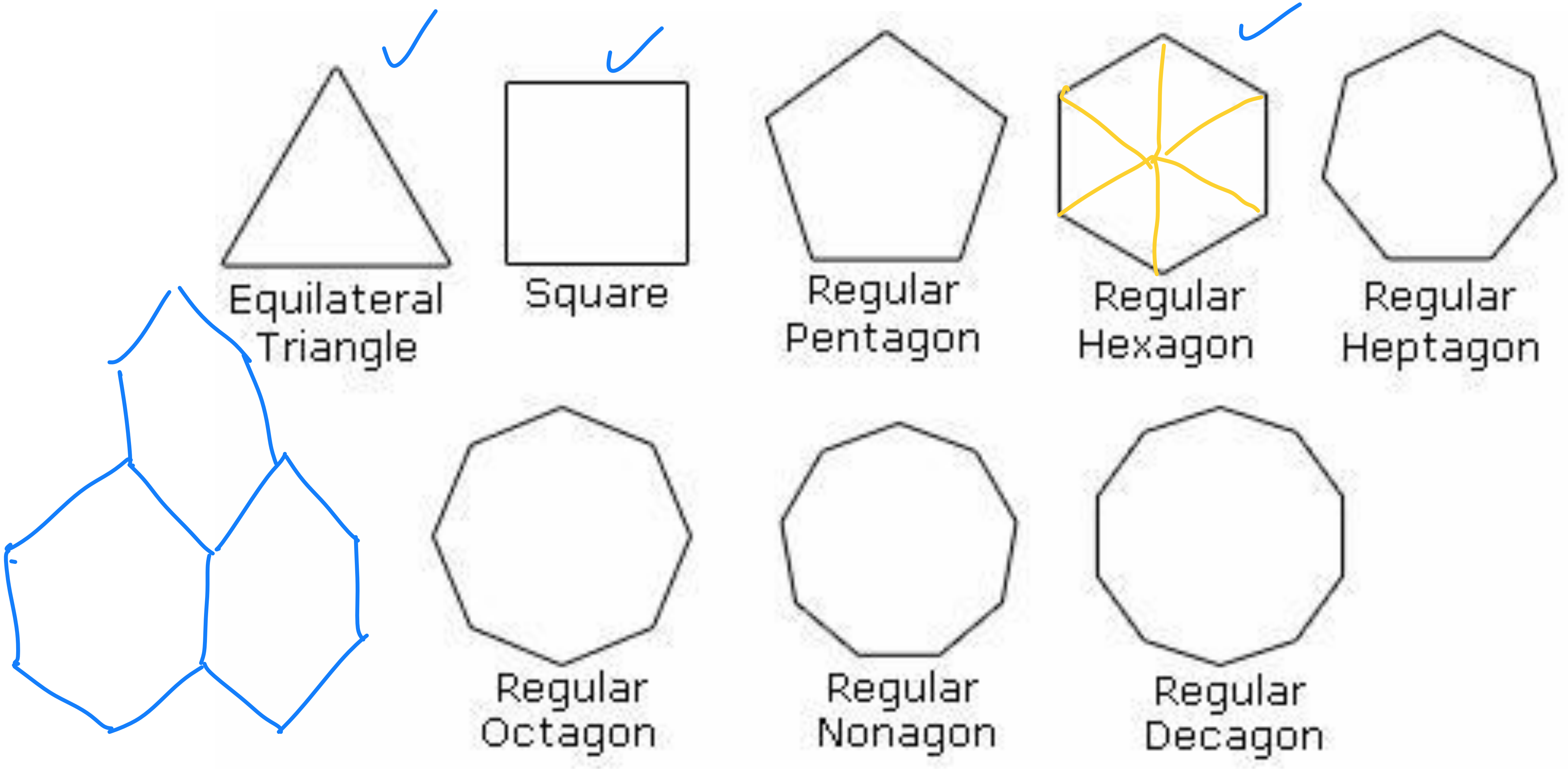


We just continue adding the smaller squares…..

Building on our success, as a fun problem see if you can tile larger and larger regions, with no gaps, with the following shapes.



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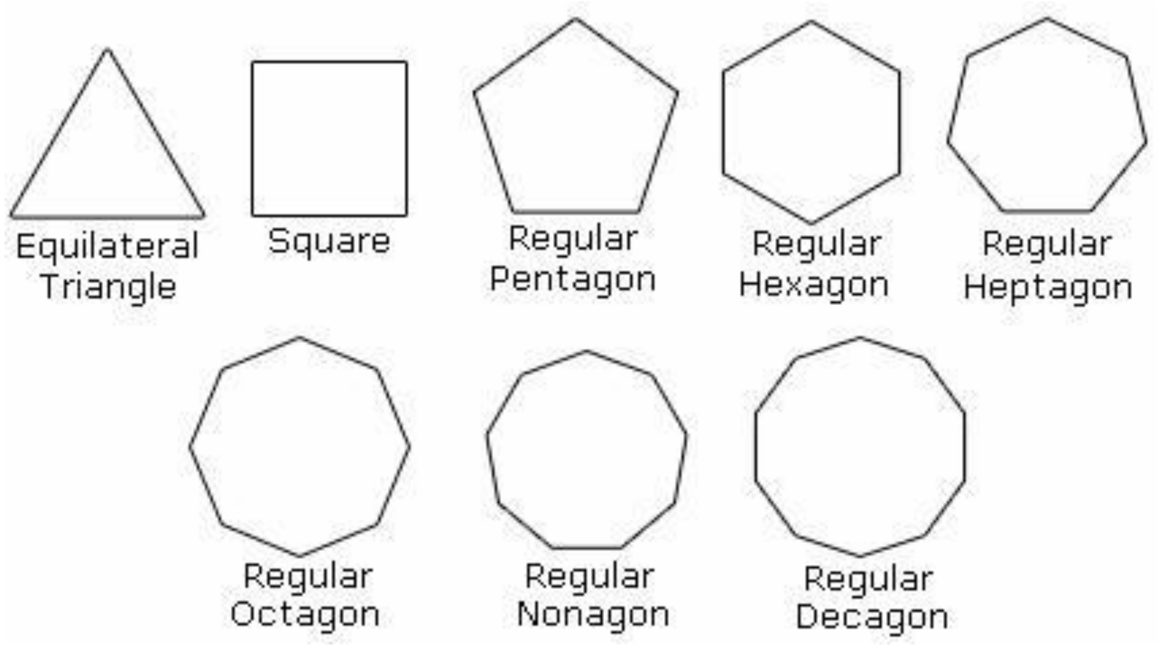


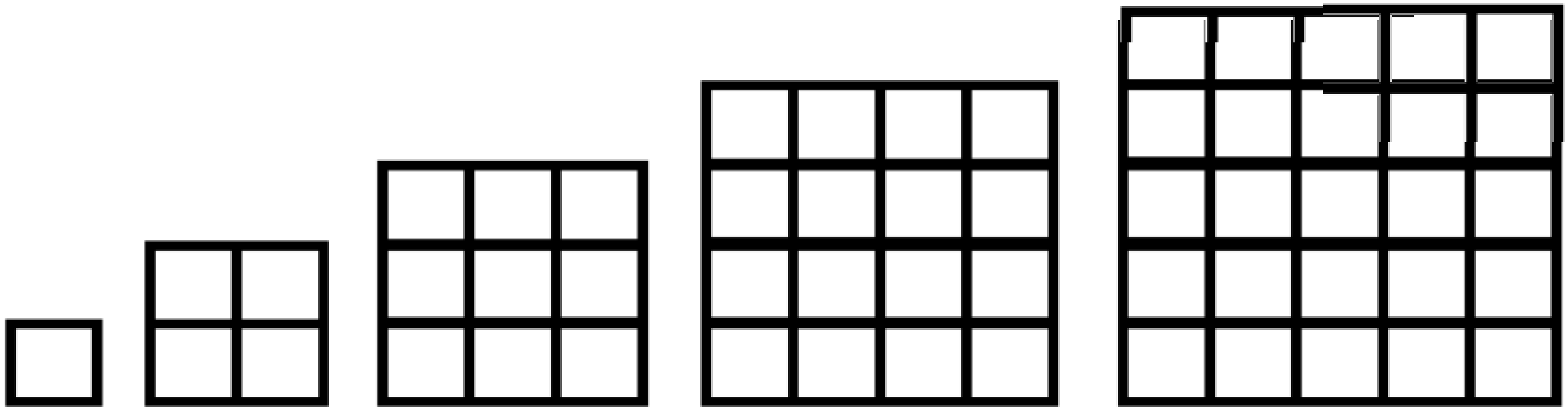
Note each shape above has all sides of the same length. We saw we can do it with the square. What about the triangle? What about the pentagon?

GOOD LUCK!

The I LOVE RECTANGLES Game

If we have an unlimited supply of 1 foot by 1 foot squares, we can cover larger and larger rectangles.

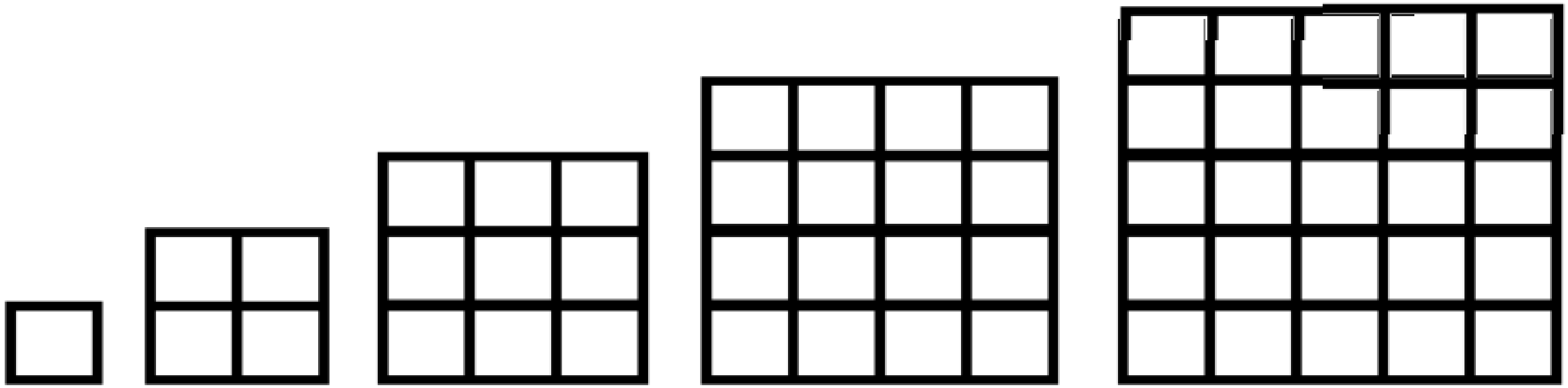
Let's make it more interesting. Imagine now we have EXACTLY ONE of each size square. We have one 1 by 1 rectangle, one 2 by 2 rectangle, one 3 by 3 rectangle, one 4 by 4 rectangle, and so on.



The I LOVE RECTANGLES Game

Let’s make it more interesting. Imagine now we have EXACTLY ONE of each size square. We have one 1 by 1 rectangle, one 2 by 2 rectangle, one 3 by 3 rectangle, one 4 by 4 rectangle, and so on.

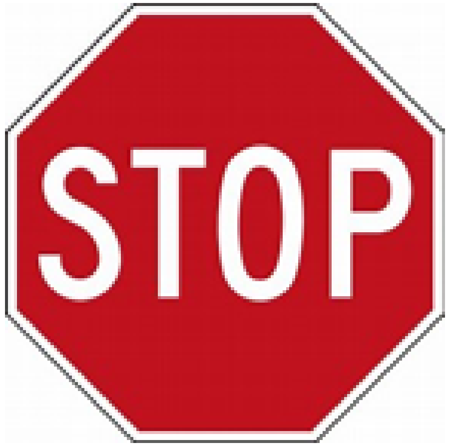
Here’s the rule: we put these squares down ONE AT A TIME, and at EVERY MOMENT IN TIME our shape MUST be a rectangle. Can it be done? Note a square IS a rectangle.



We have one 1 by 1 rectangle, one 2 by 2 rectangle, one 3 by 3 rectangle, one 4 by 4 rectangle, and so on.

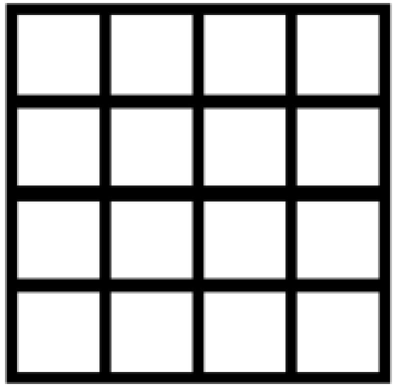
Here's the rule: we put these squares down ONE AT A TIME, and at EVERY MOMENT IN TIME our shape MUST be a rectangle. Can it be done? Note a square IS a rectangle.

SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

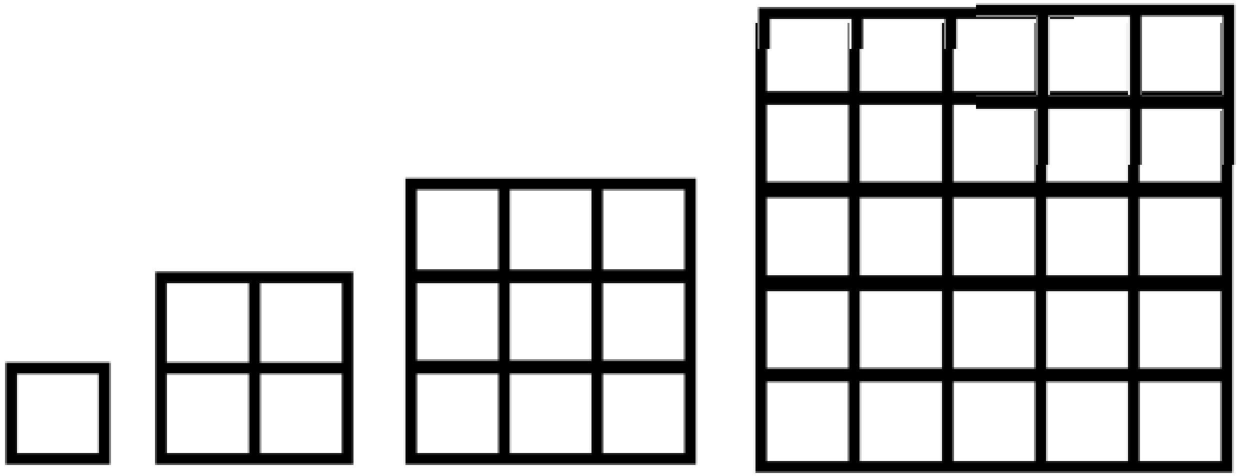


The I LOVE RECTANGLES Game

Imagine we put the 4 by 4 square down. That gives us a rectangle, so far so good. Can we put down anything else next to it and still have a rectangle?



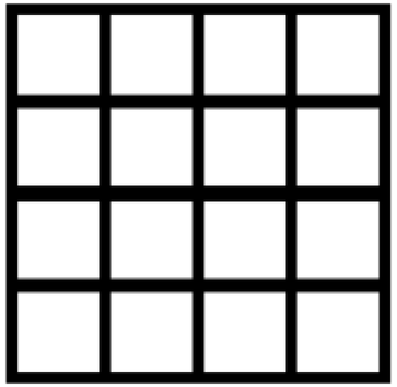
We have placed a 4 by 4 square. This is a rectangle!



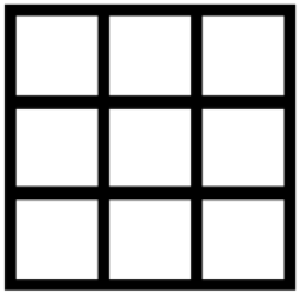
These are the squares we have left. We have a 1 by 1, a 2 by 2, a 3 by 3, a 5 by 5, a 6 by 6 (not drawn) and so on. Can we place anything next to the 4 by 4 and still have a rectangle?

The I LOVE RECTANGLES Game

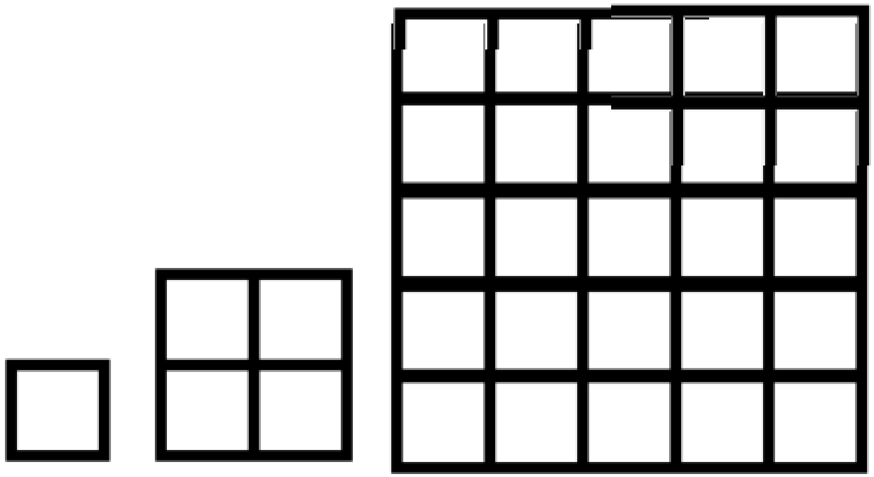
Imagine we put the 4 by 4 square down. That gives us a rectangle, so far so good. Can we put down anything else? Let's try putting down the 3 by 3.



We have placed a 4 by 4 square. This is a rectangle!



We see the 3 by 3 will not fit next to the 4 by 4 and still give a rectangle!

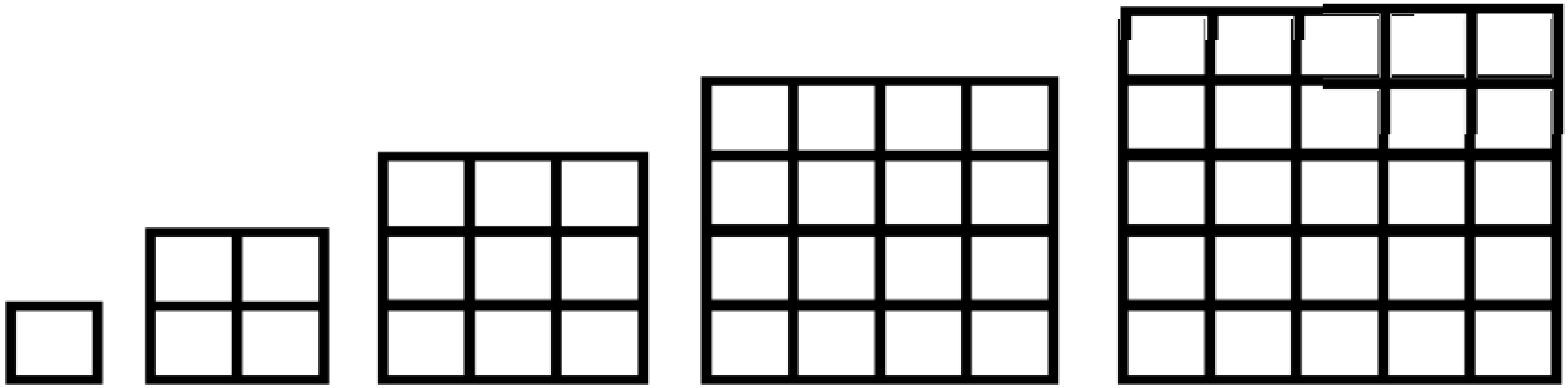


These are the squares we would have left if we try to use a 3 by 3. We would have a 1 by 1, a 2 by 2, a 5 by 5, a 6 by 6 (not drawn) and so on.

The I LOVE RECTANGLES Game

In fact, no matter WHAT square we put down first, we cannot put any more down! If we put down a 5 by 5, to keep it a rectangle we would need something that has a side of length 5, but we only have ONE of each square!

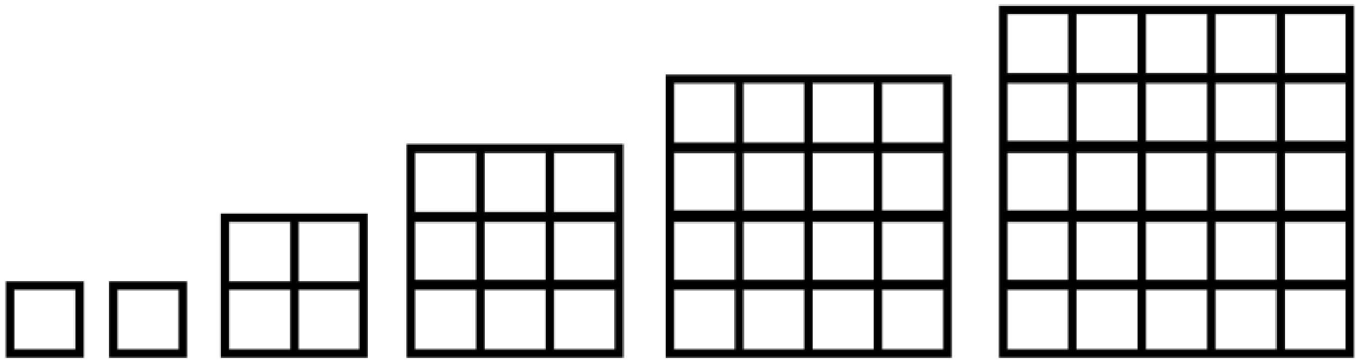
We have to modify the game. We need to give at least ONE more square. What is the smallest square we can give?



The I LOVE RECTANGLES Game

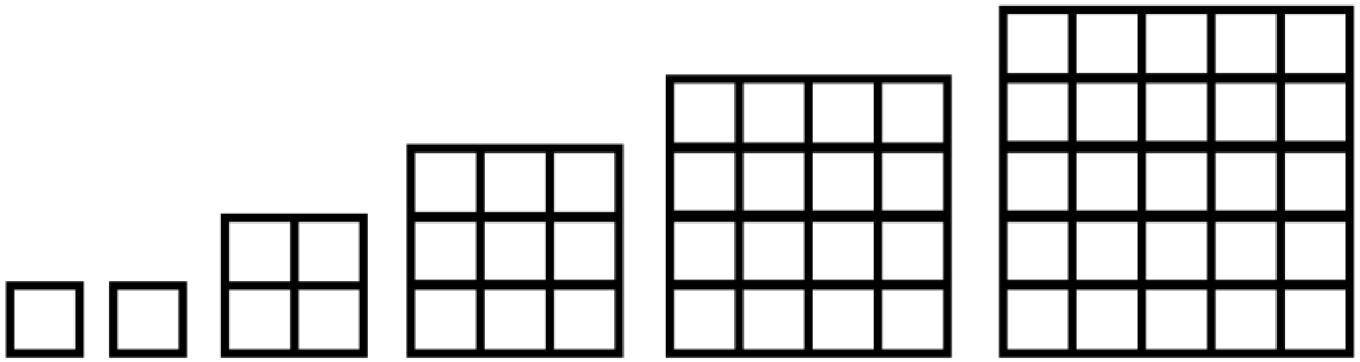
In fact, no matter WHAT square we put down first, we cannot put any more down! If we put down a 5 by 5, to keep it a rectangle we would need something that has a side of length 5, but we only have ONE of each square!

We have to modify the game. We need to give at least ONE more square. What is the smallest square we can give? Answer: a 1 by 1 square! Can we do it now?



The I LOVE RECTANGLES Game

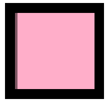
OK, we want to put the squares down one at a time so that we always have a rectangle. We cannot put a square on top of a square. Which should we put down first? Which should we put down second?

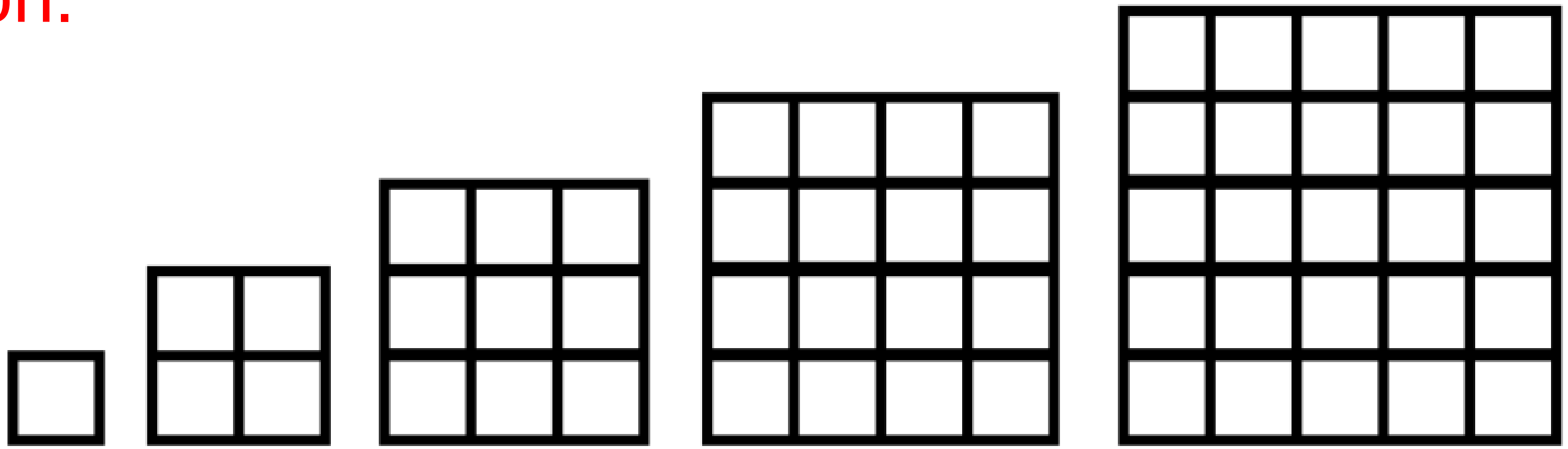


The I LOVE RECTANGLES Game

OK, we want to put the squares down one at a time so that we always have a rectangle. We cannot put a square on top of a square. Which should we put down first? Which should we put down second?

Makes sense to start with the two 1 by 1 squares, as they fit! Here is placing the first 1 by 1 square. Now we have one 1 by 1, one 2 by 2, one 3 by 3, and so on.

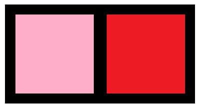


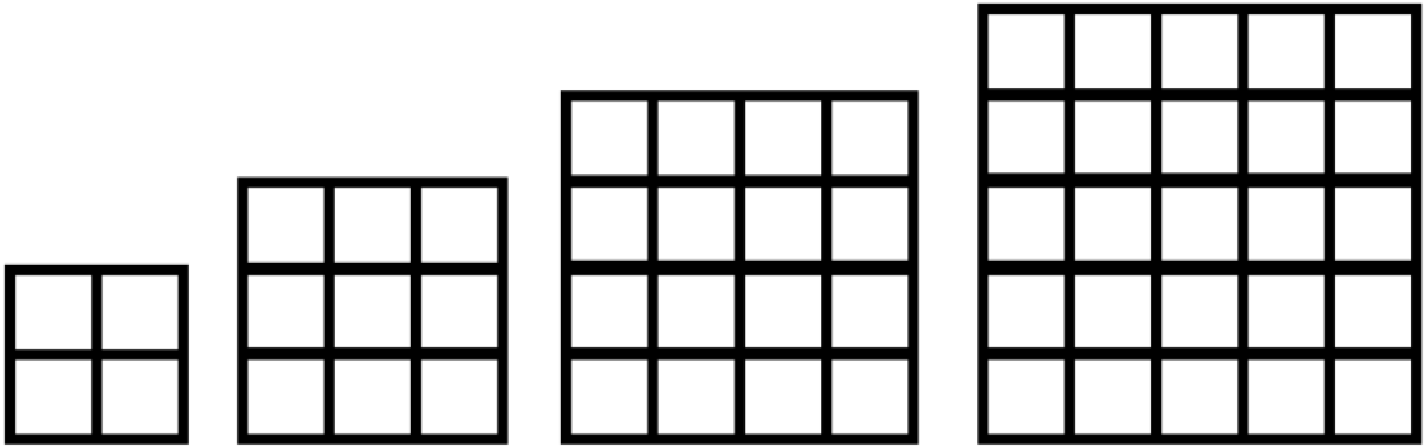


The I LOVE RECTANGLES Game

OK, we want to put the squares down one at a time so that we always have a rectangle. We cannot put a square on top of a square. Which should we put down first? Which should we put down second?

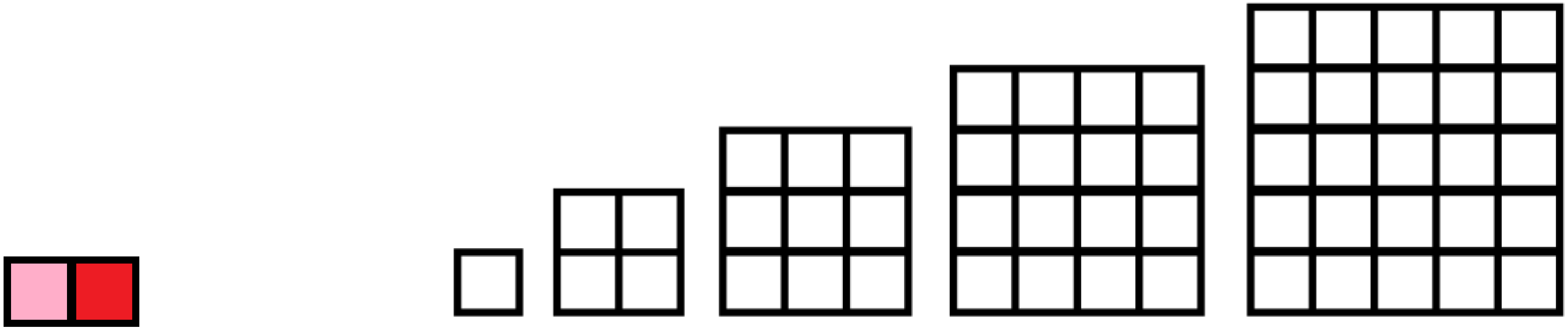
Makes sense to start with the two 1 by 1 squares, as they fit! Here is placing the second 1 by 1 next to the first 1 by 1.





The I LOVE RECTANGLES Game

We have placed the two 1 by 1 squares, we have a 2 by 2, a 3 by 3, a 4 by 4, a 5 by 5 and so on. What should we place next to the two 1 by 1 squares so that we still have a rectangle? Note the two 1 by 1 squares have formec

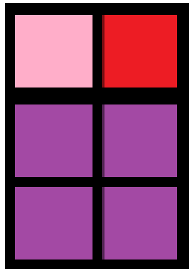


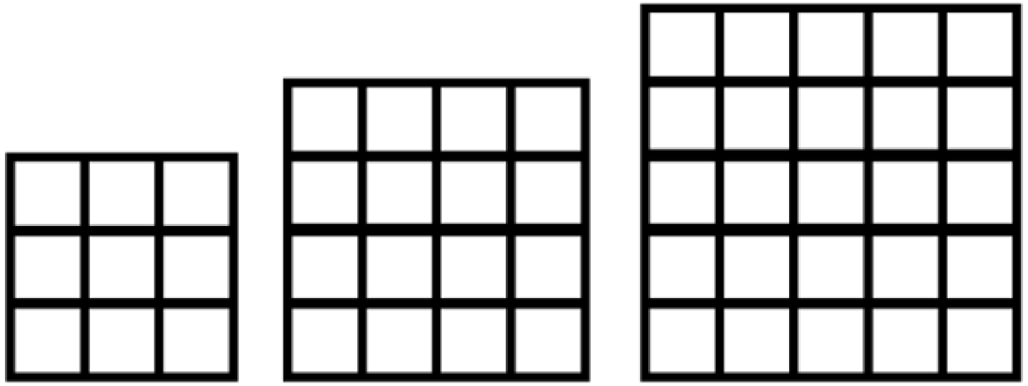
SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

We had a 1 by 2 rectangle, so we need a square that has a side of length 1 or a side of length 2. Looking at our squares, we see we can use the 2 by 2 square!

Building on this success, what should we put down next? Note we now have a rectangle that is 2 by 3….



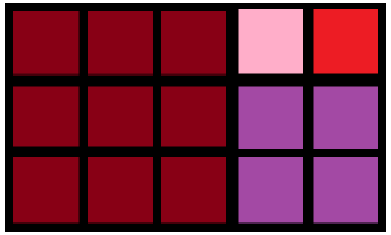


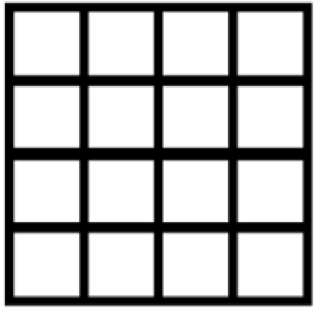
SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

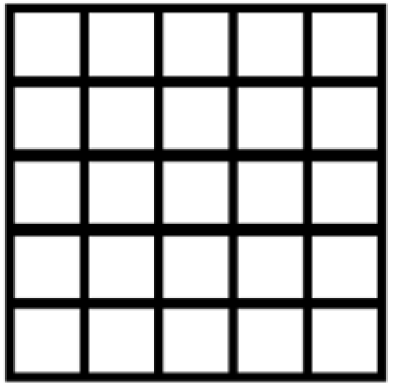
The I LOVE RECTANGLES Game

We had a 2 by 3 rectangle, so we need a square that has a side of length 2 or a side of length 3. Looking at our squares, we see we can use the 3 by 3 square!

Building on this success, what should we put down next? Note we now have a 3 by 5 rectangle.





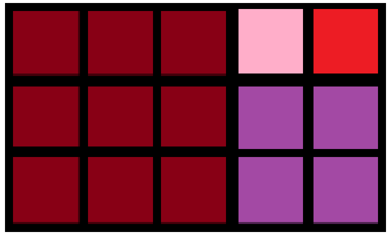


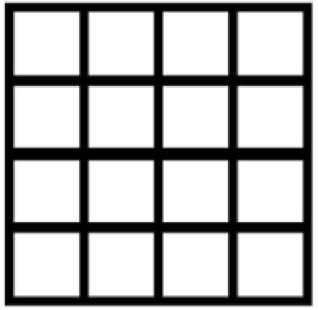
SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

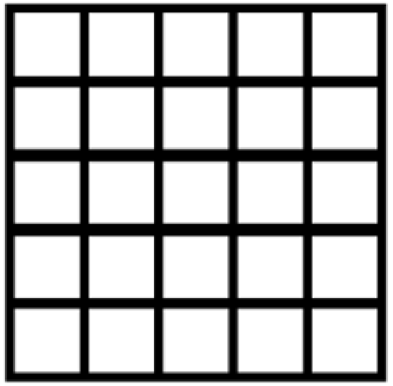
The I LOVE RECTANGLES Game

We had a 2 by 3 rectangle, so we need a square that has a side of length 2 or a side of length 3. Looking at our squares, we see we can use the 3 by 3 square!

Building on this success, what should we put down next? Note we now have a 3 by 5 rectangle. Hint: the 4 by 4 square does not fit!





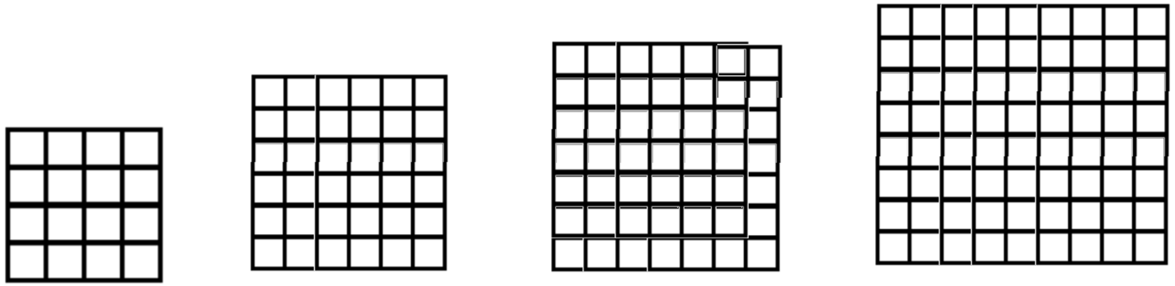


SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

We had a 3 by 5 rectangle. Looking at our squares, we see we can use the 5 by 5 square!

Building on this success, what should we put down next? Note we now have a 5 by 8 rectangle. The 4 by 4 is too small, we still have a 6 by 6, …..

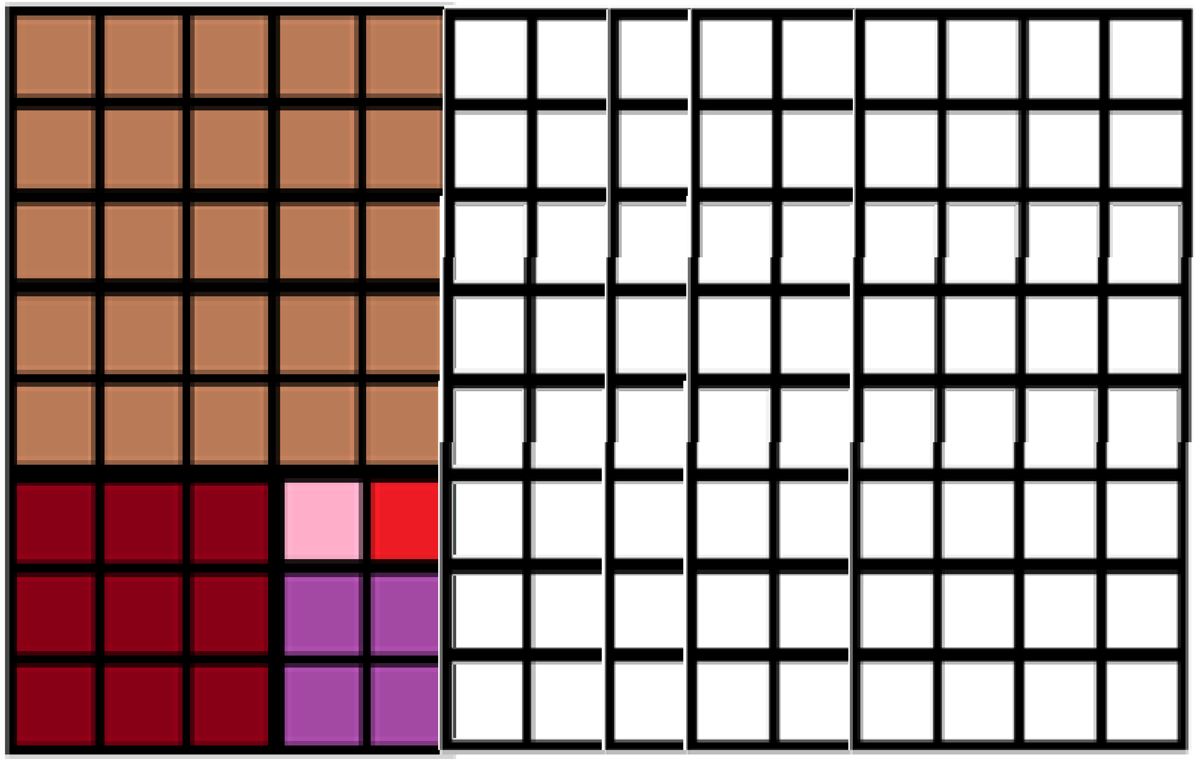


We still have a 6 by 6, a 7 by 7, an 8 by 8, a 9 by 9 (not drawn), a 10 by 10 (not drawn), and so on…..

SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

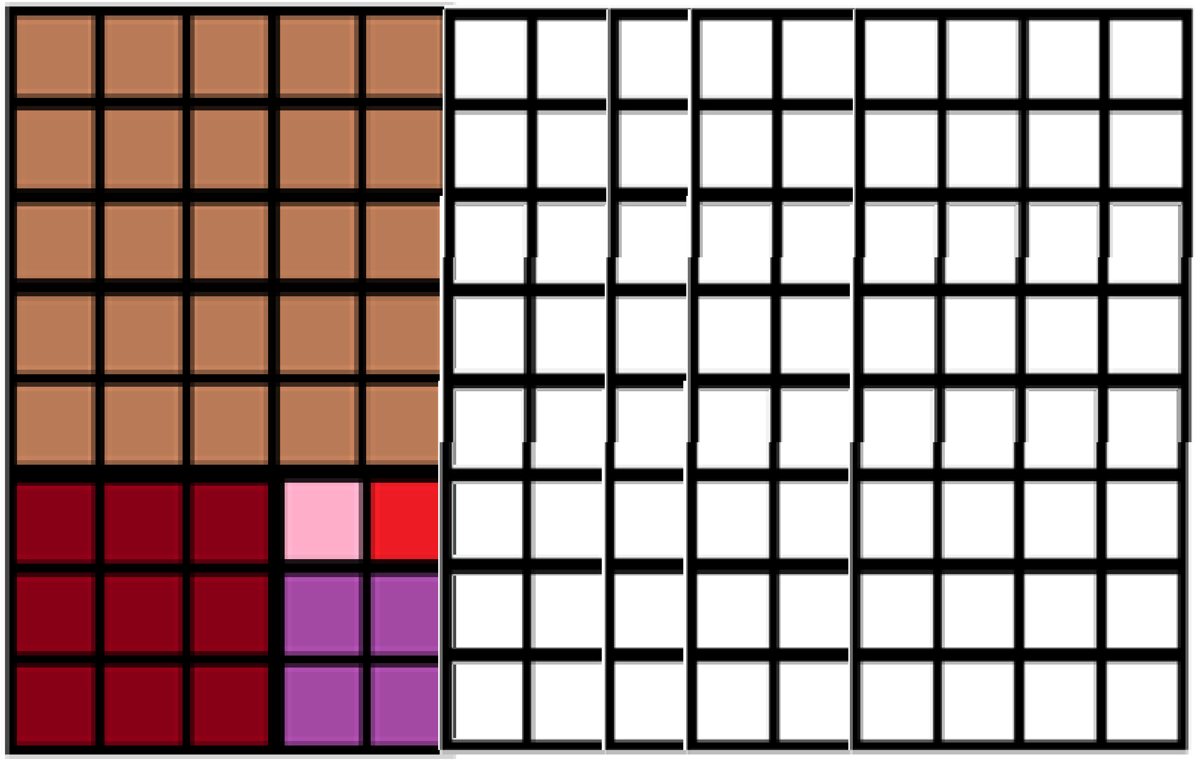
We had a 5 by 8 rectangle. We need to add something with a side of length 5 or 8. Thus we won’t use the 4 by 4, the 6 by 6 or the 7 by 7, but we will use the 8 by 8……



The I LOVE RECTANGLES Game

We write down the squares used in the order used:

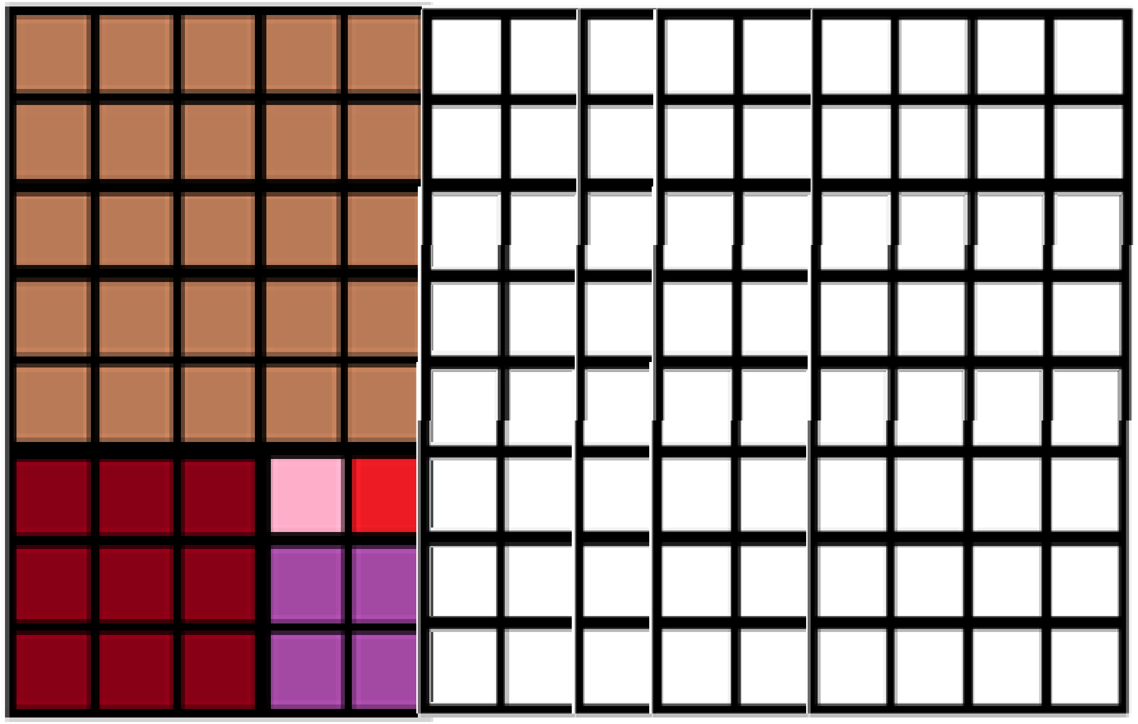
1 by 1, 1 by 1, 2 by 2, 3 by 3, 5 by 5, 8 by 8, …..



The I LOVE RECTANGLES Game

Let’s just write down the side lengths of the squares in the order used:

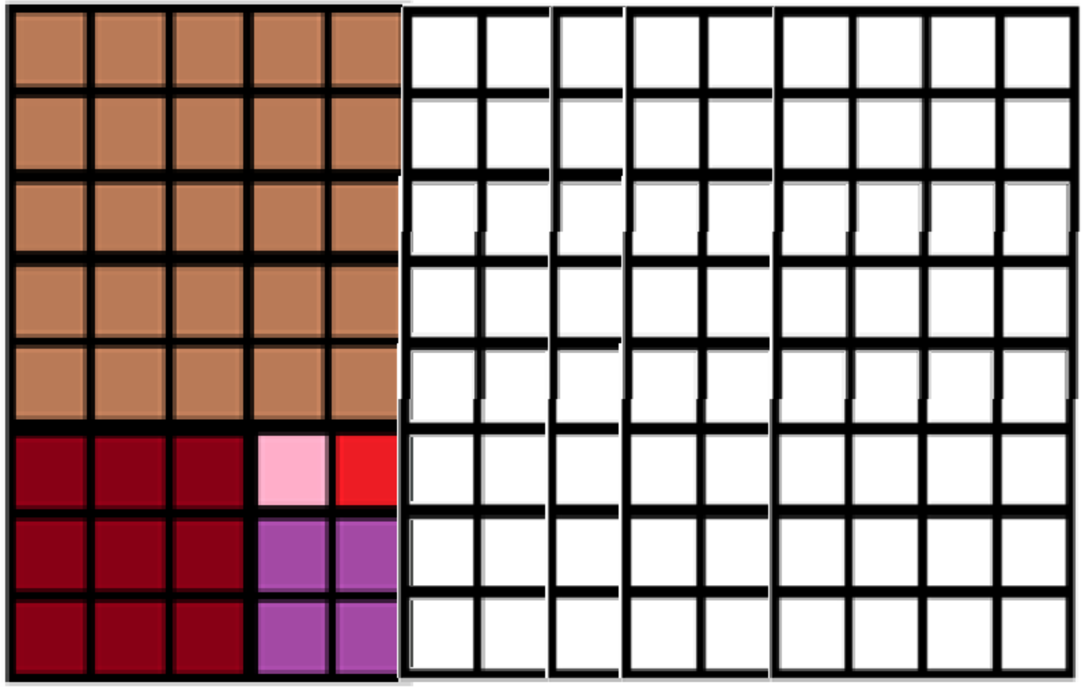
1, 1, 2, 3, 5, 8, …. DO YOU NOTICE A PATTERN?



The I LOVE RECTANGLES Game

Let's just write down the side lengths of the squares in the order used (we'll add a few more terms to the sequence):

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …. DO YOU NOTICE A PATTERN?

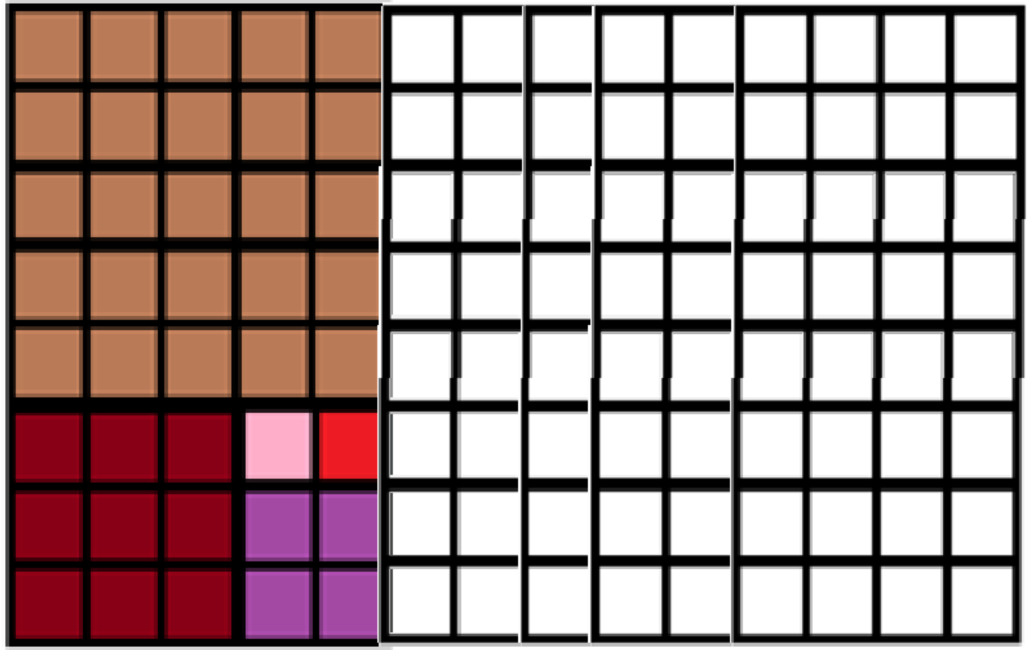


The I LOVE RECTANGLES Game

Let's just write down the side lengths of the squares in the order used:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ….

We start 1, 1, and then after that each term is the sum of the previous two terms! 2 = 1 + 1, 3 = 2 + 1, 5 = 3 + 2, 8 = 5 + 3, and so on. Can you continue the pattern?

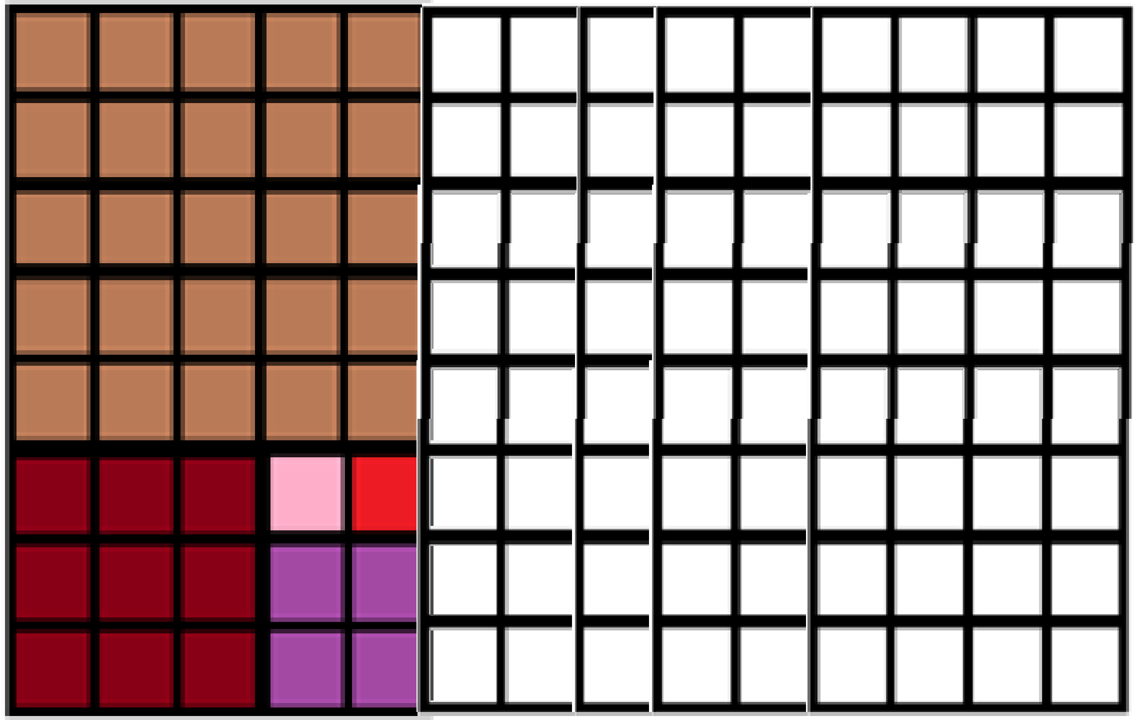


The Fibonacci Sequence

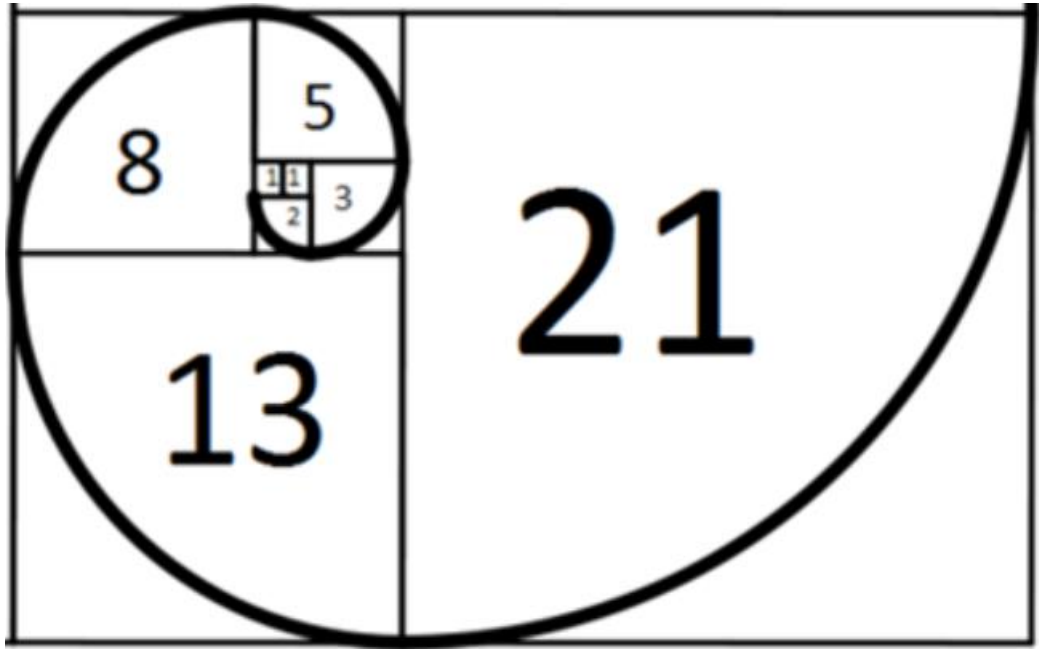
The numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ….

are called the Fibonacci numbers, and have many wondrous properties. See for example https://www.youtube.com/watch?v=me6Dnl2DOtM .



ADVANCED TOPIC!



Advanced: you can calculate area two ways. It is length times width, which here is 21 by 34. It is also the sum of the areas of each square, which is $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 + 21^2$. These are equal! You can thus prove the sum of the squares of the first n Fibonacci numbers is the n^{th} Fibonacci number times the $(n+1)^{st}$ Fibonacci number!

What is $\dfrac{100}{9801}$?

What is $\dfrac{10100}{970299}$?

What is $\dfrac{100}{9899}$?

What is $\dfrac{100}{9801}$?

= 0.01020304050607080910111213141516171819202122232425262728293031323334353637383940414243444546474849505152535455565758596061626364656667686970717273747576777879808182838485868788899091929394959697990001 0

What is $\dfrac{10100}{970299}$?

0.010409162536496482012

What is $\dfrac{100}{9899}$?

0.0101020305081321345590463 6

The Geometric Series Formula

From Shooting Hoops
to the Geometric Series Formula

$|x| < 1$ $\quad S_n = 1 + x + \cdots + x^n$

$x S_n = \quad x + \cdots + x^n + x^{n+1}$

$(1-x) S_n = 1 - x^{n+1}$

so $S_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$

if $|x| < 1$ as $n \to \infty$ $S_n \to \frac{1}{1-x}$

The Geometric Series Formula

The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

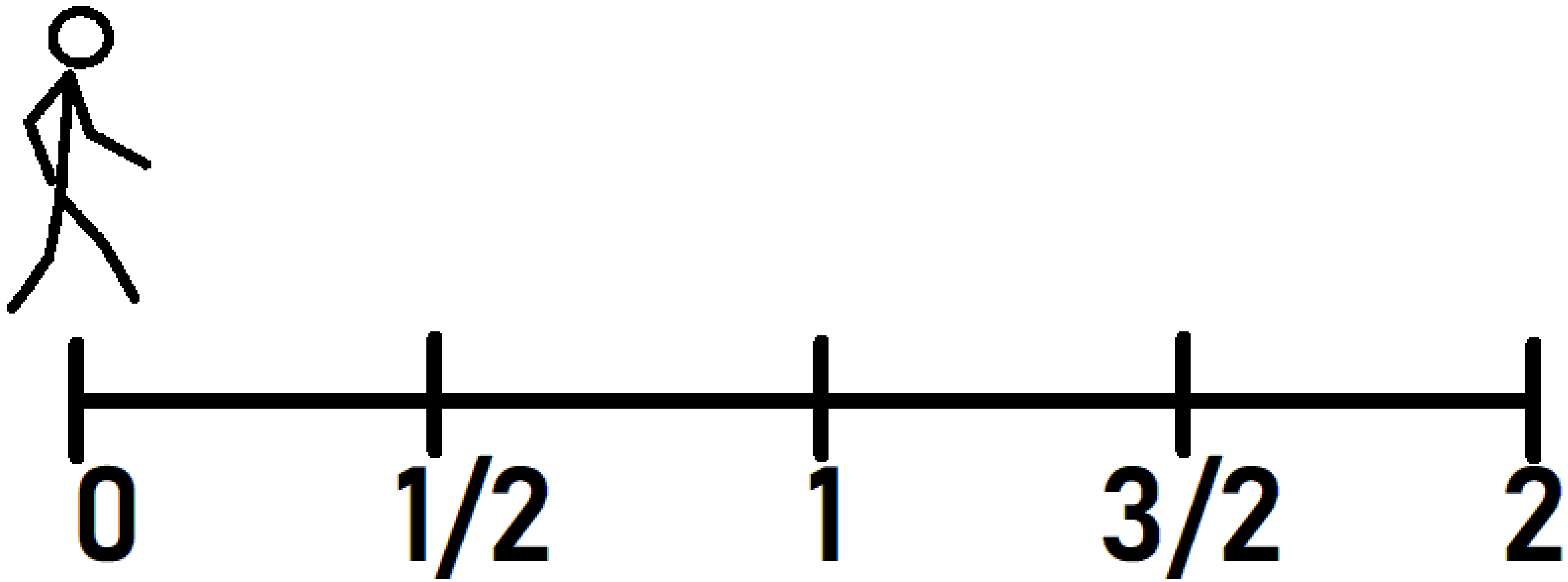
If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

This is often proved by first computing the finite sum, up to r^n, and taking a limit. Note since $|r| < 1$ that each term r^n gets small fast…..

The Geometric Series Converges if |r| < 1

$$1 + r + r2 + r3 + r4 + \cdots = \frac{1}{1-r}.$$

Why does this converge? Take r = ½. We then have 1 + ½ + ¼ + ... = $\frac{1}{1-\frac{1}{2}}$ = 2, and we can view this as we start at 0, and each step covers half the distance to 2. We thus never reach it in finitely many steps, but we cover half the ground each tir



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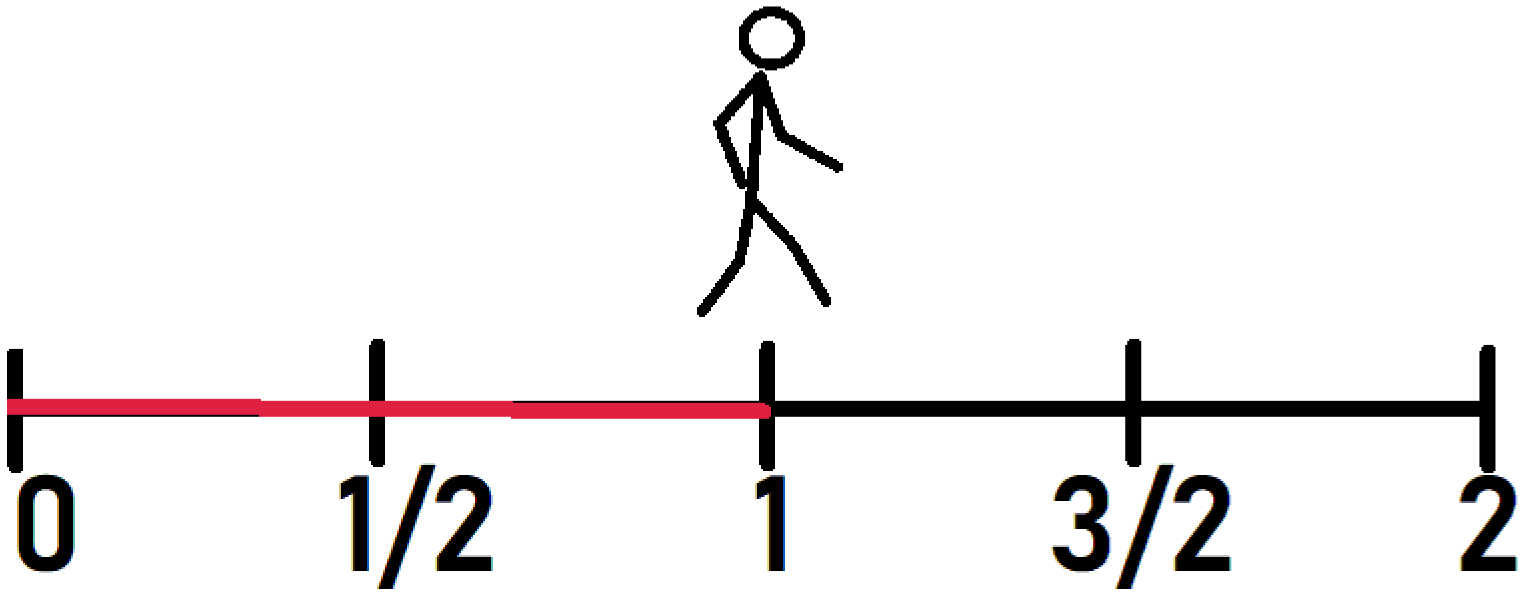

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The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

Lemma: If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots + r^n = \frac{1 - r^{n+1}}{1-r}$.

Proof: Let $S_n = 1 + r + r^2 + r^3 + r^4 + \ldots + r^n$

Then $\quad r\,S_n = \quad r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1}$

What should we do now?

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Then $\quad r S_n = \quad r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1}$

Subtract: $S_n - r S_n = 1 - r^{n+1}$,

So $(1-r) S_n = 1 - r^{n+1}$, or S_n

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So $(1-r)\, S_n = 1 - r^{n+1}$, or $S_n = \frac{1 - r^{n+1}}{1-r}$.

If we let n go to infinity, we see r^{n+1} goes to 0, so we get the infinite sum is $\frac{1}{1-r}$.

Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.

We will prove the Geometric Series Formula just by studying this basketball game!

Simpler Game: Hoops: Mathematical Formulation

Bird and Magic (I'm old!) alternate shooting; first basket wins.

- Bird always gets basket with probability p.
- Magic always gets basket with probability q.

Let x be the probability Bird wins – what is x?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

Solving the Hoop Game

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Break into cases:

- **Bird** wins on 1st shot: p.

Solving the Hoop Game

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Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1 - p)(1 - q) \cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1 - p)(1 - q) \cdot p$.
- **Bird** wins on 3rd shot: $(1 - p)(1 - q) \cdot (1 - p)(1 - q) \cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- Bird wins on 1st shot: p.
- Bird wins on 2nd shot: $(1-p)(1-q)\cdot p$.
- Bird wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- Bird wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
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- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Let $r = (1-p)(1-q)$. Then

$$\begin{aligned} x &= \text{Prob}(\textbf{Bird}\text{ wins}) \\ &= p + rp + r^2p + r^3p + \cdots \\ &= p\left(1 + r + r^2 + r^3 + \cdots\right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Solving the Hoop Game: The Power of Perspective

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$$x = \text{Prob}(\textbf{Bird}\ \text{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

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Have

$$x = \text{Prob}(\textbf{Bird}\ \text{wins}) = p +$$

Solving the Hoop Game: The Power of Perspective

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will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1 - p)(1 - q) * ???$$

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Showed

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will solve without the geometric series formula.

Have

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p + (1-p)(1-q)x$$

Solving the Hoop Game: The Power of Perspective

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Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

Solving the Hoop Game: The Power of Perspective

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$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p + (1-p)(1-q)x = p + rx.$$

Thus

$$(1-r)x \ = \ p \quad \text{or} \quad x \ = \ \frac{p}{1-r}.$$

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Thus

$$(1 - r)x \;=\; p \quad \text{or} \quad x \;=\; \frac{p}{1 - r}.$$

As $x = p(1 + r + r^2 + r^3 + \cdots)$, find

$$1 + r + r^2 + r^3 + \cdots \;=\; \frac{1}{1 - r}.$$

Advanced Geometric Series Comments

Always carefully look at what you did, and be explicit on what you proved.

The geometric series formula is:

If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

We proved this when $r = (1-p)(1-q)$, where p and q are the probabilities of making a basket for Bird and Magic. What are the ranges for p and q? We have what range of p and q?

Advanced Geometric Series Comments

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Advanced Geometric Series Comments

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Lessons from Hoop Problem

◇ Power of Perspective: Memoryless process.

◇ Can circumvent algebra with deeper understanding! (Hard)

◇ Depth of a problem not always what expect.

◇ Importance of knowing more than the minimum: connections.

◇ Math is fun!

New Sum: The Harmonic Series

The Harmonic Series $\{H_n\}$ is defined as the sequence where

$$Hn = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Thus the first few terms are

- 1,
- 1 + 1/2 = 3/2 = 1.5,
- 1 + 1/2 + 1/3 = 11/6 or about 1.83,
- 1 + 1/2 + 1/3 + 1/4 = 25/12 or about 2.08
- H_{100} = or about 5.18
- H_{10000} is $\frac{14\,466\,636\,279\,520\,351\,160\,221\,518\,043\,104\,131\,447\,711}{2\,788\,815\,009\,188\,499\,086\,581\,352\,357\,412\,492\,142\,272}$
- $H_{1000000}$ is about 14.3927; the terms are growing but VERY slowly…..

The Harmonic Series Diverges!

The Harmonic Series $\{H_n\}$ is the sequence where $Hn = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Let H be the limit as n goes to infinity of H_n, thus it is the sum of the reciprocals of integers. We claim H = ∞, *so the sum diverges*

Proof: Assume H is finite, let H_{even} be the sum of the reciprocals of even numbers, H_{odd} the sum of the odd terms.

$Hodd = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ $\quad Heven = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$

As 1/1 > 1/2, 1/3 > 1/4, what can you say about the size of H_{odd} versus the size of H_{even}?

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Thus $H = H_{even} + H_{odd} > H_{even} + H_{even} = 2H_{even}$.

Note however that $H_{even} = 1/2 + 1/4 + 1/6 + 1/8 + \ldots = \frac{1}{2}(1 + 1/2 + 1/3 + 1/4 + \ldots) = \frac{1}{2}$ H.

Why is this true?

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So $H > 2\ H_{even} = 2 * \frac{1}{2}$ H = H; why is this a contradiction?

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So $H > 2\ H_{even} = 2 * \frac{1}{2}$ H = H; but H cannot be larger than H, contradiction, thus our assumption that H converges is false!

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The Harmonic Series $\{H_n\}$ is the sequence where $Hn = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.

The divergence of this sum is so important we give another proof.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \cdots$$

If we group terms together, we can get infinitely many sums that are more than 1/2, so it diverges.

What should we group with 1/3 to get terms that sum to more than 1/2?

The Harmonic Series Diverges!

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$$\frac{1}{1} + \frac{1}{2} + \boxed{\frac{1}{3} + \frac{1}{4}} + \boxed{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \boxed{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}} + \cdots$$

If we group terms together, we can get infinitely many sums that are more than 1/2, so it diverges.

Note 1/3 and 1/4 are each at least 1/4, so their sum is at least 2 * 1/2 = 1/2.

Note 1/5, …, 1/8 are each at least 1/8, so their sum is at least 4 * 1/8 = 1/2.

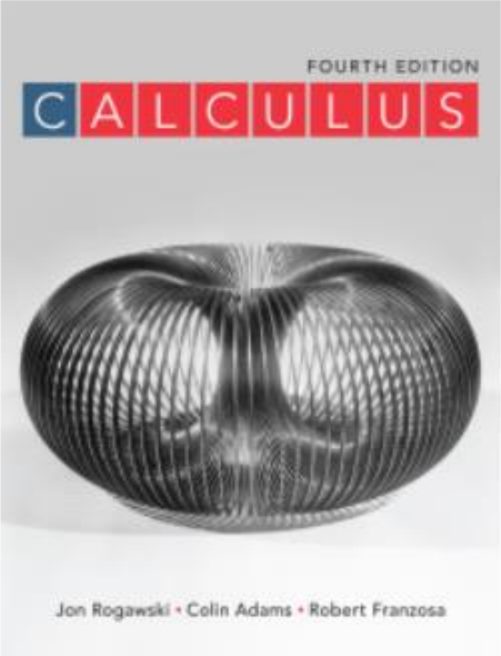
Note 1/9, …, 1/16 are each at least 1/16, so their sum is at least 8 * 1/16 = 1/2.

Math 150: Multivariable Calculus: Spring 2023:
Lecture 05: Sequences and Series: https://youtu.be/gtLVCKB32B8

Plan for the day.

- Understanding finite and infinite sums.
- Limit Laws.
- Convergence / Divergence Tests.

Note: all quoted text taken from the textbook for the class:



Calculus 4th Edition

Author(s)	Jon Rogawski; Colin Adams; Robert Franzosa
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DEFINITION

Sequence

A **sequence** $\{a_n\}$ is an ordered collection of numbers defined by a function f on a set of sequential integers. The values $a_n = f(n)$ are called the **terms** of the sequence, and n is called the **index**. Informally, we think of a sequence $\{a_n\}$ as a list of terms:

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \ldots$$

The sequence does not have to start at $n = 1.$ It can start at $n = 0, n = 2,$ or any other integer.

DEFINITION

Limit of a Sequence

We say $\{a_n\}$ **converges to a limit** L and write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L$$

if, for every $\epsilon > 0,$ there is a number M such that $|a_n - L| < \epsilon$ for all $n > M$.

- If no limit exists, we say that $\{a_n\}$ **diverges**.
- If the terms increase without bound, we say that $\{a_n\}$ **diverges to infinity**.

Limit is unique!

DEFINITION

Limit of a Sequence

We say $\{a_n\}$ **converges to a limit** L and write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L$$

if, for every $\epsilon > 0,$ there is a number M such that $|a_n - L| < \epsilon$ for all $n > M.$

- If no limit exists, we say that $\{a_n\}$ **diverges**.
- If the terms increase without bound, we say that $\{a_n\}$ **diverges to infinity**.

THEOREM 2

Limit Laws for Sequences

Assume that $\{a_n\}$ and $\{b_n\}$ are convergent sequences with

$$\lim_{n\to\infty} a_n = L, \qquad \lim_{n\to\infty} b_n = M$$

L, M are finite

Avoid

0/0

$\infty \cdot 0$

∞/∞

$\infty - \infty$

Then

i. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$

ii. $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM$

iii. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{L}{M} \quad \text{if } M \neq 0$

iv. $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL \quad \text{for any constant } c$

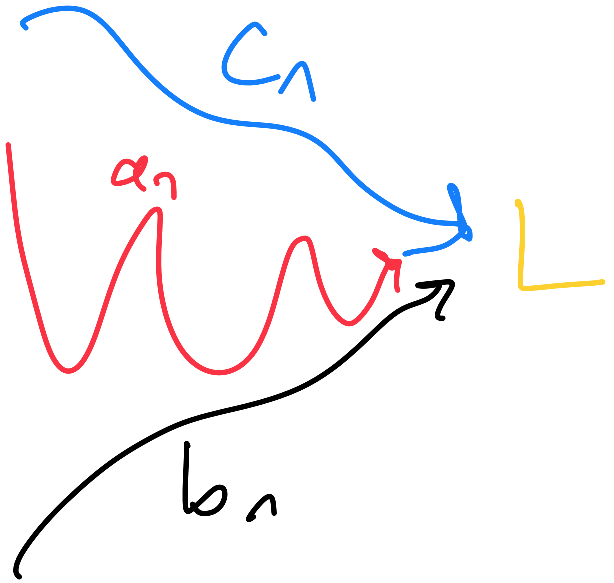
THEOREM 3

Squeeze Theorem for Sequences

Let $\{a_n\}, \; \{b_n\}, \; \{c_n\}$ be sequences such that for some number M,

$$b_n \le a_n \le c_n \qquad \text{for } n > M \qquad \text{and} \qquad \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$$

Then $\lim_{n\to\infty} a_n = L.$

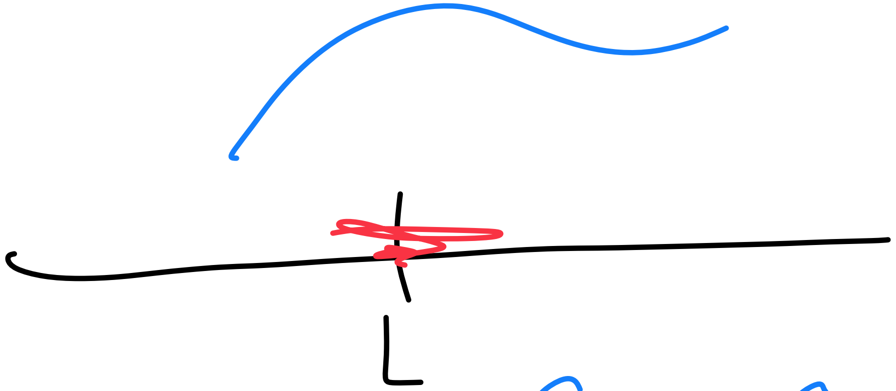


THEOREM 4

If f is continuous and $\lim_{n\to\infty} a_n = L,$ then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

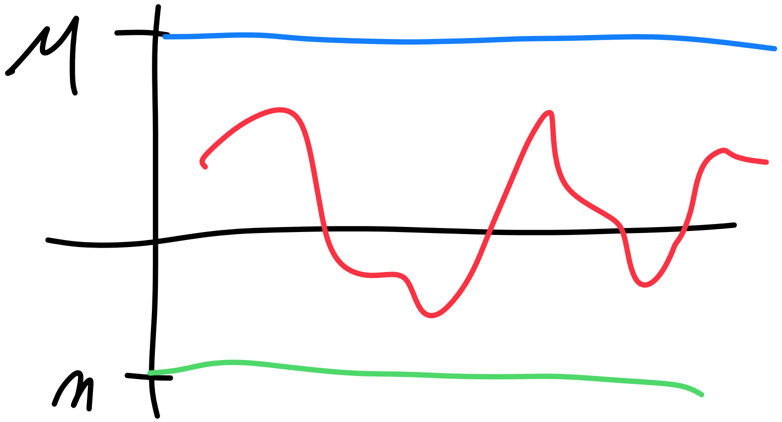
In other words, we may pass a limit of a sequence inside a continuous function.



Cont if $\lim_{x\to L} f(x) = f(L)$

DEFINITION

Bounded Sequences



A sequence $\{a_n\}$ is

- **Bounded from above** if there is a number M such that $a_n \leq M$ for all $n.$ The number M is called an *upper bound*.
- **Bounded from below** if there is a number *m* such that $a_n \geq m$ for all $n.$ The number *m* is called a *lower bound*.

The sequence $\{a_n\}$ is called **bounded** if it is bounded from above and below. A sequence that is not bounded is called an **unbounded sequence**.

THEOREM 6

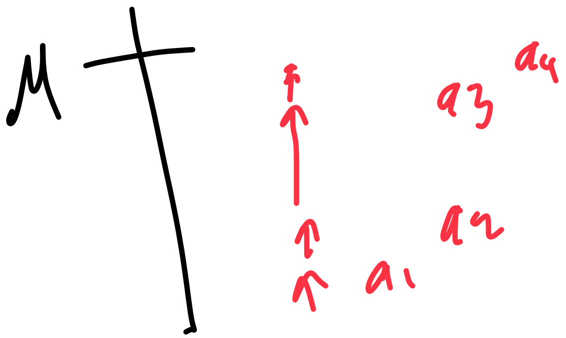
Bounded Monotonic Sequences Converge

- If $\{a_n\}$ is increasing and $a_n \leq M$, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \leq M$.
- If $\{a_n\}$ is decreasing and $a_n \geq m$, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \geq m$.

Say $\{a_n\}$ is bounded from above but not increasing: Must it converge? NO

↳ consider $\{a_n\} = (-1)^n$: doesn't converge

Proof



Convergence of an Infinite Series

An infinite series $\sum_{n=k}^{\infty} a_n$ *converges* to the sum S if the sequence of its partial sums $\{S_N\}$ converges to S:

$$\lim_{N\to\infty} S_N = S$$

In this case, we write $S = \sum_{n=k}^{\infty} a_n .$

- If the limit does not exist, we say that the infinite series diverges.
- If the limit is infinite, we say that the infinite series diverges to infinity.

Geometric

$$S_N = 1+x+x^2+\cdots+x^N$$

$$= \frac{1-x^{N+1}}{1-x}$$

$$= \frac{1}{1-x} - \frac{x^{N+1}}{1-x}$$

$$\lim_{N\to\infty} S_N = \frac{1}{1-x} \quad \text{if } |x|<1$$

THEOREM 1

Linearity of Infinite Series

If $\sum a_n$ and $\sum b_n$ converge, then $\sum (a_n + b_n)$, $\sum (a_n - b_n)$, and $\sum c a_n$ also converge, the latter for any constant c. Furthermore,

$$\begin{aligned} \sum (a_n + b_n) &= \sum a_n + \sum b_n \\ \sum (a_n - b_n) &= \sum a_n - \sum b_n \\ \sum c a_n &= c \sum a_n \qquad \text{(} c \text{ any constant)} \end{aligned}$$

assume all sums are finite!

Partial Sums of a Geometric Series

For the geometric series $\sum_{n=0}^{\infty} cr^n$ with $r \neq 1,$

$$S_N = c + cr + cr^2 + cr^3 + \cdots + cr^N = \frac{c\left(1 - r^{N+1}\right)}{1 - r} \tag{3}$$

THEOREM 3

Sum of a Geometric Series

Let $c \neq 0.$ If $|r| < 1,$ then

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \cdots = \frac{c}{1-r} \tag{4}$$

If $|r| \geq 1,$ then the geometric series diverges.

THEOREM 4

nth Term Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $a_n \not\to 0$ then there is an ε such that there are infinitely many n with $|a_n| > \varepsilon$

Example: $a_n = \frac{1}{1000} + \frac{(-1)^n}{2000}$

Notation: $\exists$ there exists and $\forall$ for all

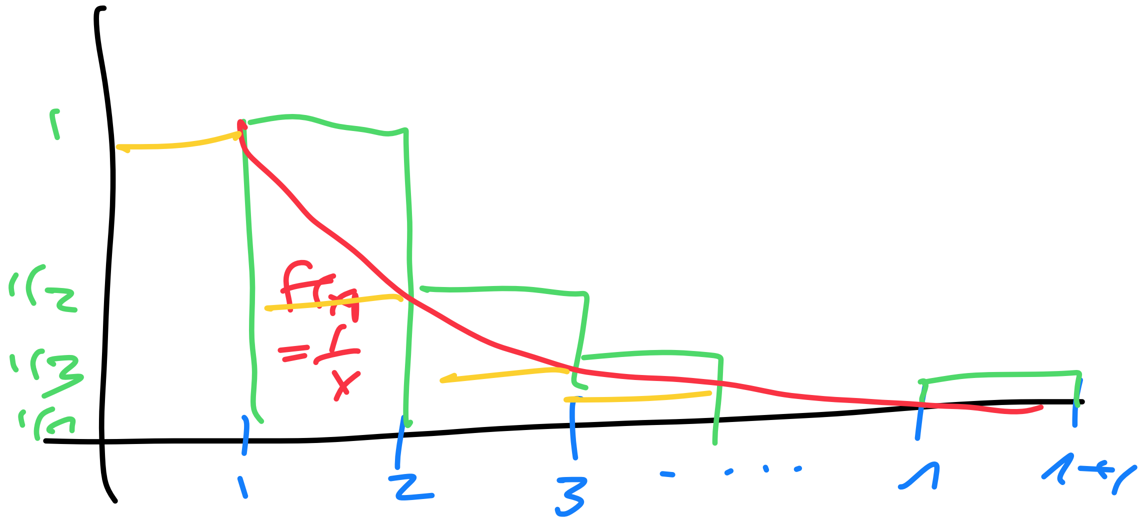
THEOREM 2

Integral Test

Let $a_n = f(n),$ where f is a positive, decreasing, and continuous function of x for $x \geq 1.$

i. If $\int_1^{\infty} f(x)\ dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. If $\int_1^{\infty} f(x)\ dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.



$a_n = 1/n \qquad f(x) = 1/x$

replace n with x

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}}$$

$$\int_1^n \frac{1}{x}\,dx \le \sum_{m=1}^{n} a_m \cdot 1$$

$$\int_1^n \frac{1}{x}\,dx + 1 \ge \sum_{m=1}^{n} a_m \cdot 1$$

$$\sum_{m=1}^{n} \frac{1}{m} \approx \int_1^n \frac{1}{x}\,dx$$

$$= \ln(x)\Big|_1^n = \ln(n)$$

or

$$\log(x)\Big|_1^n = \log(n)$$

THEOREM 4

Direct Comparison Test

only need for all but finitely many n

Assume that there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for $n \geq M$.

i. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

ii. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

if just had $a_n \leq b_n$

false:

let $a_n = -1$

If Σb_n diverges: nothing

if Σa_n converges: nothing

1.1. **10.1: Sequences – Problems.** #1: Exercise 10.1.24: Determine the limit of $a_n = \frac{n}{\sqrt{n^3+1}}$. #2: Exercise 10.1.62. Find the limit of $b_n = n!/\pi^n$. #3: Exercise 10.1.68. Use L'Hopital's Rule to find the limit of $b_n = \sqrt{n}\ln\left(1+\frac{1}{n}\right)$.

1.2. **10.2: Summing an Infinite Series – Problems.** #1: Exercise 10.2.15: Find the sum of $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots$.#2: Exercise 10.2.18: Use the nth Term Divergence Test to prove that $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^2+1}}$ diverges. #3: Exercise 10.2.27: Evaluate $\sum_{n=3}^{\infty}\left(\frac{3}{11}\right)^{-n}$. #4: Exercise 10.2.37: Evaluate $\frac{7}{8}-\frac{49}{64}+\frac{343}{512}-\frac{2401}{4096}+\cdots$.

1.3. **10.3: Convergence of Series with Positive Terms – Problems.** #1: Exercise 10.3.10: Use the Integral Test to determine whether $\sum_{n=1}^{\infty} ne^{-n^2}$ is a convergent infinite series. #2: Exercise 10.3.25. Use the Direct Comparison Test to determine whether $\sum_{n=1}^{\infty}\frac{2}{3^n+3^{-n}}$ is a convergent infinite series. #3: Exercise 10.3.57: Determine convergence or divergence for $\sum_{k=1}^{\infty}4^{1/k}$. #4: Exercise 10.3.68: Determine convergence or divergence for $\sum_{n=1}^{\infty}\frac{\sin(1/n)}{\sqrt{n}}$.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if $0/0$ or ∞/∞

L'Hopital

Proof: Say $0/0$

$f(a)=g(a)=0$

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$\frac{\frac{1}{x-a}}{\frac{1}{x-a}} = 1$$

Math 150: Multivariable Calculus: Spring 2023:
Lecture 06: Sequences and Series: https://youtu.be/kOIOjyHQtNc

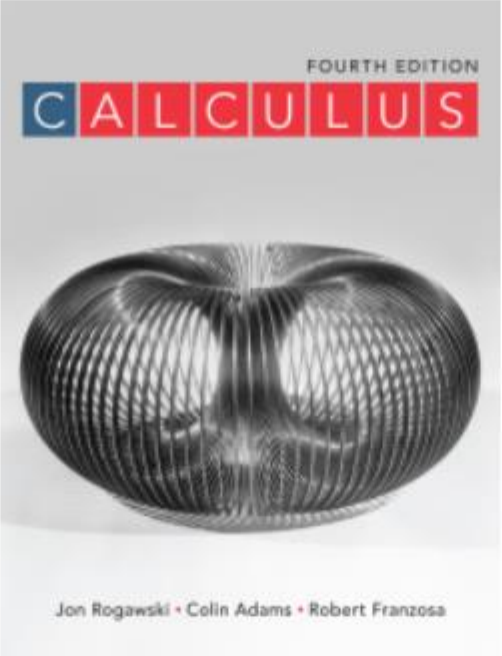
Plan for the day.

- Absolute and Conditional Convergence.

- Alternating Series.

- Convergence / Divergence Tests.

Note: all quoted text taken from the textbook for the class:



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DEFINITION

Absolute Convergence

The series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

THEOREM 1

Absolute Convergence Implies Convergence

If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

DEFINITION

Conditional Convergence

An infinite series $\sum a_n$ **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Ex:

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$$

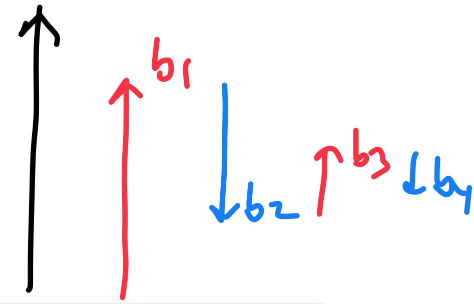
$$\frac{1}{2}\left(\underbrace{\frac{1}{1} - \frac{1}{3}}_{\frac{2}{1\cdot 3}} + \underbrace{\frac{1}{3} - \frac{1}{5}}_{\frac{2}{3\cdot 5}} + \underbrace{\frac{1}{5} - \frac{1}{7}}_{\frac{2}{5\cdot 7}} + \cdots\right)$$

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$$

diverges $b_n = \frac{1}{2n-1}$

$a_n = \frac{1}{2n}$ here $b_n \geq a_n$

Alternating Series Test



Assume that $\{b_n\}$ is a positive sequence that is decreasing and converges to 0:

$$b_1 > b_2 > b_3 > b_4 > \cdots > 0, \qquad \lim_{n\to\infty} b_n = 0$$

Then the following alternating series converges:

Ex:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 \underbrace{- b_2}_{\text{neg}} \overbrace{+ b_3}^{\text{pos}} \underbrace{- b_4}_{\text{neg}} \overbrace{+ b_5}^{\text{pos}} - b_6 + \cdots$$

Ex: $a_n = \frac{(-1)^n}{n}$ or $\frac{(-1)^{n+1}}{n}$

Furthermore, if $S = \sum_{n-1}^{\infty} (-1)^{n-1} b_n,$ then

$$0 < S < b_1 \quad \text{and} \quad S_p < S < S_q \quad \text{for } p \text{ even and } q \text{ odd}$$

THEOREM 1

Ratio Test

Assume that the following limit exists:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

i. If $\rho < 1$, then $\sum a_n$ converges absolutely.

ii. If $\rho > 1$, then $\sum a_n$ diverges.

iii. If $\rho = 1$, the test is inconclusive.

Ex: $a_n = r^n$

$$\rho = \lim_{n\to\infty} \left| \frac{r^{n+1}}{r^n} \right| = \lim_{n\to\infty} |r|$$

converge if $|r| < 1$, diverge if $|r| > 1$

Ex: $a_n = 1/n$

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

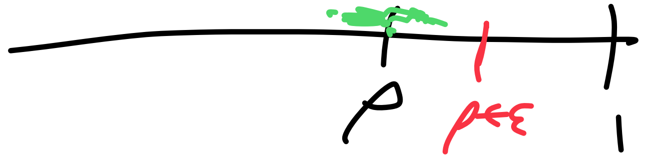
no information

Ex: $a_n = 1/n^2$ converges by the integral test

$$\rho = \lim_{n\to\infty} \left| \frac{1/(n+1)^2}{1/n^2} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

No information

Sketch of Proof



Assume $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\exists\, \varepsilon > 0$ such that $\forall n$ sufficiently large $\left|\frac{a_{n+1}}{a_n}\right| < \rho+\varepsilon < 1$

$|a_{n+1}| \le (\rho+\varepsilon)\,|a_n|$ for n big

$|a_{n+2}| \le (\rho+\varepsilon)\,|a_{n+1}| \le (\rho+\varepsilon)^2\,|a_n|$

$|a_{n+3}| \le (\rho+\varepsilon)\,|a_{n+2}| \le (\rho+\varepsilon)^3\,|a_n|$

$$|a_n| + |a_{n+1}| + |a_{n+2}| + \cdots \le |a_n|\Big(\underbrace{1 + (\rho+\varepsilon) + (\rho+\varepsilon)^2 + \cdots}_{\frac{1}{1-(\rho+\varepsilon)}}\Big)$$

as $|\rho+\varepsilon| < 1$

converges

Secret comparison test with a geometric series

THEOREM 2

Root Test

Assume that the following limit exists:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

i. If $L < 1$, then $\sum a_n$ converges absolutely.

ii. If $L > 1$, then $\sum a_n$ diverges.

iii. If $L = 1$, the test is inconclusive.

$L = \lim_{n\to\infty} 1/\sqrt[n]{n!}$

$= \lim_{n\to\infty} \dfrac{1}{\left(\underbrace{n\cdot(n-1)\cdots 3\cdot 2\cdot 1}\right)^{1/n}}$

$\geq \dfrac{1}{\left(\left(\frac{2}{2}\right)^{n/2}\right)^{1/n}} = \dfrac{1}{\sqrt{\frac{1}{2}}}$

Ex: $a_n = 1/n!$ Consider $\sum_{n=1}^{\infty} 1/n!$

$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots$

Comparison test:

n big, $1/n! \leq 1/n^2$ converges

$\dfrac{1}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$ vs $\dfrac{1}{n^2}$

Comparison $1/n! \leq 1/2^n$ if n is big

Ratio: $\rho = \lim_{n\to\infty} \dfrac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \dfrac{n!}{(n+1)!}$

$= \lim_{n\to\infty} \dfrac{1}{n+1} = 0$ Converges

1.4. **10.4: Absolute and Conditional Convergence – Problems.** #1: Exercise 10.4.7: Determine whether the series converges absolutely, conditionally, or not at all: $\sum_{n=1}^{\infty} (-1)^n e^{-n}/n^2$. #2: Exercise 10.4.26: Determine whether the series converges absolutely, conditionally, or not at all: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$. #3: Exercise 10.4.36: Determine whether the following series converges conditionally: $1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{7} + \frac{1}{4} - \frac{1}{9} + \frac{1}{5} - \frac{1}{11} + \cdots$.

1.5. **10.5: The Ratio and Root Tests and Strategies for Choosing Tests – Problems.** #1: Exercise 10.5.18: Use the Ratio Test to evaluate $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$. #2: Exercise 10.5.25: Show that $\sum_{n=1}^{\infty} \frac{r^n}{n}$ converges if $|r| < 1$. #3: Exercise 10.5.40: Use the Root Test to evaluate $\sum_{n=1}^{\infty} \left(2 + \frac{1}{n}\right)^{-n}$. #4: Exercise 10.5.60: Evaluate $\sum_{n=1}^{\infty} (-1)^n \cos(\frac{1}{n})$.

Consider $e^x := \sum_{n=0}^{\infty} x^n/n! = 1 + x + x^2/2! + x^3/3! + \cdots$

Ratio: $\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \frac{x^{n+1}}{x} \right| = \lim_{n\to\infty} \frac{1}{n+1} |x| = 0$

converges absolutely $\forall x$

$e^x e^y = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{y^m}{m!}\right) \overset{\text{yes it is}}{=} \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$

Math 150: Multivariable Calculus: Spring 2023:
Lecture 07: Taylor Series: https://youtu.be/pLqCQFS9KMM

Plan for the day.

- Taylor Series.
- Errors in Taylor Expansions.
- Famous Taylor Series and Applications.

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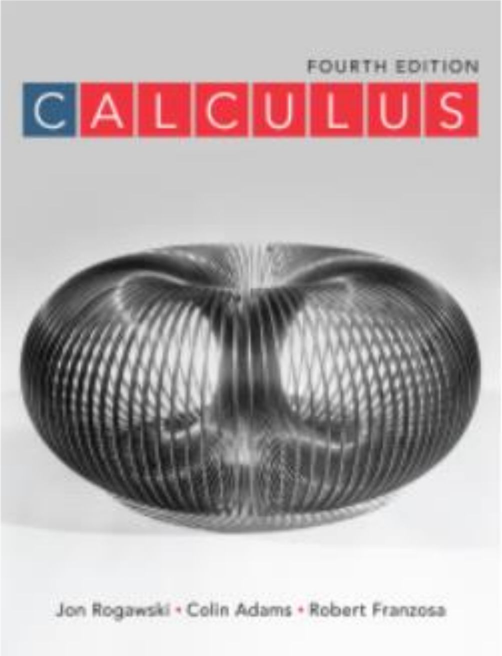

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With series we can make sense of the idea of a polynomial of infinite degree:

$$F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Specifically, a **power series** with center c is an infinite series

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \cdots$$

THEOREM 1

Radius of Convergence

Every power series

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

has a radius of convergence R, which is either a nonnegative number $(R \geq 0)$ or infinity $(R = \infty)$. If R is finite, $F(x)$ converges absolutely when $|x-c| < R$ and diverges when $|x-c| > R$. If $R = \infty$, then $F(x)$ converges absolutely for all x.

HAVE $|x-3| < 2$

Either $x-3 < 2$ or $-(x-3) < 2$

Get $x < 5$ or $-x+3 < 2$

$-x < -1$

$x > 1$

Ex: $a_n = 1/2^n \qquad c = 3$

$$b_n = a_n(x-c)^n = \frac{1}{2^n}(x-3)^n = \left(\frac{x-3}{2}\right)^n$$

$$\rho = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left| \left(\frac{x-3}{2}\right)^{n+1} \Big/ \left(\frac{x-3}{2}\right)^n \right|$$

$$= \lim_{n\to\infty} \left|\frac{x-3}{2}\right| = \left|\frac{x-3}{2}\right|$$

want $\rho < 1$ to converge

so need $\left|\frac{x-3}{2}\right| < 1$

so $|x-3| < 2$

so $1 < x < 5$

THEOREM 2

Term-by-Term Differentiation and Integration

Assume that

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence $R > 0$. Then F is differentiable on $(c-R, c+R)$. Furthermore, we can integrate and differentiate term by term. For $x \in (c-R, c+R)$,

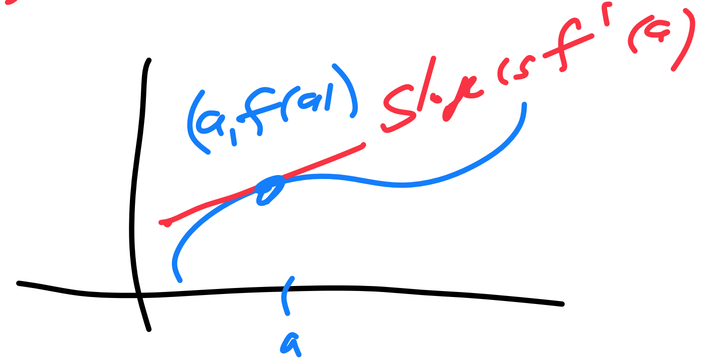
$$\begin{aligned} F'(x) &= \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \\ \int F(x)\,dx &= A + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1} \qquad (A \text{ any constant}) \end{aligned}$$

For both the derivative series and the integral series the radius of convergence is also R.

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Deg n Taylor Series

Zeroth Order: $T_0(x) = f(a)$ note $T_0(a) = f(a)$



1st Order: $T_1(x) = f(a) + f'(a)(x-a)$

note: $T_1'(x) = 0 + f'(a) * 1$

so $T_1'(a) = f'(a)$

tangent line:

$y - f(a) = f'(a)(x-a)$

$\Rightarrow y = f(a) + f'(a)(x-a)$

2nd Order: $T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$T_2'(x) = 0 + f'(a) * 1 + \frac{f''(a)}{2!} 2(x-a)'$

$T_2''(x) = 0 + 0 + \frac{f''(a)}{2!} 2 \cdot 1$

$T_2(a) = f(a)$, $T_2'(a) = f'(a)$, $T_2''(a) = f''(a)$

For n^{th} order, $T(a) = f(a), \ldots, T^{(n)}(a) = f^{(n)}(a)$

DEFINITION

Taylor Series

If f is infinitely differentiable at $x = c$, then the Taylor series for $f(x)$ centered at c is the power series

$$T(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Questions: 1) Does it converge, and if so for what x?

2) What does it converge to? Does it converge to $f(x)$.

Answers: 1) NOT ALWAYS GOING TO CONVERGE

2) CAN CONVERGE TO SOMETHING OTHER THAN $f(x)$

THEOREM 1

The polynomial T_n centered at a agrees with f to order n at $x = a$, and it is the only polynomial of degree at most n with this property.

Assume that $f^{(n+1)}$ exists and is continuous. Let *K* be a number such that $\left|f^{(n+1)}(u)\right| \le K$ for all u between a and x. Then

$$\boxed{|f(x) - T_n(x)| \le K \frac{|x-a|^{n+1}}{(n+1)!}}$$

K is $K_{f,n+1}$

or $K_{f^{(n+1)}}$

where T_n is the nth Taylor polynomial centered at $x = a$.

For a proof of a weaker error bound, using the MVT and the IVT (with proofs of each), see

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp23/handouts/MVT_TaylorSeries.pdf

THEOREM 2

Let $I=(c-R,c+R)$, where $R>0$, and assume that f is infinitely differentiable on I. Suppose there exists $K>0$ such that all derivatives of f are bounded by K on I:

$$\left|f^{(k)}(x)\right|\leq K \quad \text{for all } k\geq 0 \quad \text{and} \quad x\in I$$

Then f is represented by its Taylor series in I:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \quad \text{for all } x\in I$$

Note K is indep of n!

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp23/mathematicaprograms/ComparingTaylorSeries.nb

Taylor Series

Goal is to see how well Taylor Series approximate functions,

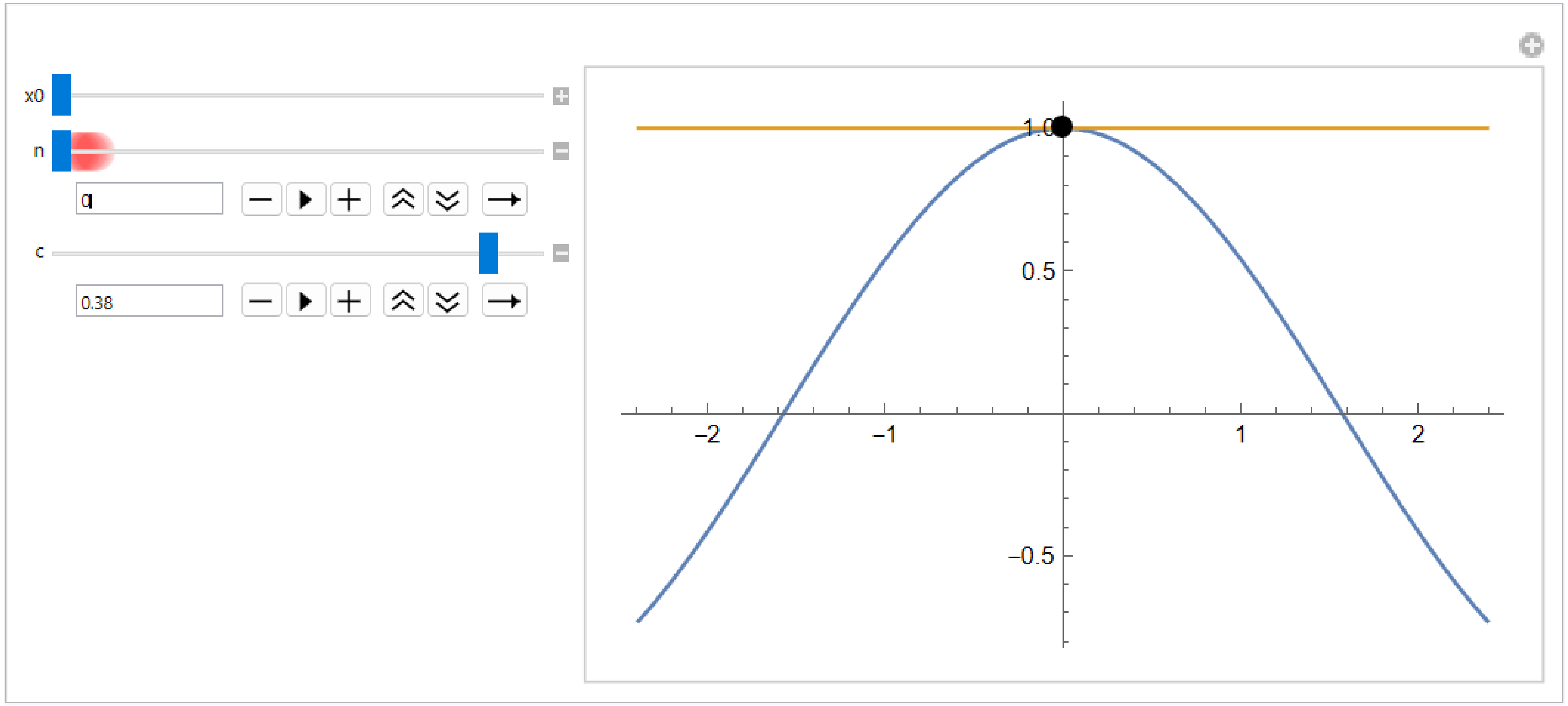
how listtle later terms change approximation ....

For definiteness, will do Cos[x]

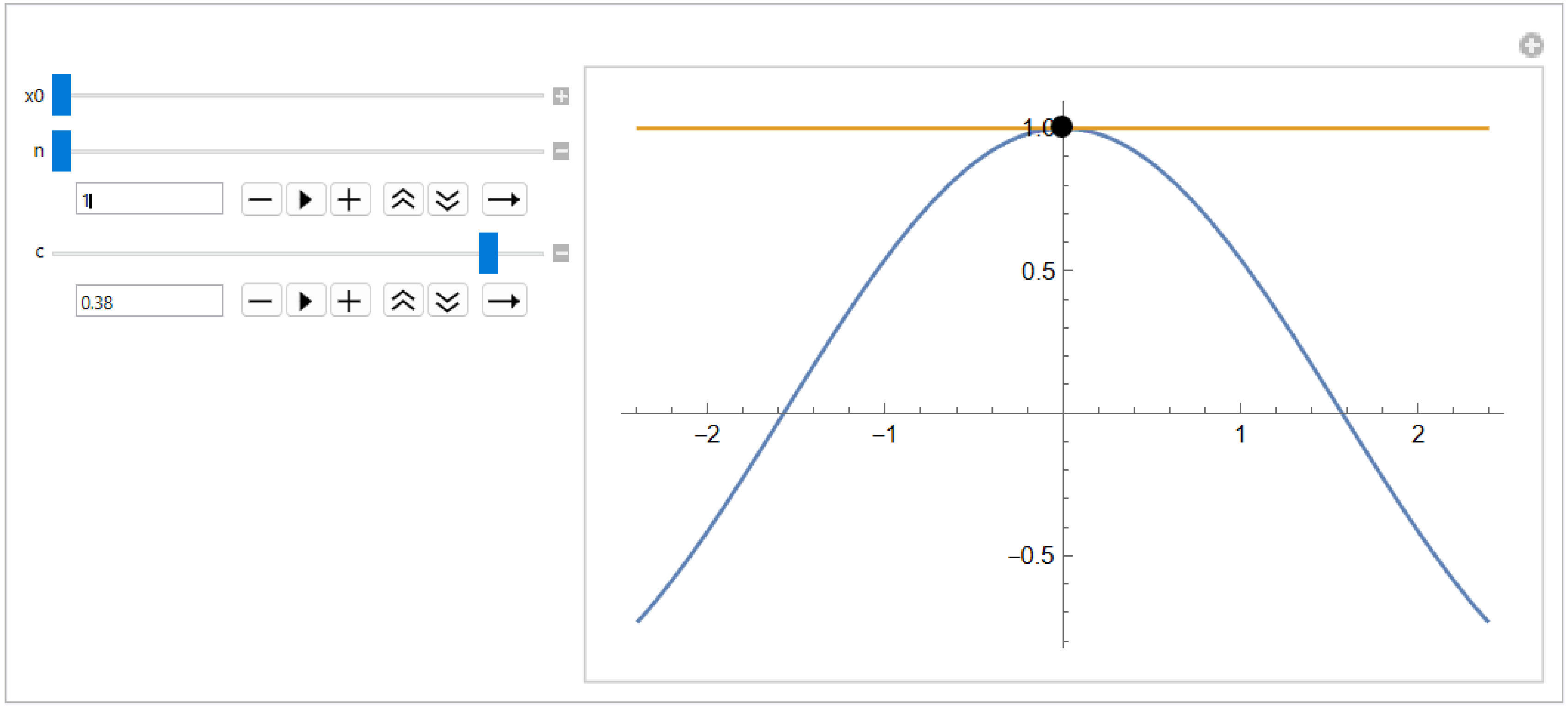
```
In[ ]:= coeff[x0_, n_] := If[Mod[n, 4] == 0, Cos[x0],
    If[Mod[n, 4] == 1, - Sin[x0],
     If[Mod[n, 4] == 2, - Cos[x0], Sin[x0]]
    ]];
approx[x_, x0_, n_] := Sum[coeff[x0, nn] (x - x0) ^ nn / nn!, {nn, 0, n}]

Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 - 2 Pi c, x0 + 2 Pi c},
     Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi / 2}, {n, 1, 40},
  {c, 4, .01}]
```

```
Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 - 2 Pi c, x0 + 2 Pi c},
    Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi/2}, {n, 1, 40},
  {c, 4, .01}]
```



```
Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 - 2 Pi c, x0 + 2 Pi c},
    Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi/2}, {n, 1, 40},
  {c, 4, .01}]
```



$f(x) = \cos x$

$f(0) = 1$

$f'(x) = -\sin x$

$f'(0) = 0$

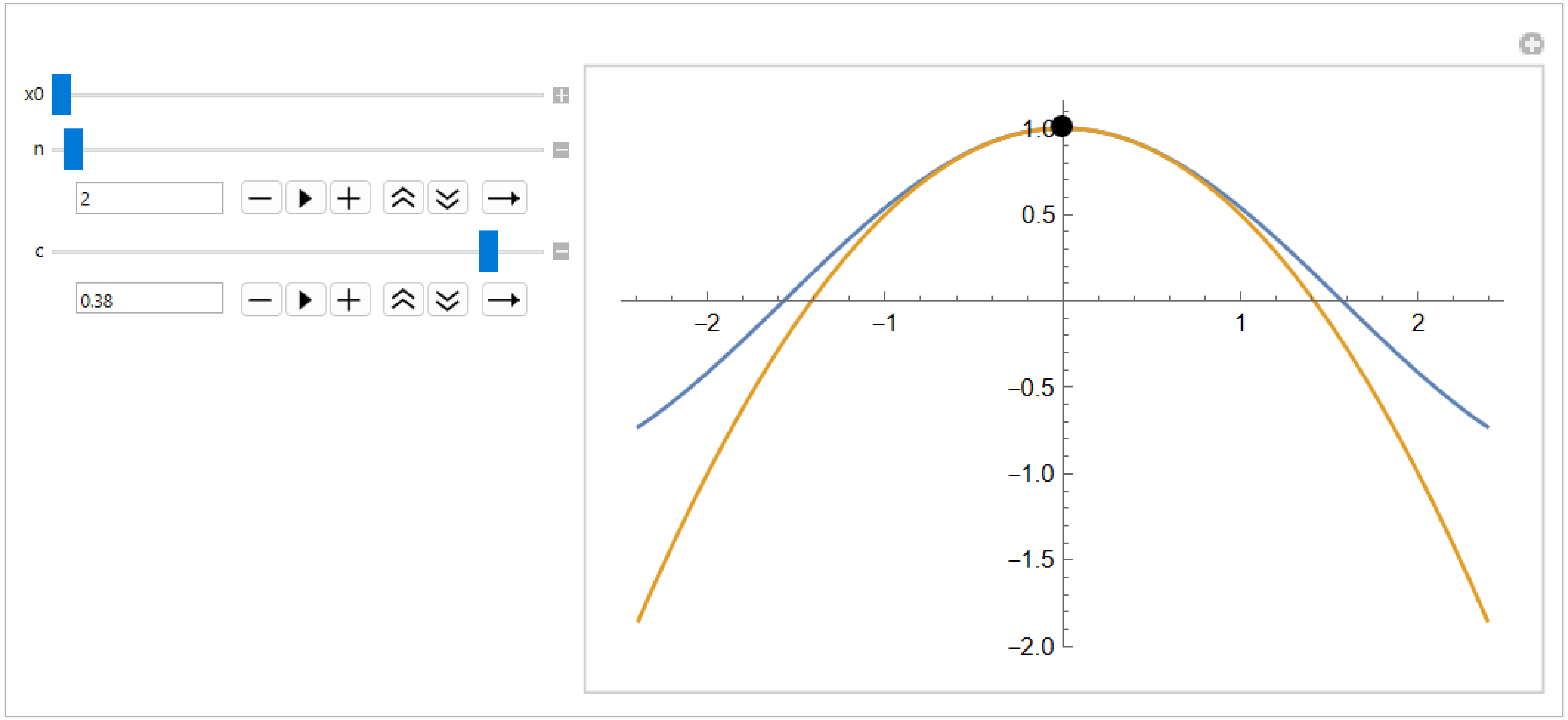
$f''(x) = -\cos x$

$f''(0) = -1$

$f'''(x) = \sin x$

$f''''(x) = \cos x$

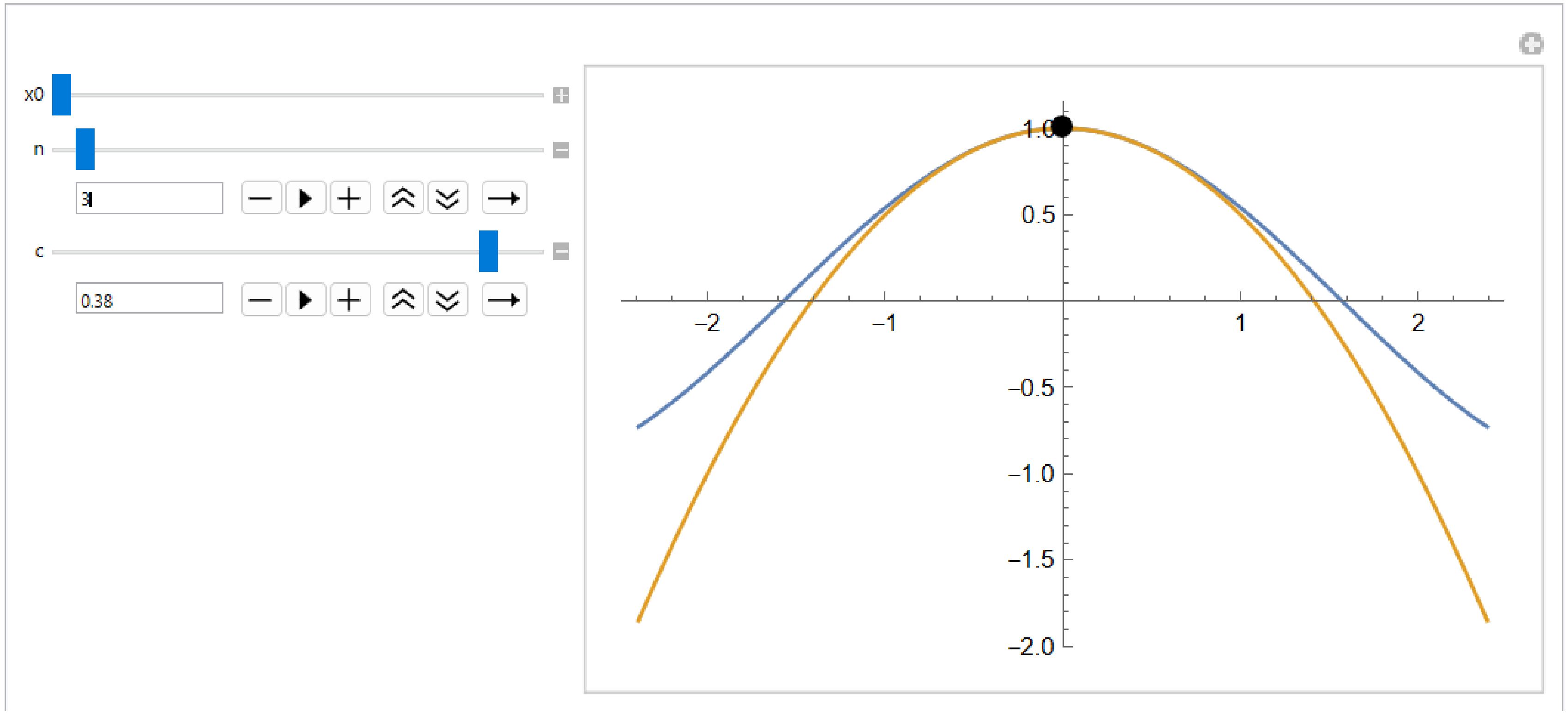
```
Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 - 2 Pi c, x0 + 2 Pi c},
    Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi/2}, {n, 1, 40},
  {c, 4, .01}]
```



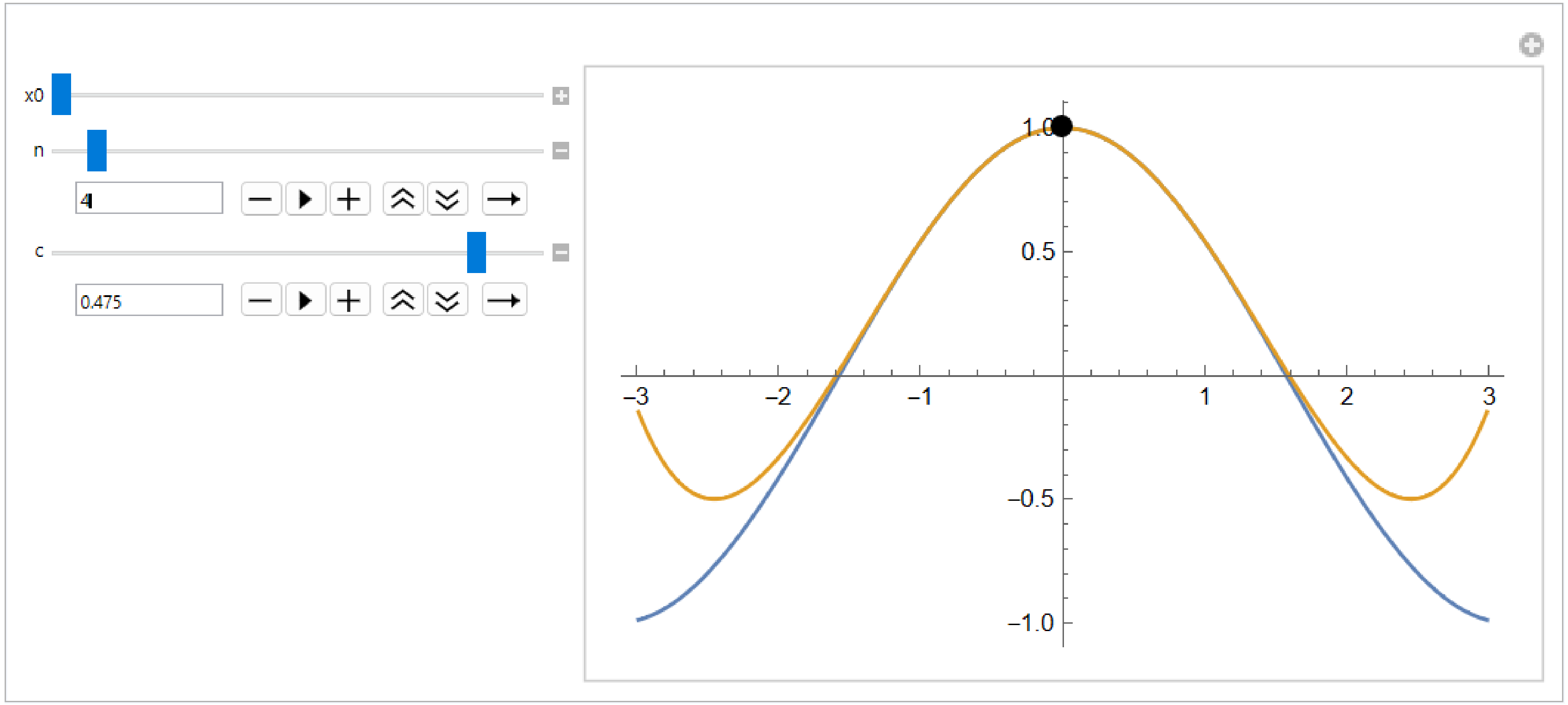
$T_2(x)$

$= 1 - \frac{x^2}{2!}$

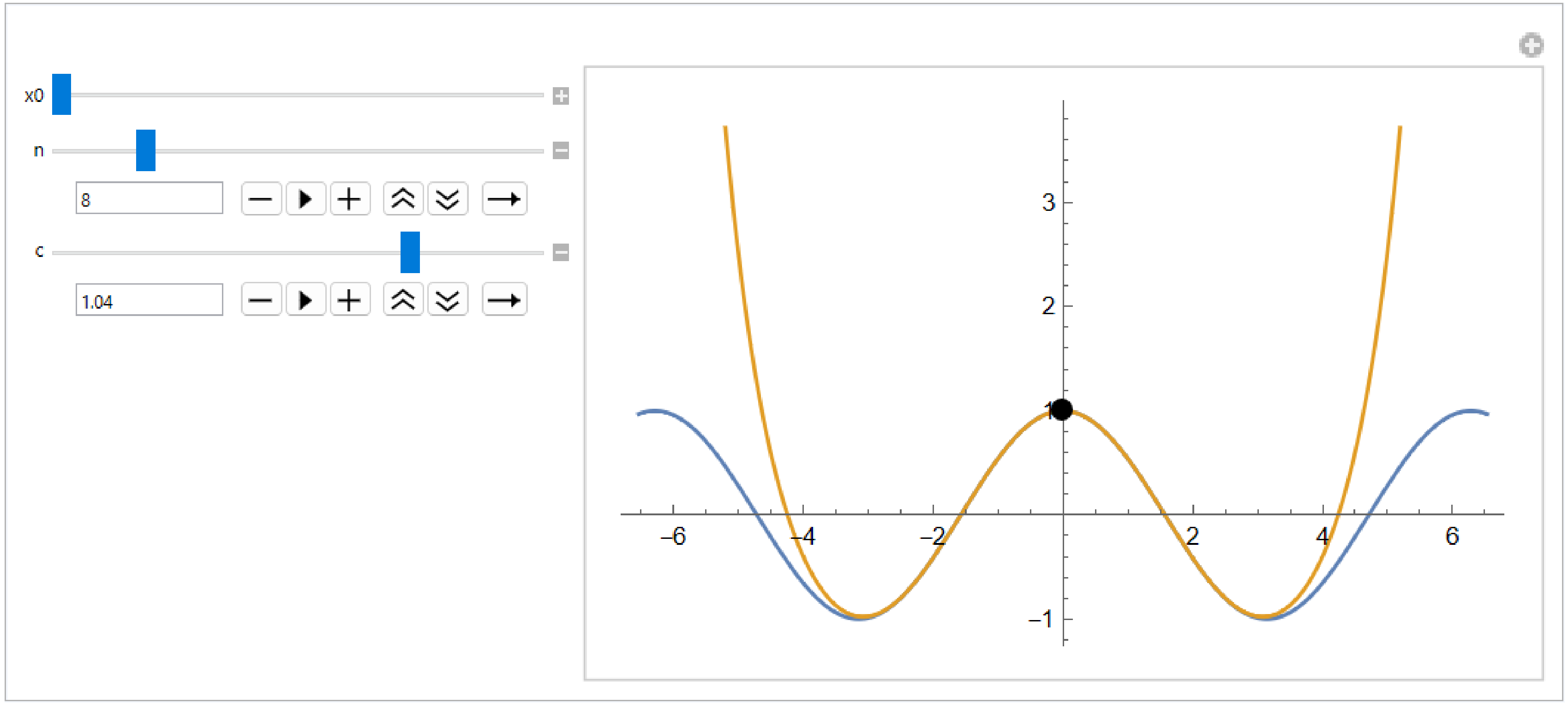
```
Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 − 2 Pi c, x0 + 2 Pi c},
   Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi/2}, {n, 1, 40},
 {c, 4, .01}]
```



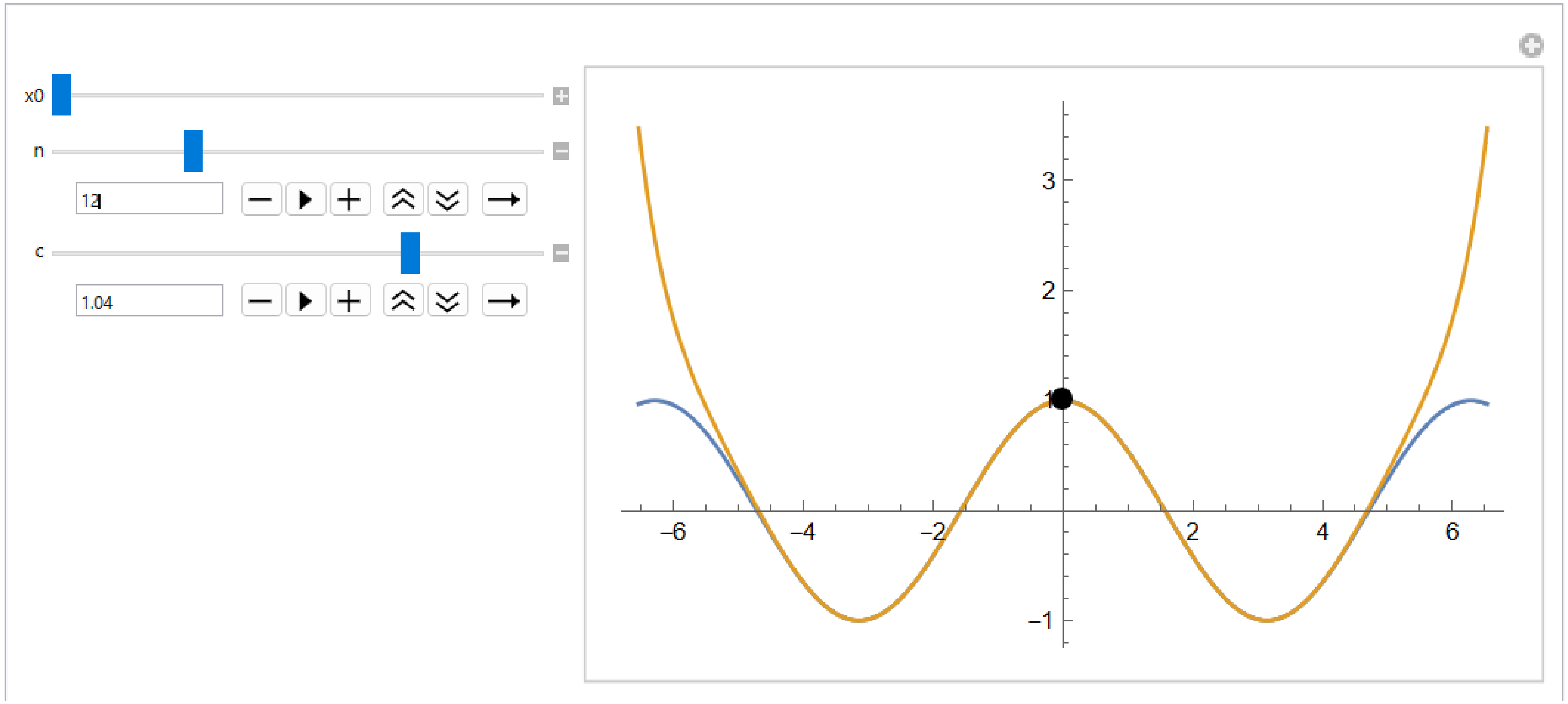
```
Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 − 2 Pi c, x0 + 2 Pi c},
    Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi/2}, {n, 1, 40},
  {c, 4, .01}]
```



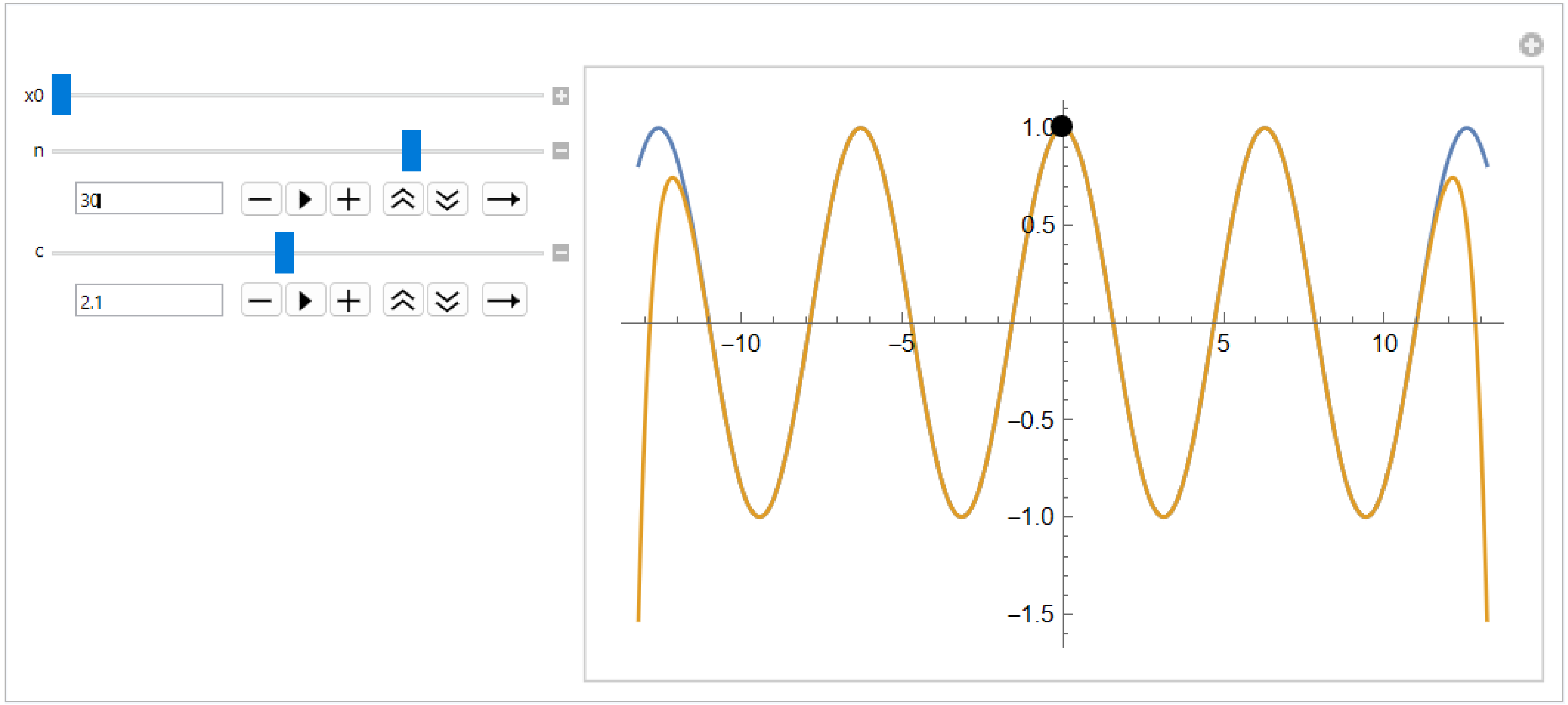
```
Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 - 2 Pi c, x0 + 2 Pi c},
    Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi/2}, {n, 1, 40},
  {c, 4, .01}]
```



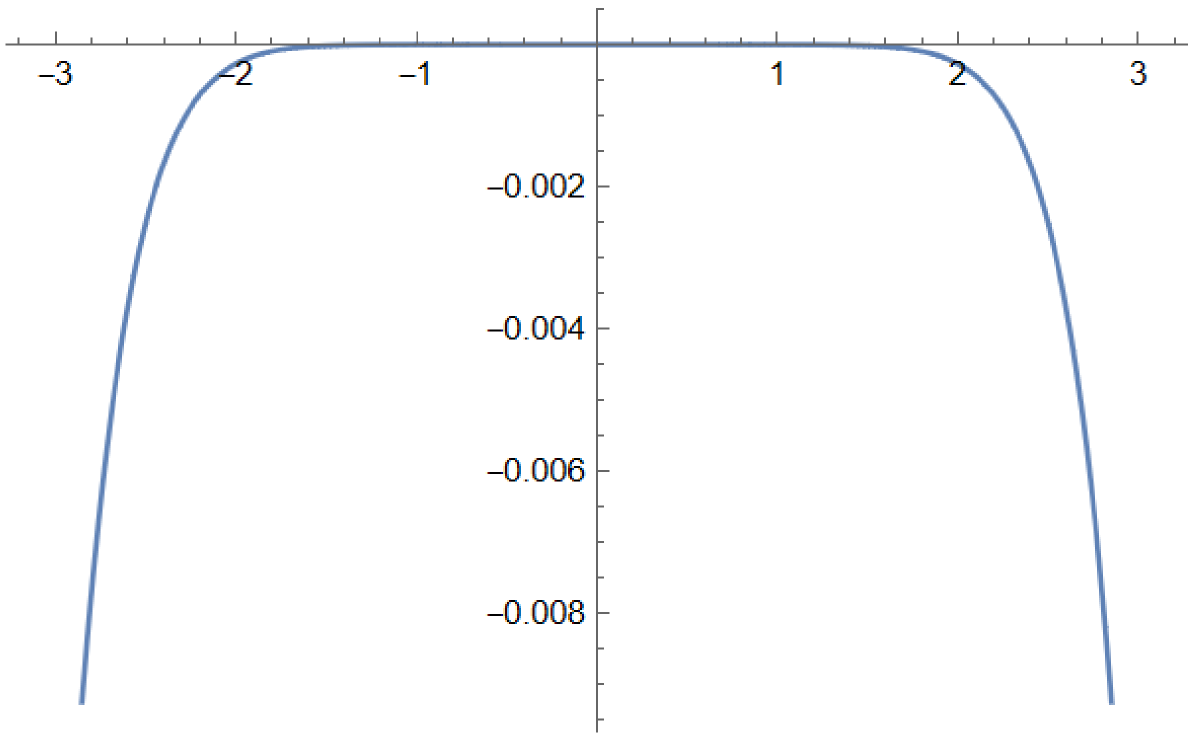
```
Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 − 2 Pi c, x0 + 2 Pi c},
   Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi/2}, {n, 1, 40},
 {c, 4, .01}]
```



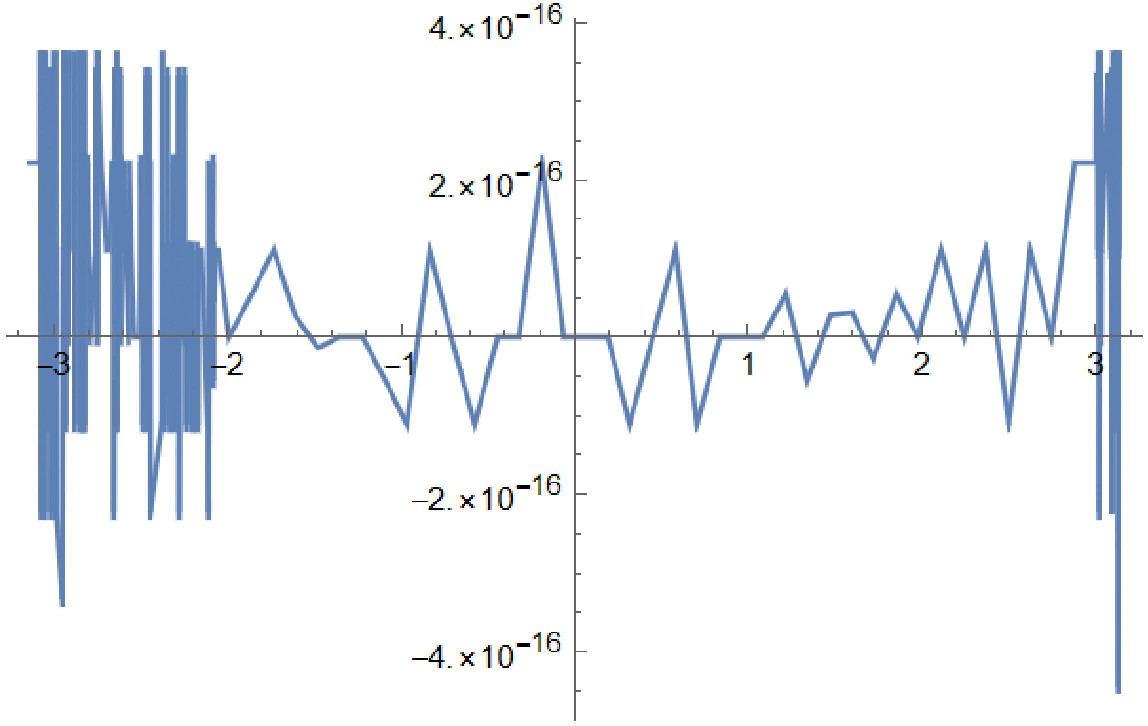
```
Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 - 2 Pi c, x0 + 2 Pi c},
    Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi/2}, {n, 1, 40},
  {c, 4, .01}]
```



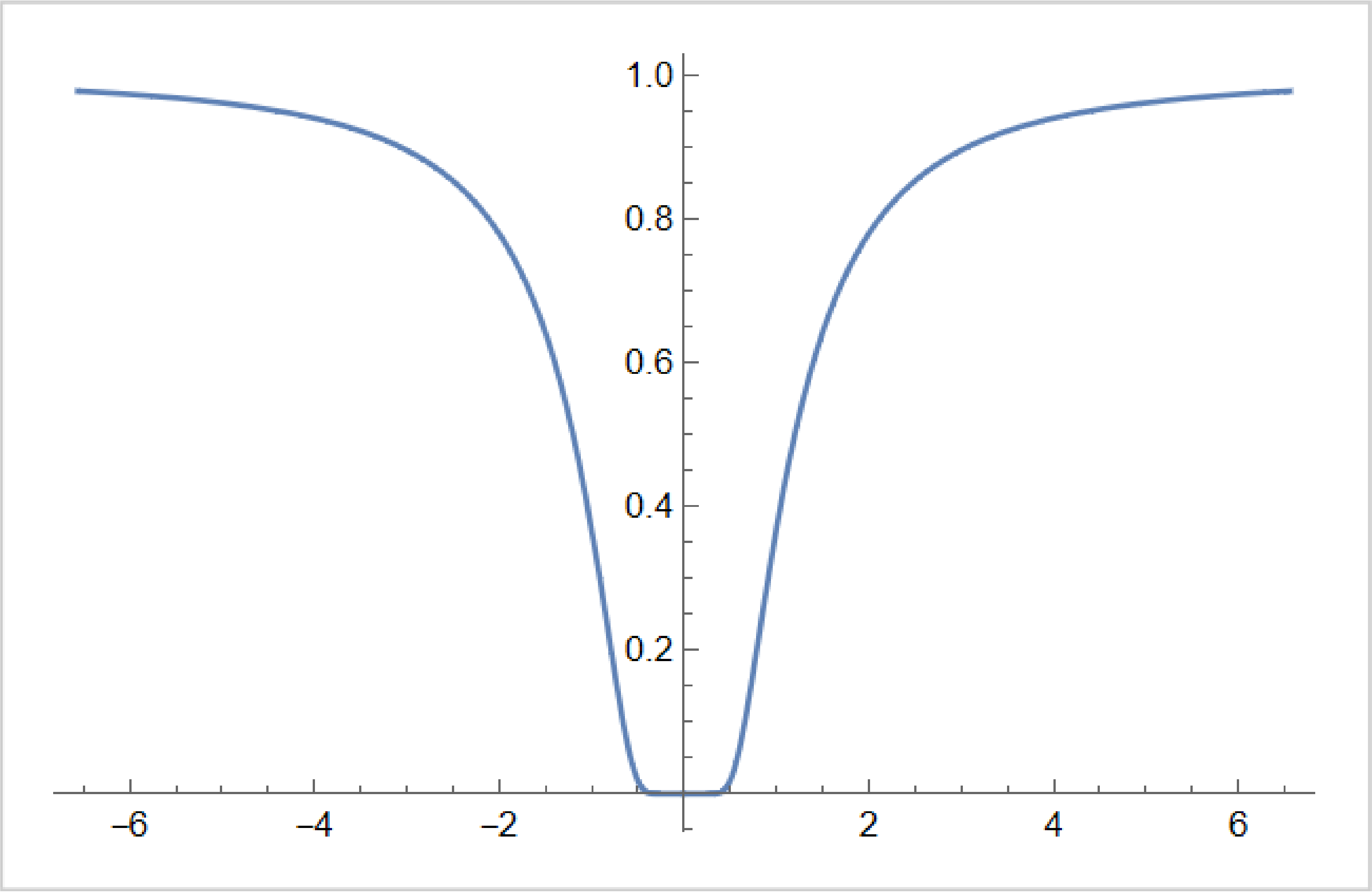
```
Plot[Cos[x] - Sum[x^(2 k) (-1)^k / Factorial[2 k],
   {k, 0, 4}], {x, -Pi, Pi}]
```




```
Plot[Cos[x] - Sum[x^(2 k) (-1)^k / Factorial[2 k],
   {k, 0, 15}], {x, -Pi, Pi}]
```

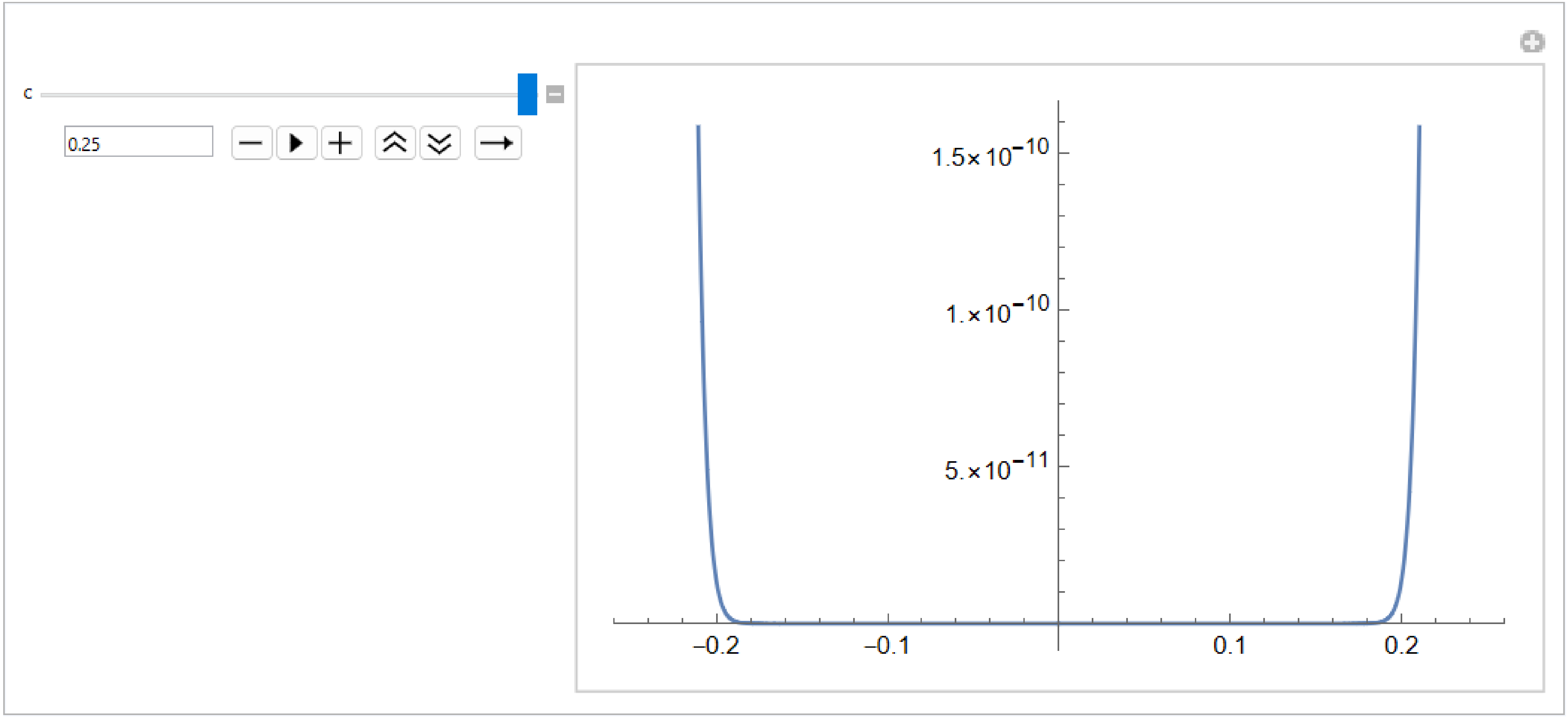


```
Manipulate[Plot[Exp[-1/x^2], {x, -c, c}], {c, 10, .25}]
```



$$\begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

```
Manipulate[Plot[Exp[-1/x^2], {x, -c, c}], {c, 10, .25}]
```



$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$i^2 = -1$ where $i = \sqrt{-1}$

$e^{ix} = \cos x + i \sin x$, $e^{ix} e^{iy} = e^{i(x+y)}$

$$\binom{a}{n} = \frac{a\,(a-1)\,(a-2)\cdots(a-n+1)}{n!}, \quad \binom{a}{0} = 1$$

THEOREM 3

The Binomial Series

For any exponent a and for $|x| < 1$,

$$(1+x)^a = 1 + \tfrac{a}{1!}x + \tfrac{a(a-1)}{2!}x^2 + \tfrac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n}x^n + \cdots$$

1.6. **10.6: Power Series – Problems.** #1: Exercise 10.6.14: Find the interval of convergence: $\sum_{n=8}^{\infty} n^7 x^n$. #2: Exercise 10.6.29: Find the interval of convergence: $\sum_{n=1}^{\infty} \frac{2^n}{3n} (x+3)^n$. #3: Exercise 10.6.59: Find all values of x such that $\sum_{n=1}^{\infty} \frac{x^{n^2}}{n!}$ converges.

1.7. **10.7: Taylor Polynomials – Problems.** #1: Exercise 10.7.9: Calculate the Taylor polynomials T_2 and T_3 for $f(x) = \tan(x)$ centered at $x = 0$. #2: Exercise 10.7.29: Find T_n for all n for $f(x) = \cos x$ centered at $x = \frac{\pi}{4}$. #3: Exercise 10.7.33: Find T_2 and use a calculator to compute the error $|f(x) - T_2(x)|$ for $a = 1$, $x = 1.2$, and $f(x) = x^{-2/3}$.

1.8. **10.8: Taylor Series – Problems.** #1: Exercise 10.8.13: Find the Maclaurin series and find the interval on which the expansion is valid for $f(x) = e^{x-2}$. #2: Exercise 10.8.15: Find the Maclaurin series and find the interval on which the expansion is valid for $f(x) = \ln(1-5x)$. #3: Exercise 10.8.37: Find the Taylor series centered at $c = 4$ and the interval on which the expansion is valid for $f(x) = 1/x^2$. #4: Exercise 10.8.70: Find the function with $f(x) = x^4 - \frac{x^{12}}{3} + \frac{x^{20}}{5} - \frac{x^{28}}{7} + \cdots$ as its Maclaurin series. #5: Exercise 10.8.90: Use Euler's Formula to demonstrate $\cos z = \left(e^{iz} + e^{-iz}\right)/2$.

Math 150: Multivariable Calculus: Spring 2023:
Lecture 08: Taylor Series II: https://youtu.be/KevnjvST4Kg

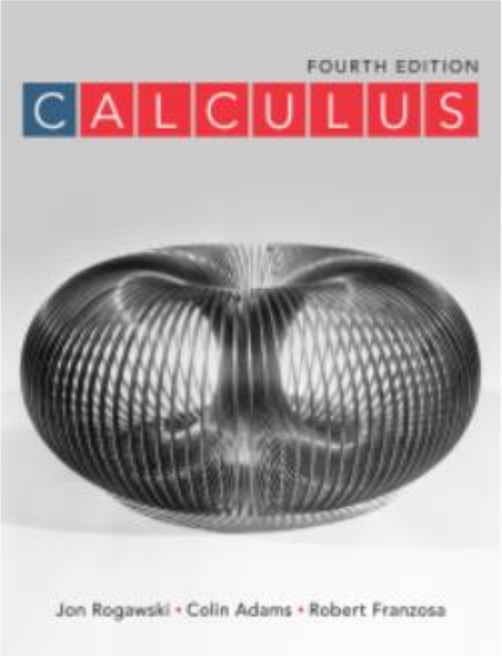
Plan for the day.

- Taylor Series Computations.

- Taylor Series and Trigonometric Identities.

- Multivariable Taylor Series.

Note: all quoted text taken from the textbook for the class:



Calculus 4th Edition

Author(s)	Jon Rogawski; Colin Adams; Robert Franzosa
Publisher	W.H. Freeman & Company
Format	Reflowable
Print ISBN	9781319050733, 1319050735
eText ISBN	9781319055844, 1319055842
Edition	4th
Copyright	2019

What is the Taylor Series of f(x) = cos(x) sin(x)?

Taylor Series of $f(x)$ is

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad \text{note } T_f^{(n)}(0) = f^{(n)}(0)$$

$f(0) = 0$

$f'(x) = -\sin x \sin x + \cos x \cos x = \cos^2 x - \sin^2 x$

$f'(0) = 1$

$f''(0) = 2\cos x(-\sin x) - 2\sin x \cos x$

$= -4\cos x \sin x = -4 f(x)$

$f''(0) = 0$

$f'''(0) = -4, \quad f^{iv}(0) = 16, \ldots$

Get

$T_f(x) =$

$x - \frac{4}{3!} x^3 + \ldots$

$f(x) = \cos x \sin x$

Math is lazy: reduce to known

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x \sin x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

$$= x + \left(-\frac{1}{2!} * 1 + 1 * \frac{-1}{3!}\right) x^3 + \dots$$

$$= x - \frac{2}{3} x^3 + \dots \quad \text{or} \quad x - \frac{4}{3!} x^3 + \dots$$

Aside: Feynman: $U = 0$

Unworldliness: $(F - ma)^2 + (E - mc^2)^2 + \cdots$

$f(x) = \cos x \sin x$

$f(x) = \sin x \cos x$

$= \frac{1}{2} 2 \sin x \cos x = \frac{1}{2} \sin(2x)$

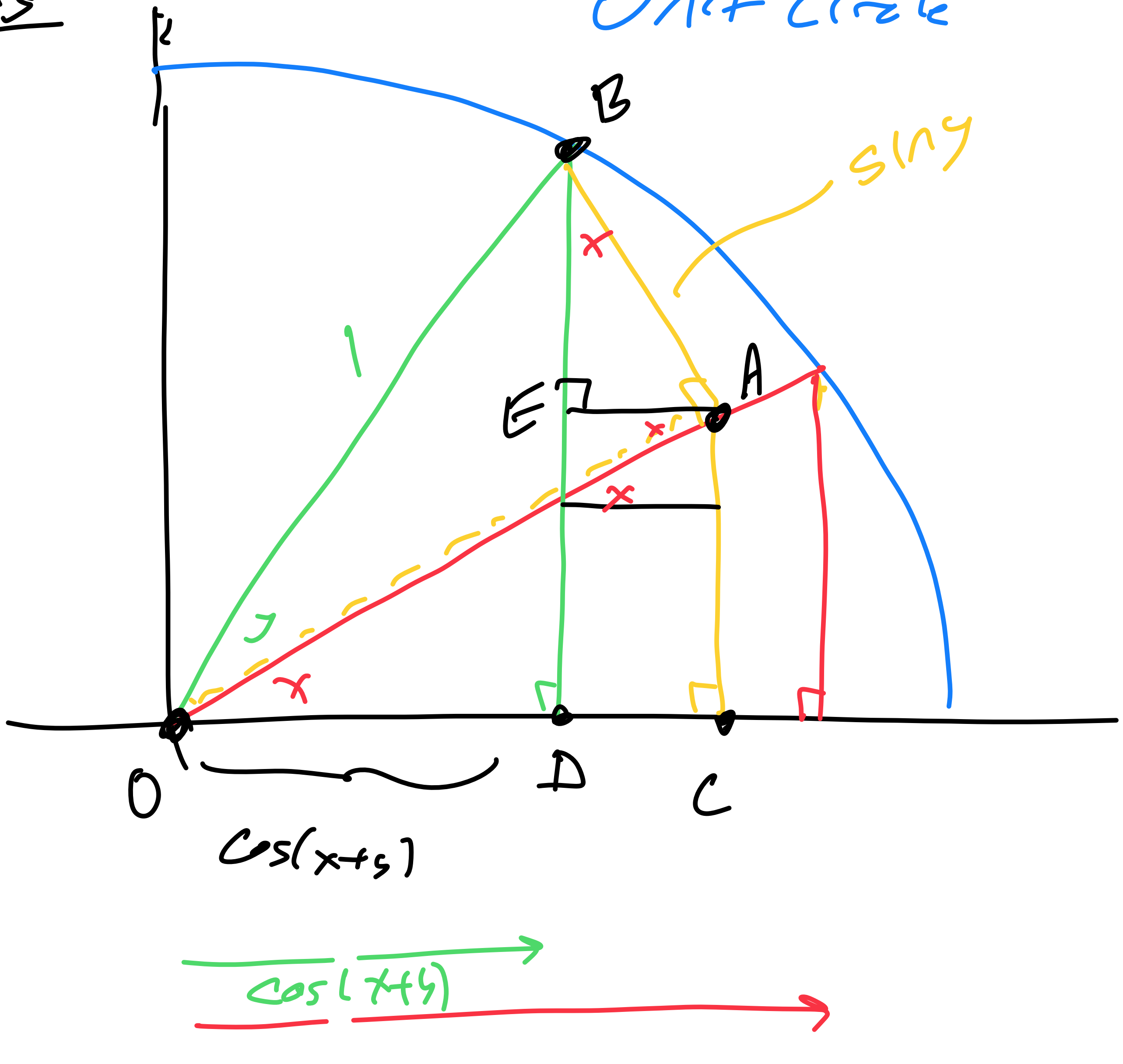
Know $\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots$

let $u = 2x$

Thus $\frac{1}{2} \sin(2x) = \frac{1}{2}\left(2x - \frac{(2x)^3}{3!} + \cdots\right) = \frac{1}{2}\left(2x - \frac{8x^3}{3!} + \cdots\right)$

$= x - 4x^3/3! + \cdots$

Trig:



$\overline{OA} = \cos y$
from $\triangle OAB$

Look at $\triangle OAC$
$\overline{OA}$ is $\cos y$
so $\overline{OC}$ is $\cos y * \cos(x)$
$\overline{OC}$ is $\cos x \cos y$
overshot $\overline{OD}$ by $\overline{DC}$
$\overline{DC} = \overline{EA} = hyp * \sin(x)$
$= \sin y \sin x$

$\overline{OD} = \overline{OC} - \overline{DC} = \cos x \cos y - \sin x \sin y$

Trig by Calculus

$$e^x = 1 + x + x^2/2! + x^3/3! + \cdots = \sum_{n=0}^{\infty} x^n/n!$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Identity: $e^x e^y = e^{x+y}$

Pythagoras: $e^{ix} e^{-ix} = e^0 = 1$

But this is $(\cos x + i\sin x)(\cos(-x) + i\sin(-x))$

Using $e^{ix} = \cos x + i\sin x$, note $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$

$$e^{-ix} = \cos x - i\sin x = \cos(-x) + i\sin(-x)$$

$$(\cos x + i\sin x)(\cos x - i\sin x) = \cos^2 x - i^2 \sin x$$

$$= \cos^2 x + \sin^2 x \quad \blacksquare$$

Angle Addition

$$e^{ix}e^{iy} = e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$$

$$(\cos x + i\sin x)(\cos y + i\sin y)$$

$$= (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y)$$

if $a+ib = c+id$ then $a=c$, $b=d$ if $a,b,c,d \in \mathbb{R}$

recall $i=\sqrt{-1}$ so $i^2=-1$

Big Items: $e^{ix} = \cos x + i\sin x$

$e^{ix}e^{iy} = e^{i(x+y)}$ with $i^2=-1$

$$f(x,y) = \sin(y)\cos(x+y)$$

$$\sin(y) = y - y^3/3! + \cdots$$

$$\cos(u) = 1 - u^2/2! + u^4/4! - \cdots \qquad \text{take } u = x+y$$

$$\cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \cdots$$

$$= 1 - \frac{x^2+2xy+y^2}{2!} + \frac{x^4+4x^3y+\cdots+y^4}{4!} - \cdots$$

$$\sin(y)\cos(x+y) = 0 + 0x + 1y + 0x^2 + 0xy + 0y^2$$

$$- \frac{1}{2!}x^2y - \frac{2}{2!}xy^2 - \frac{1}{2!}y^3 + \cdots$$

Partial Derivatives

$\dfrac{\partial f}{\partial x}$ means take the deriv with respect to x, holding all other variables constant.

$$f(x) = 3x^2 + 17x + 8 \qquad f'(x) = \frac{df}{dx} = 6x + 17$$

$3 \cdot 2$

$$f(x,y) = x^2y + 17x + 8y^3 \qquad \frac{\partial f}{\partial x} = 2xy + 17 + 0$$

$$\frac{\partial f}{\partial y} = x^2 + 0 + 24y^2$$

read as partial f, partial x

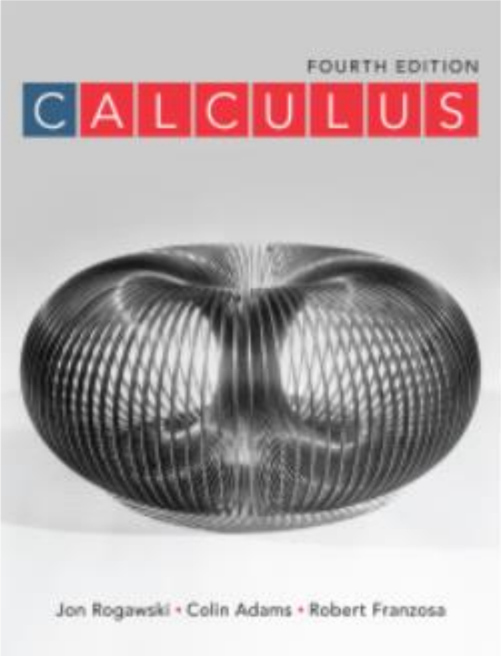
Math 150: Multivariable Calculus: Spring 2023:
Lecture 09: Introduction to Vectors: https://youtu.be/K0J6WHQwLQQ

Plan for the day.

- Definition of Vectors.
- Vector Algebra and Properties, unit vectors, i, j, k....
- Distance Formula.
- Equations of Lines.

Note: all quoted text taken from the textbook for the class:

Calculus 4th Edition



Author(s)	Jon Rogawski; Colin Adams; Robert Franzosa
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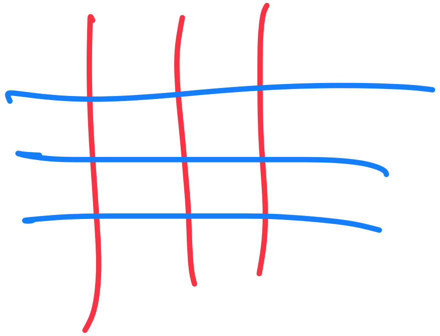
Print ISBN	9781319050733, 1319050735
eText ISBN	9781319055844, 1319055842
Edition	4th
Copyright	2019

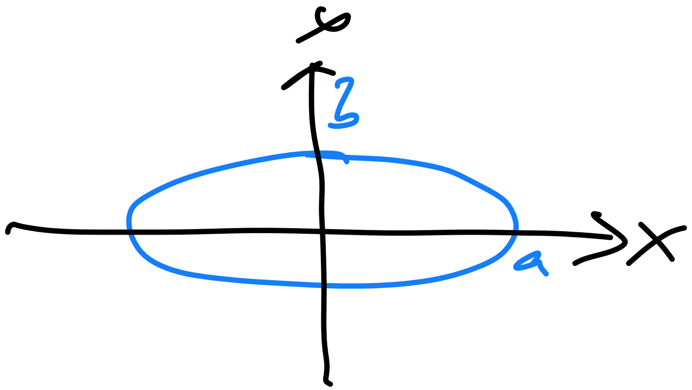
Vectors: Magnitude and Direction

Work in $\mathbb{R}^3$ or $\mathbb{R}^n$

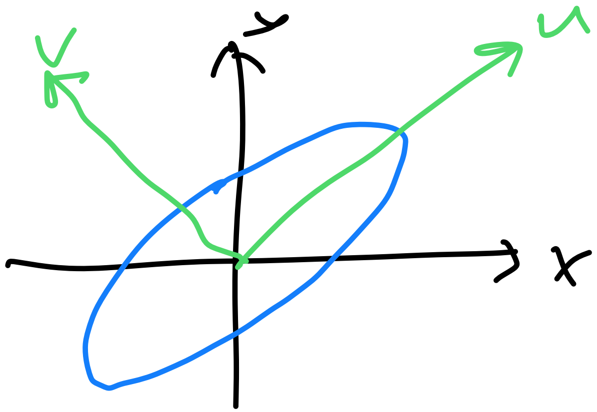
$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$

$\vec{v}$ for a vector $\quad \mathbf{v}$ vs $\vec{v}$





$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

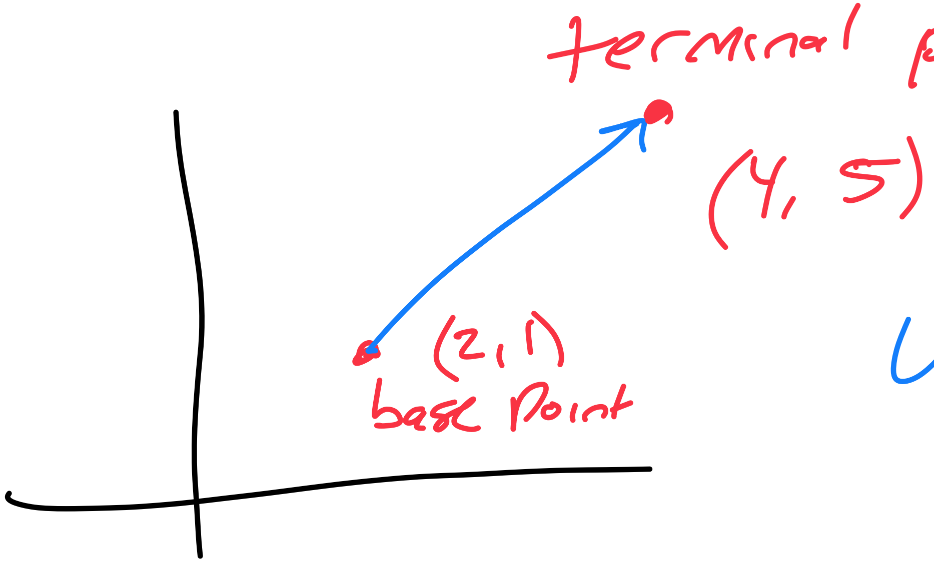


$$\vec{e}_1 = (1, 0, 0, \ldots, 0)$$
$$\vec{e}_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$
$$\vec{e}_n = (0, 0, 0, \ldots, 0, 1)$$

$$\hat{i} = (1, 0, 0)$$
$$\hat{j} = (0, 1, 0)$$
$$\hat{k} = (0, 0, 1)$$

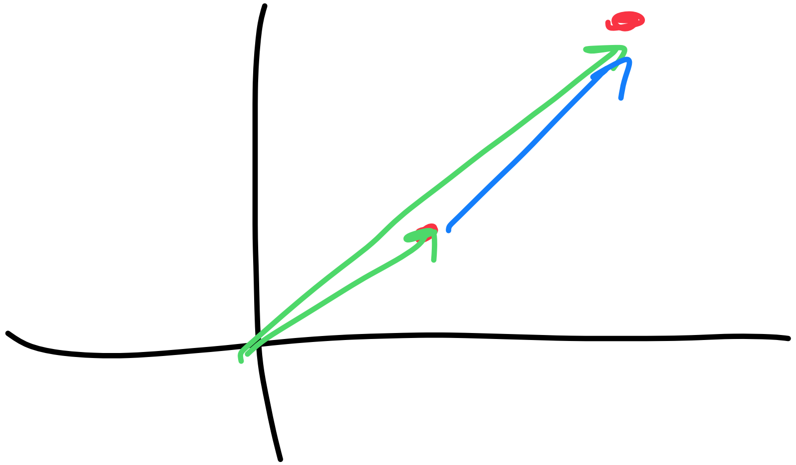
"e" for Euclidean Space in $\mathbb{R}^n$

hat means unit vector : length 1



Vector is $\langle 4-2, 5-1 \rangle$

$$= \langle 2, 4 \rangle$$



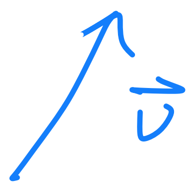
$$\langle 4, 5 \rangle - \langle 2, 1 \rangle = \langle 2, 4 \rangle$$

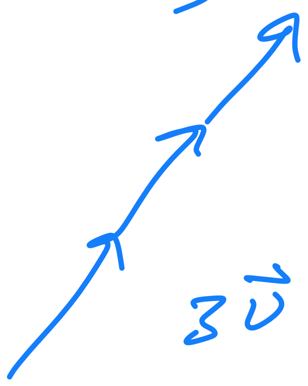
Vectors have nice properties

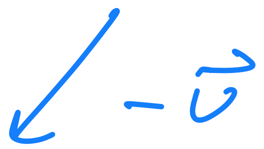
$\vec{u}, \vec{v}, \vec{w}$ vectors a, b are scalars (think $\mathbb{R}$ or $\mathbb{C}$)

1) $a\vec{v}$ is same dir as $\vec{v}$ and a times as far
(if a is neg, it is $180°$ the other direction)

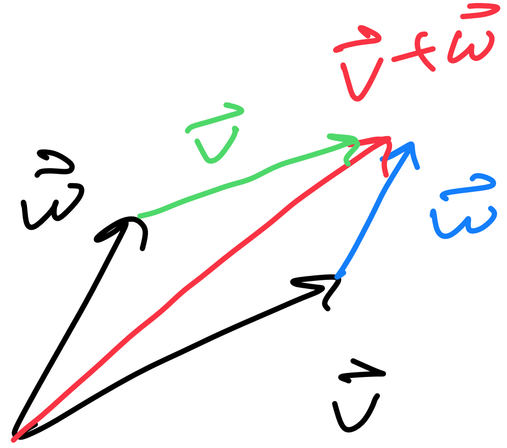
Ex: $\vec{v} = (1, 2)$ $a = 3$ $a = -1$







2) $\vec{v} + \vec{w}$: Parallelogram Rule $= \vec{w} + \vec{v}$



Commutivity

3) $\vec{v} - \vec{w} = \vec{v} + (-1)\vec{w}$

4) $\vec{u} + \vec{v} + \vec{w}$ is $(\vec{u} + \vec{v}) + \vec{w}$ or $\vec{u} + (\vec{v} + \vec{w})$

associativity

Ex: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$
$\vec{u}$ $\vec{v}$ $\vec{w}$

then $\vec{u}+\vec{v} = \langle 1,1,0 \rangle$

$(\vec{u}+\vec{v})+\vec{w} = \langle 1,1,0 \rangle + \langle 0,0,1 \rangle$

$= \langle 1,1,1 \rangle$

Ex: $\langle 1,4,8 \rangle$ and $\langle 2,-3,8 \rangle$

Then $\langle 1,4,8 \rangle + \langle 2,-3,8 \rangle = \langle 3,1,16 \rangle$

$$\vec{v}+\vec{w} = \langle v_1, v_2, \dots, v_n \rangle + \langle w_1, w_2, \dots, w_n \rangle$$

$$= \langle v_1+w_1, \dots, v_n+w_n \rangle$$

$$= \langle w_1+v_1, \dots, w_n+v_n \rangle \quad \text{normal commutativity}$$

$$= \langle w_1, \dots, w_n \rangle + \langle v_1, \dots, v_n \rangle$$

$$= \vec{w}+\vec{v}$$

$\blacksquare$

5) $a(\vec{v}+\vec{w}) = a\vec{v} + a\vec{w}$ Distributive law

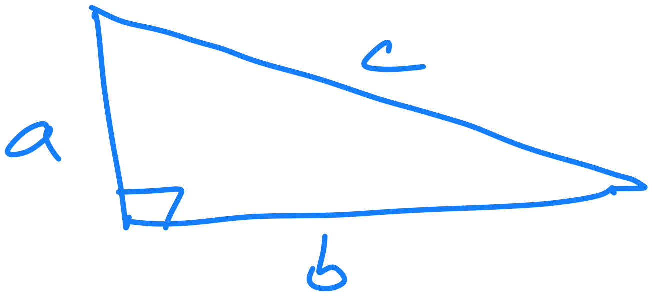
6) Special Vector:

Zero vector $\vec{0}$

note: $\vec{0}+\vec{v} = \vec{v}+\vec{0} = \vec{v}$

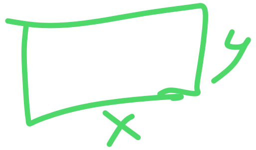
Length of a Vector?

Pythagorean Formula

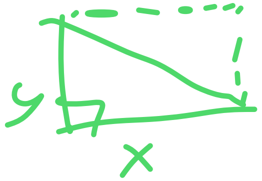


$$a^2+b^2=c^2$$

Assume

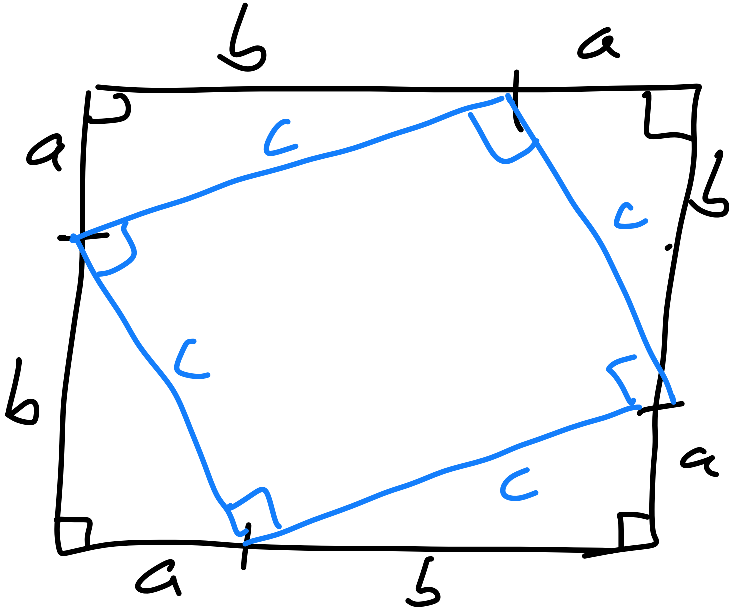


area is xy



area is $\frac{1}{2}xy$

Proof



Big Square area is $(a+b)^2$

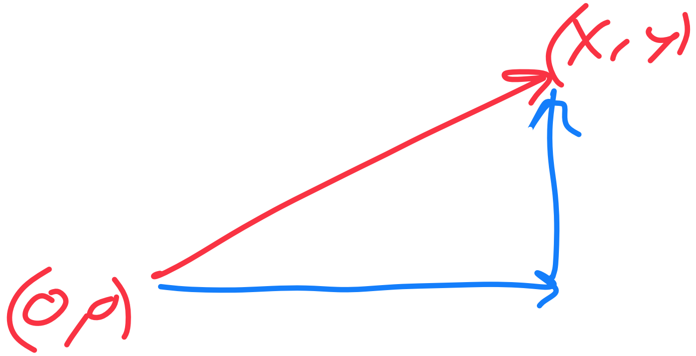
Little Square is c^2

each triangle is $\frac{1}{2}ab$

so $(a+b)^2 = c^2 + 4 * \frac{1}{2}ab$

$a^2 + \cancel{2ab} + b^2 = c^2 + \cancel{2ab}$ ∎

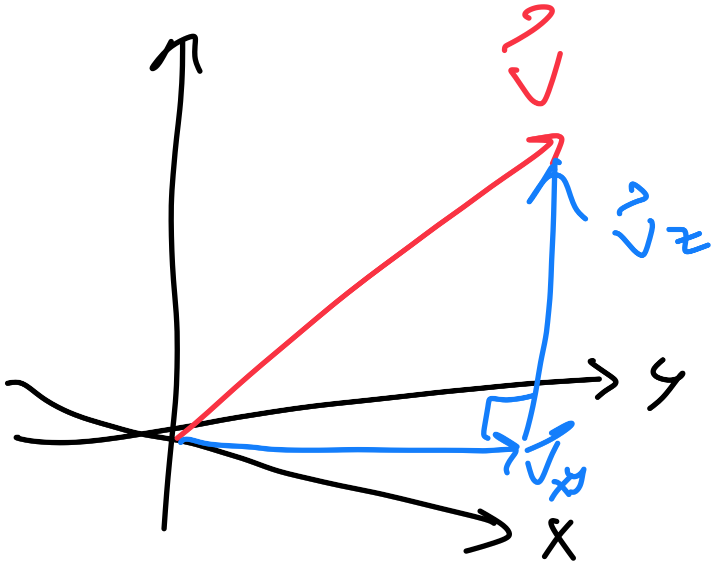
Length in the Plane



length is $\sqrt{x^2+y^2}$

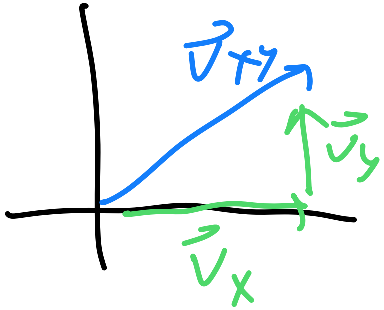
$\vec{v} = \langle x,y \rangle$ $\quad \|\vec{v}\|$ or $|\vec{v}|$ is $\sqrt{x^2+y^2}$
(notation for length)

3-Space: Pythagoras Twice



$$\|\vec{v}\|^2 = \|\vec{v}_z\|^2 + \underbrace{\|\vec{v}_{xy}\|^2}$$

$$= \|\vec{v}_x\|^2 + \|\vec{v}_y\|^2$$

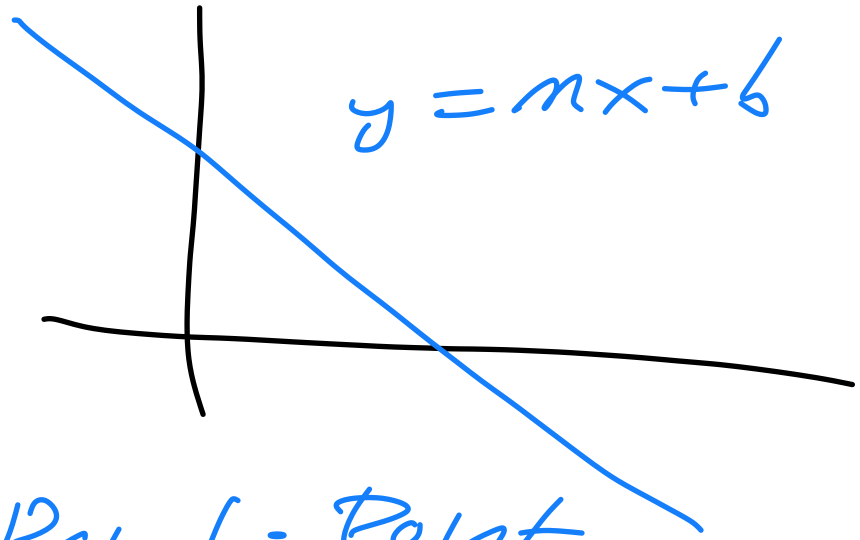


If $\vec{v} = \langle x,y,z \rangle$

$$\|\vec{v}\| = \sqrt{x^2+y^2+z^2}$$

$\vec{v} = \langle x_1, \dots, x_n \rangle$ then $\|\vec{v}\| = \sqrt{x_1^2 + \dots + x_n^2}$

Eq of a Line



$y = mx + b$

Slope-Intercept

$\hookrightarrow m$ slope

b is the y-intercept

Point: Slope

Slope m, Point (x_0, y_0)

Get $y - y_0 = m(x - x_0)$

or $y = y_0 + m(x - x_0)$

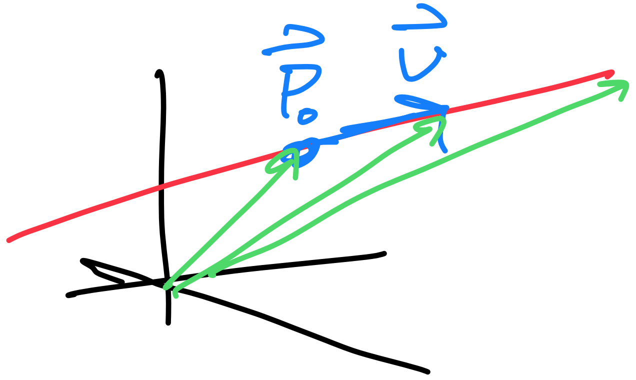
Point-Point

$m = \dfrac{y_1 - y_0}{x_1 - x_0}$

now use this

Lines in higher dimension

Point $\vec{P}_0 = (x_0, y_0, z_0)$ direction $\vec{v}$



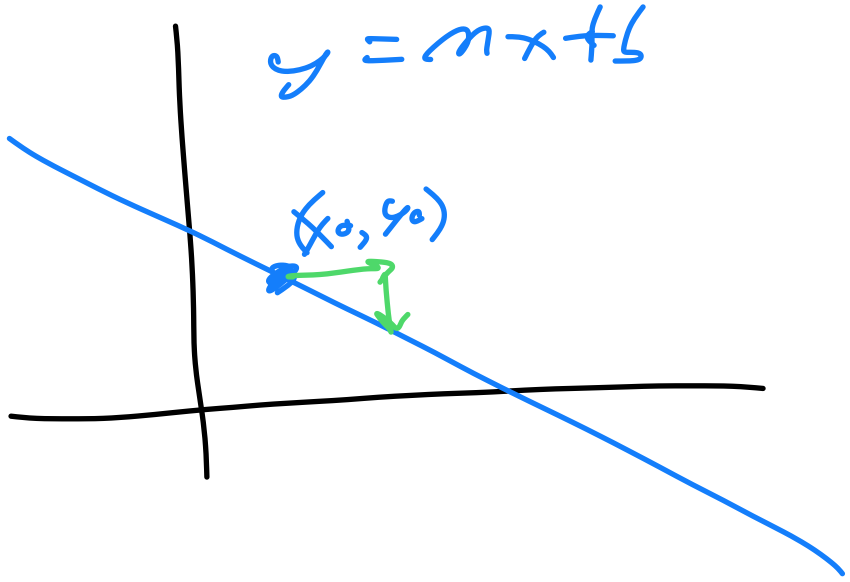
All points of the form

$$\vec{P}_t = \vec{P}_0 + t\vec{v} \qquad t \in \mathbb{R}$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

or

$$x(t) = x_0 + t v_x$$
$$y(t) = y_0 + t v_y$$
$$z(t) = z_0 + t v_z$$



Slope is $\dfrac{\Delta y}{\Delta x} = \dfrac{y - y_0}{x - x_0}$

Write this as a vector

$\langle 1, m \rangle$

$y - y_0 = m(x - x_0)$

3.1. **12.1: Vectors in the Plane – Problems.** #1: Exercise 12.1.44: Determine the unit vector e_w, where $w = \langle 24, 7 \rangle$. #2: Exercise 12.1.49: Determine the unit vector that makes an angle of $4\pi/7$ with the x-axis. #3: Exercise 12.1.52: Determine the unit vector that points in the direction from $(-3, 4)$ to the origin.

3.2. **12.2: Three-Dimensional Space: Surfaces, Vectors, and Curves – Problems.** #1: Exercise 12.2.34: Describe the surface given by the equation $x^2 + y^2 + z^2 = 9$, with $x, y, z \geq 0$. #2: Exercise 12.2.38: Give an equation for the sphere centered at the origin passing through $(1, 2, -3)$. #3: Exercise 12.2.50: Find a vector parametrization for the line passing through $(1, 1, 1)$ which is parallel to the line passing through $(2, 0, -1)$ and $(4, 1, 3)$.

Math 150: Multivariable Calculus: Spring 2023:
Lecture 10: Vectors: https://youtu.be/0keO_ByxMEE

Plan for the day.

- Equations of Lines.
- Equations of Planes.
- Dot Product.

Note: all quoted text taken from the textbook for the class.

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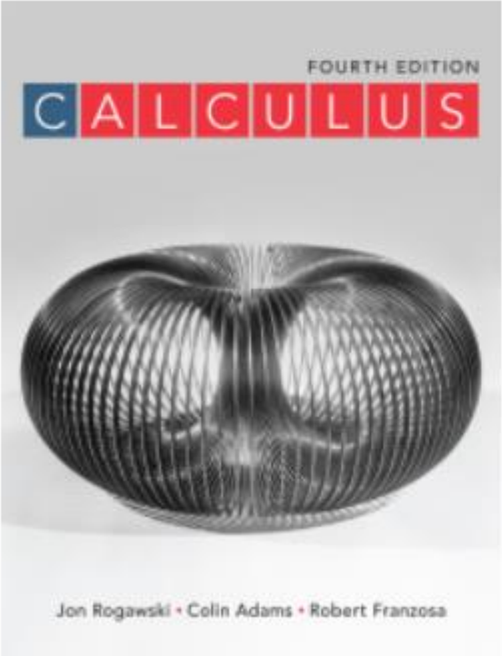

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Print ISBN	9781319050733, 1319050735
eText ISBN	9781319055844, 1319055842
Edition	4th
Copyright	2019

3.3. **12.3: Dot Product and the Angle Between Two Vectors – Problems.** #0: Exercise 12.3.13: Determine whether $\langle 1,1,1\rangle$ and $\langle 1,-2,-2\rangle$ are orthogonal, and, if not, whether the angle between them is acute or obtuse. #1: Exercise 12.3.25: Find the angle between $\langle 1,1,1\rangle$ and $\langle 1,0,1\rangle$. #2: Exercise 12.3.57: Find the projection of $u = \langle -1,2,0\rangle$ along $v = \langle 2,0,1\rangle$. #3: Exercise 12.3.64: Compute the component of $u = \langle 3,0,9\rangle$ along $v = \langle 1,2,2\rangle$.

Eq of a line

$y - y_0 = m(x - x_0)$

Let $x = x_0 + t$

$y = y_0 + m(x - x_0)$

$= y_0 + mt$

$\begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 + t \\ y_0 + mt \end{pmatrix}$

$= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 1 \\ m \end{pmatrix} t$

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \vec{v} t$

Ex:

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2m \end{pmatrix} s$

$= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 1 \\ m \end{pmatrix} 2s$

Let $t = 2s$ or $s = t/2$

Trying to make x-coordinate 1

Ex: $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$

Same direction

$\vec{v} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Danger: $\vec{v} = \begin{pmatrix} 0 \\ m \end{pmatrix} \rightarrow$ Cannot Do!

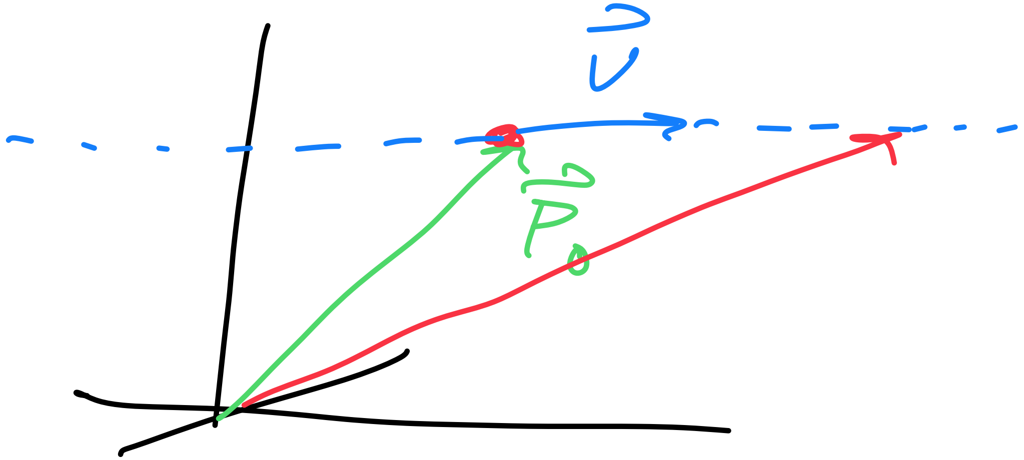
General Eq of a line:

Point $\vec{P}_0$ and direction $\vec{v}$

Line is $\vec{P}(t) = \vec{P}_0 + t\vec{v}$ or

$$x(t) = x_0 + t v_x$$
$$\vdots$$
$$z(t) = z_0 + t v_z$$

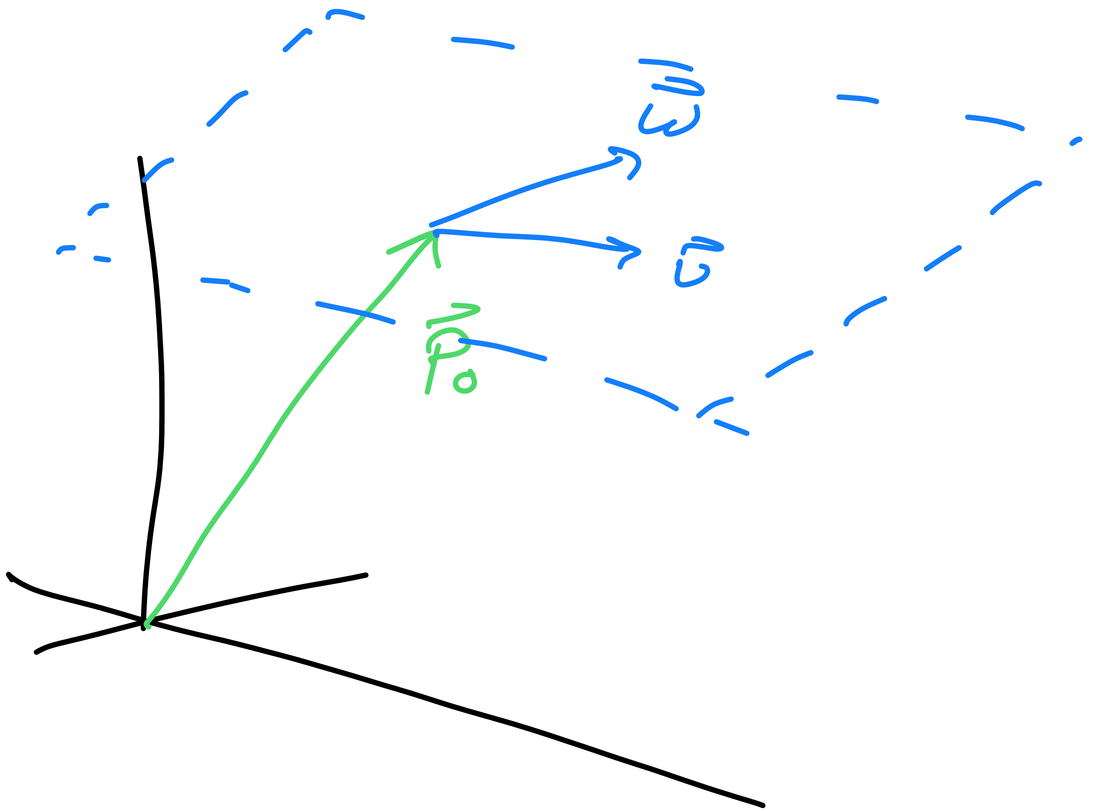
One free variable: 1-dim



Eq of a non-degenerate plane

Input: Point $\vec{P_0}$, two independent dirs $\vec{v}$ and $\vec{w}$

Output: $\vec{P}(t,s) = \vec{P_0} + t\vec{v} + s\vec{w}$



two-dimensional

vars: t, s

Ex: $\vec{p}_0 + t\vec{v} + s\vec{w}$ and $\vec{p}_0 + u\vec{v} + q(\vec{v}+\vec{w})$

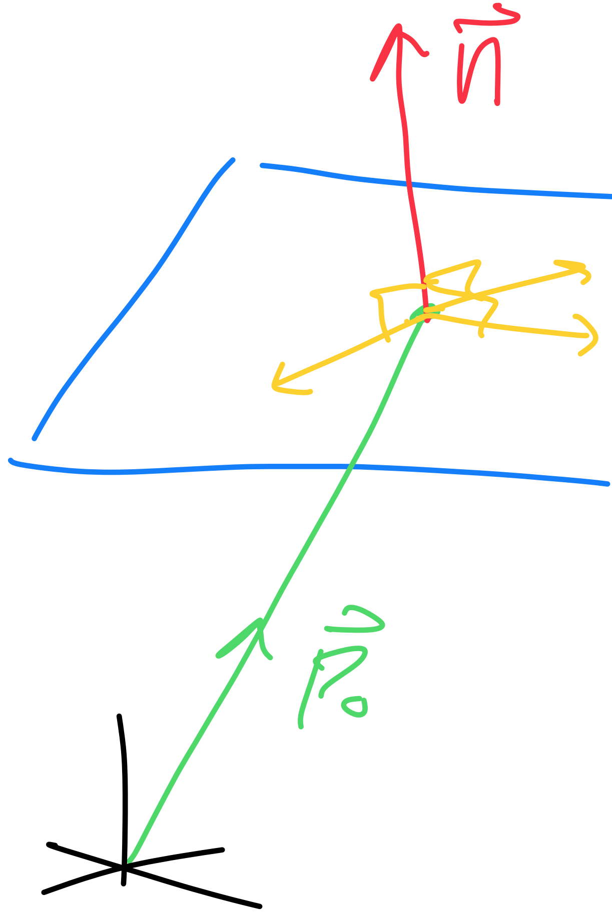
say $\cancel{\vec{p}_0} + t\vec{v} + s\vec{w} = \cancel{\vec{p}_0} + u\vec{v} + q(\vec{v}+\vec{w})$

$$\boxed{t\vec{v} + s\vec{w} = (u+q)\vec{v} + q\vec{w}}$$

Given $t, s \longrightarrow u = t - s \quad q = s$

Given $u, q \longrightarrow t = u+q \quad s = q$

Normal Approach



Plane is all points $\vec{P}$ such that $\vec{P} - \vec{P}_0$ is perpendicular (orthogonal, $\perp$) to the normal direction $\vec{n}$: $(\vec{P} - \vec{P}_0) \perp \vec{n}$

Dot Product / Inner Product

$\vec{v} \cdot \vec{w}$ or $(\vec{v}, \vec{w})$ is $v_1 w_1 + \cdots + v_n w_n$

Recall $\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$ or $\|\vec{v}\|^2 = v_1^2 + \cdots + v_n^2$

$= \vec{v} \cdot \vec{v}$

THEOREM 2

Dot Product and the Angle

Let θ be the angle between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$. Then

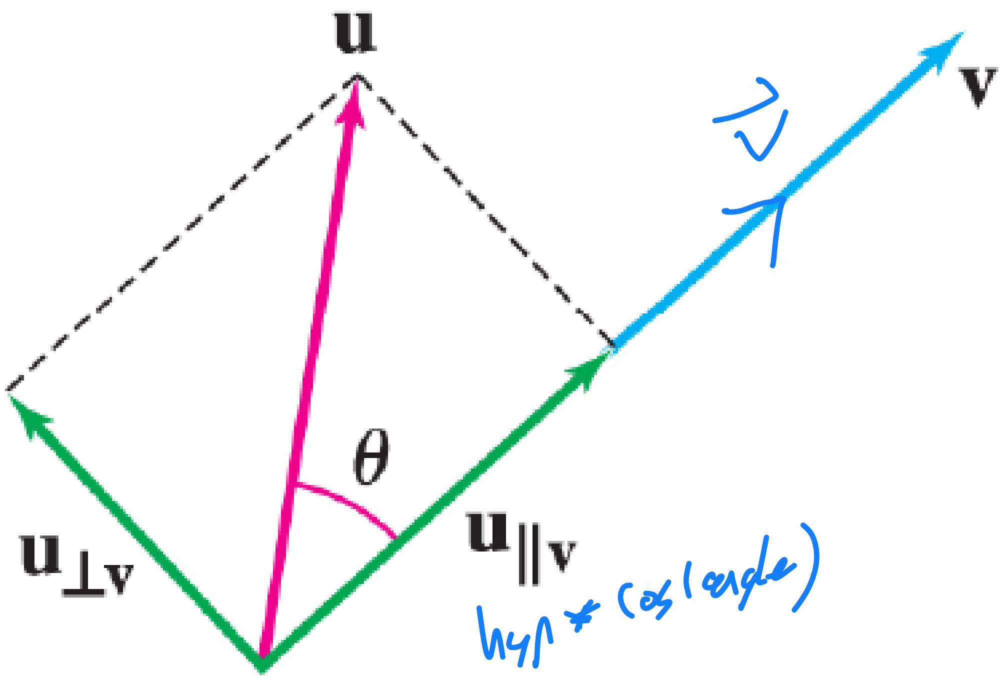
$$\mathbf{v}\cdot\mathbf{w} = ||\mathbf{v}||\ ||\mathbf{w}|| \cos\theta \quad \text{or} \quad \cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\ ||\mathbf{w}||}$$

$$\mathbf{v}\perp\mathbf{w} \quad \text{if and only if} \quad \mathbf{v}\cdot\mathbf{w} = 0$$

Projection of u along v

Assume $\mathbf{v} \neq \mathbf{0}$. The **projection** of $\mathbf{u}$ along $\mathbf{v}$ is the vector

$$\mathbf{u}_{||\mathbf{v}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||^2}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||}\right)\mathbf{e}_{\mathbf{v}}$$



Rogawski et al., *Multivariable Calculus*, 4e, © 2019 W. H. Freeman and Company

This is sometimes denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$. The scalar $\dfrac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||}$ is called the **component** or the **scalar component** of $\mathbf{u}$ along $\mathbf{v}$ and is sometimes denoted $\text{comp}_{\mathbf{v}}\mathbf{u}$.

Reasonable?

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta_{vw}$ $\left(= v_1 w_1 + \cdots + v_n w_n \right)$

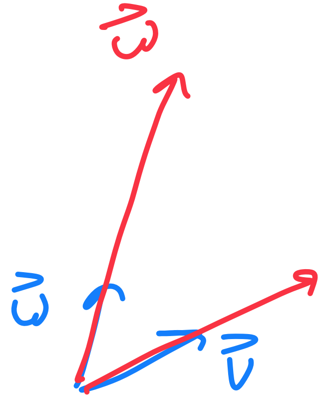
double $\vec{v}$, triple $\vec{w}$

↳ LHS: each term ↑ by a factor of 6

RHS: $\|\vec{v}\|$ is ↑ by factor of 2

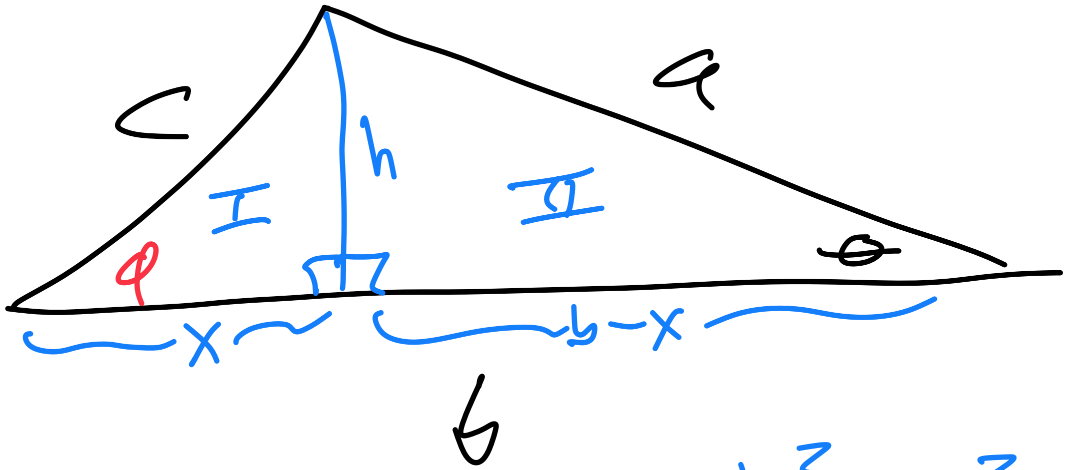
$\|\vec{w}\|$ is ↑ by a factor of 3

$\cos\theta$ unchanged



Reasonable! Scales correctly!

Law of Cosines



$c^2 = a^2 + b^2 - 2ab\cos\theta$

$\theta = 90 \to c^2 = a^2 + b^2$ known!

$a^2 = c^2 + b^2 - 2bc\cos\varphi$

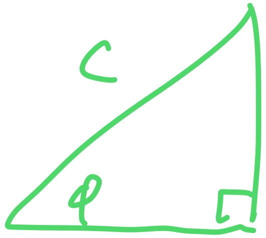
I: $c^2 = x^2 + h^2$ or $h^2 = c^2 - x^2$

II: $a^2 = (b-x)^2 + h^2$ or $h^2 = a^2 - (b-x)^2$

$\Rightarrow h^2 = c^2 - x^2 = a^2 - (b-x)^2$

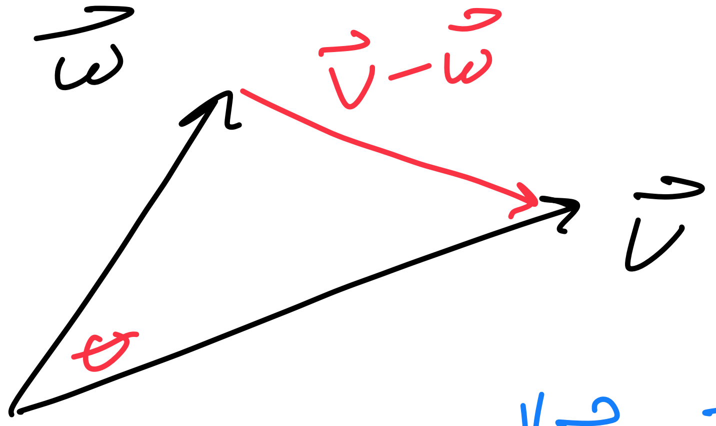
so $c^2 = a^2 + x^2 - (b^2 - 2bx + x^2) = a^2 - b^2 + 2bx$

so $c^2 + b^2 - 2bx = a^2$



$c * \cos\varphi = x$

done as $c * \cos\varphi = x$



$2+(5-2) = 5$

$17+(11-17) = 11$

Law of Cosines: $\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\,\|\vec{w}\|\cos\theta$

$\|\vec{v}-\vec{w}\|^2 = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$

$a\vec{v}\cdot\vec{w} = a(\vec{v}\cdot\vec{w})$ and $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$

$(v_1-w_1, v_2-w_2, \dots)\cdot(v_1-w_1, \dots)$

$(v_1-w_1)^2 + \dots$

$$\|\vec{v}-\vec{w}\|^2 = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v} - \vec{w}\cdot\vec{v} - \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$$

$$= \cancel{\|\vec{v}\|^2} - \cancel{2}\,\vec{v}\cdot\vec{w} + \cancel{\|\vec{w}\|^2} = \cancel{\|\vec{v}\|^2} + \cancel{\|\vec{w}\|^2}$$

$$\cancel{-2}\|\vec{v}\|\|\vec{w}\|\cos\theta$$

by Law of Cosines

$$\Longrightarrow \vec{v}\cdot\vec{w} = \|\vec{v}\|\ \|\vec{w}\|\cos\theta$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\ \|\vec{w}\|}$$

if $\theta = \pi/2$ then $\vec{v}\cdot\vec{w} = 0$

perpendicular test!

Plane: $\vec{P}_0$ normal (a,b,c) then eq of the plane is

$$((x,y,z)-(x_0,y_0,z_0))\cdot(a,b,c) = 0$$

or $\quad a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$

or $\quad ax+by+cz = \vec{P}_0\cdot\vec{n} = d$

3.3. **12.3: Dot Product and the Angle Between Two Vectors – Problems.** #0: Exercise 12.3.13: Determine whether $\langle 1, 1, 1\rangle$ and $\langle 1, -2, -2\rangle$ are orthogonal, and, if not, whether the angle between them is acute or obtuse. #1: Exercise 12.3.25: Find the angle between $\langle 1, 1, 1\rangle$ and $\langle 1, 0, 1\rangle$. #2: Exercise 12.3.57: Find the projection of $u = \langle -1, 2, 0\rangle$ along $v = \langle 2, 0, 1\rangle$. #3: Exercise 12.3.64: Compute the component of $u = \langle 3, 0, 9\rangle$ along $v = \langle 1, 2, 2\rangle$.

Math 150: Multivariable Calculus: Spring 2023:
Lecture 11: Cross Product: https://youtu.be/KpJmKkFqJe0

Plan for the day.

- Cross Product.
- Coordinate Systems.

Note: all quoted text taken from the textbook for the class.

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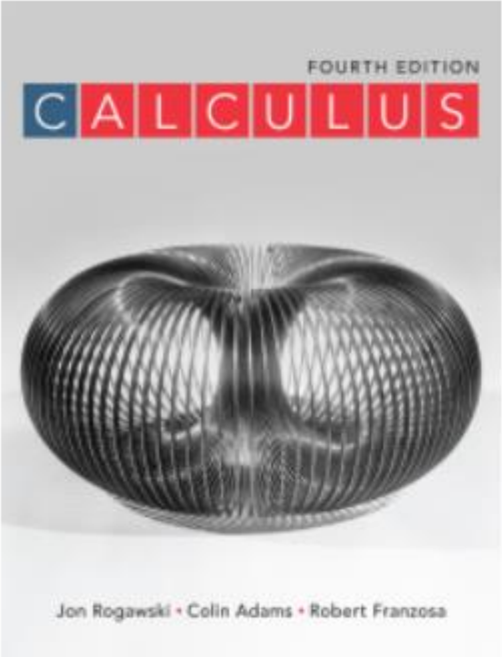

Author(s)	Jon Rogawski; Colin Adams; Robert Franzosa
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Format	Reflowable
Print ISBN	9781319050733, 1319050735
eText ISBN	9781319055844, 1319055842
Edition	4th
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7.7. **12.4: The Cross Product – Problems.** #0: Preliminary Question 12.4.6: When is the cross product $v \times w$ equal to zero? #1: Exercise 12.4.16: Calculate $(j - k) \times (j + k)$. #2: Exercise 12.4.30: What are the possible angles θ between two unit vectors e and f if $\|e \times f\| = 1/2$?

7.9. **12.5: Planes in 3-Space – Problems.** #1: Exercise 12.5.13: Find a vector normal to the plane specified by $9x - 4y - 11z = 2$. #2: Exercise 12.5.18: Find the equation of the plane that passes through $(4, 1, 9)$ and is parallel to $x + y + z = 3$. #3: Exercise 12.5.48: Find the trace of the plane specified by $3x + 4z = -2$ in the xz coordinate plane. #4: Exercise 12.5.63: Find an equation of a plane making an angle of $\pi/2$ with the plane $3x + y - 4z = 2$.

Do all but one!

Given vectors $\vec{a}$ and $\vec{b}$, where

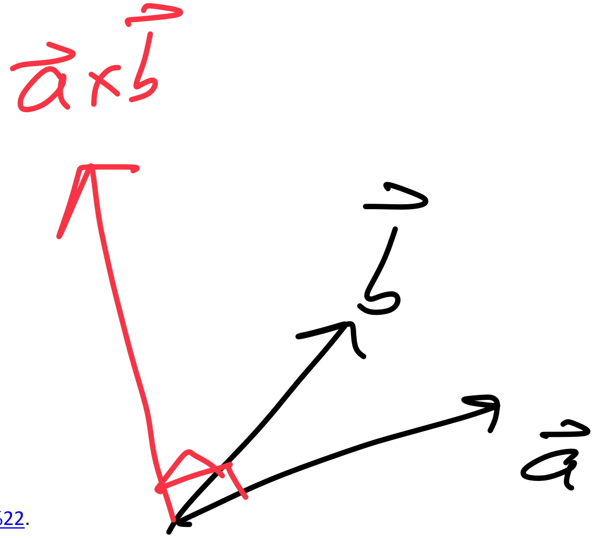
$\vec{a} = \langle a_1, a_2, a_3 \rangle$

and

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

Then the cross product of $\vec{a}$ and $\vec{b}$ is:

$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$.



https://simple.wikipedia.org/wiki/Cross_product#:~:text=The%20cross%20product%20is%20a%20mathematical%20operation%20which,to%20both%20of%20the%20vectors%20which%20were%20%22crossed%22.

Show $\vec{a} \times \vec{b}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$

Proof: show $(\vec{a} \times \vec{b}) \cdot \vec{a}$ is zero (the number)

Ex: $\vec{i} \times \vec{k} = \langle 1,0,0 \rangle \times \langle 0,0,1 \rangle = \langle 0,-1,0 \rangle = -\vec{j}$

- Determinants of sizes 2×2 and 3×3:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A) = ad - bc$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- The *cross product* of $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ is the determinant

$+(v_3w_1 - v_1w_3)$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \underbrace{\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}}_{(v_2w_3 - v_3w_2)} \mathbf{i} - \underbrace{\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}}_{(v_1w_3 - v_3w_1)} \mathbf{j} + \underbrace{\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}}_{(v_1w_2 - v_2w_1)} \mathbf{k}$$

$\vec{i} = (1,0,0)$

$\vec{j} = (0,1,0)$

$\vec{k} = (0,0,1)$

- The cross product $\mathbf{v} \times \mathbf{w}$ is the unique vector with the following three properties:
  i. $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$.
  ii. $\mathbf{v} \times \mathbf{w}$ has length $||\mathbf{v}|| \, ||\mathbf{w}|| \sin\theta$ (where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$).
  iii. $\{\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w}\}$ is a right-handed system.
- Properties of the cross product:
  i. $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$
  ii. $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{w} = \lambda\mathbf{v}$ for some scalar or $\mathbf{v} = \mathbf{0}$
  iii. $(\lambda\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda\mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w})$
  iv. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$ and $\mathbf{v} \times (\mathbf{u} + \mathbf{w}) = \mathbf{v} \times \mathbf{u} + \mathbf{v} \times \mathbf{w}$

$\vec{i} \times \vec{i} = \vec{0}$ $\vec{j} \times \vec{i} = -\vec{k}$ $\vec{k} \times \vec{i} = \vec{j}$

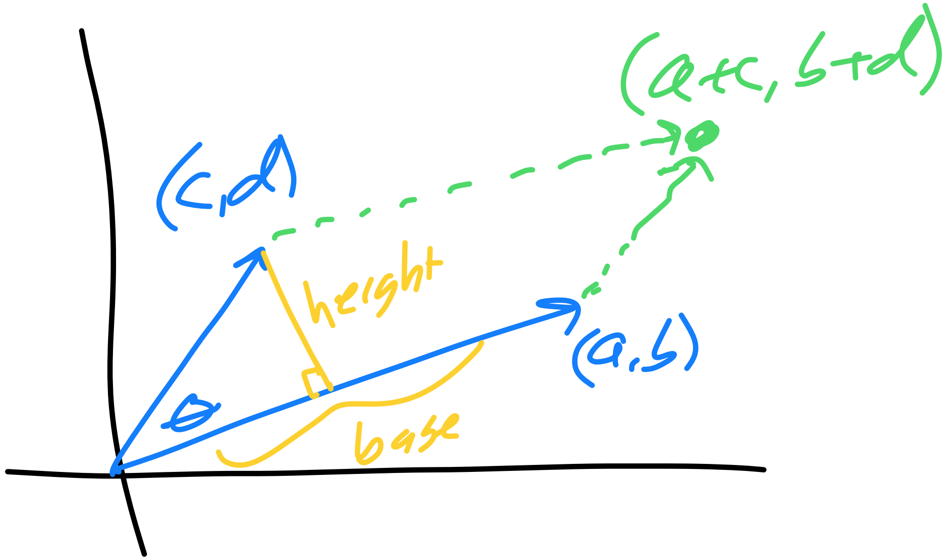
$\vec{i} \times \vec{j} = \vec{k}$ $\vec{j} \times \vec{j} = \vec{0}$ $\vec{k} \times \vec{j} = -\vec{i}$

$\vec{i} \times \vec{k} = -\vec{j}$ $\vec{j} \times \vec{k} = \vec{i}$ $\vec{k} \times \vec{k} = \vec{0}$

$$\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle$$

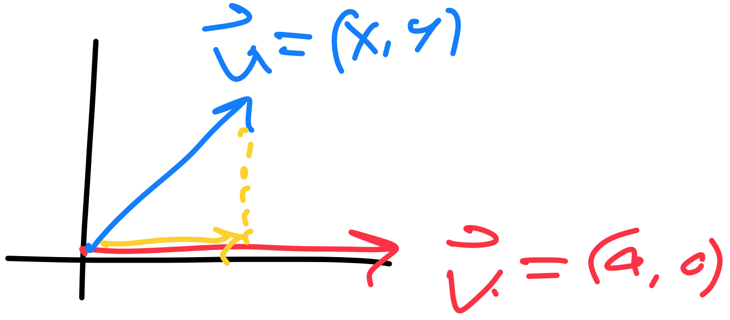
$$= (a_1 \vec{i} + \underline{a_2 \vec{j}} + \underline{a_3 \vec{k}}) \times (b_1 \vec{i} + \underline{b_2 \vec{j}} + \underline{b_3 \vec{k}})$$

$$= (a_3 b_2 - a_2 b_3)\vec{i} + \cdots$$



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Aside: Dot Product



$$\vec{u} \cdot \vec{v} = x \cdot a + y \cdot 0 = x \cdot a$$

Area is base * height

base is $\|\langle a, b\rangle\|$

height is $\|\langle c, d\rangle\| \sin\theta$

Area is $(a^2+b^2)^{\frac{1}{2}}(c^2+d^2)^{\frac{1}{2}} \sin\theta$

$\text{Area}^2 = (a^2+b^2)(c^2+d^2)\sin^2\theta$

use $\sin^2\theta = 1 - \cos^2\theta$

$$\text{Area}^2 = (a^2+b^2)(c^2+d^2)(1-\cos^2\theta)$$

$$\langle a,b\rangle \cdot \langle c,d\rangle = \|\langle a,b\rangle\| \; \|\langle c,d\rangle\| \cos\theta$$

$$(ac+bd)^2 = (a^2+b^2)(c^2+d^2)\cos^2\theta$$

$$\text{Area}^2 = (a^2+b^2)(c^2+d^2) - (ac+bd)^2$$

$$= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

$$-a^2c^2 \qquad -b^2d^2 \quad -2abcd$$

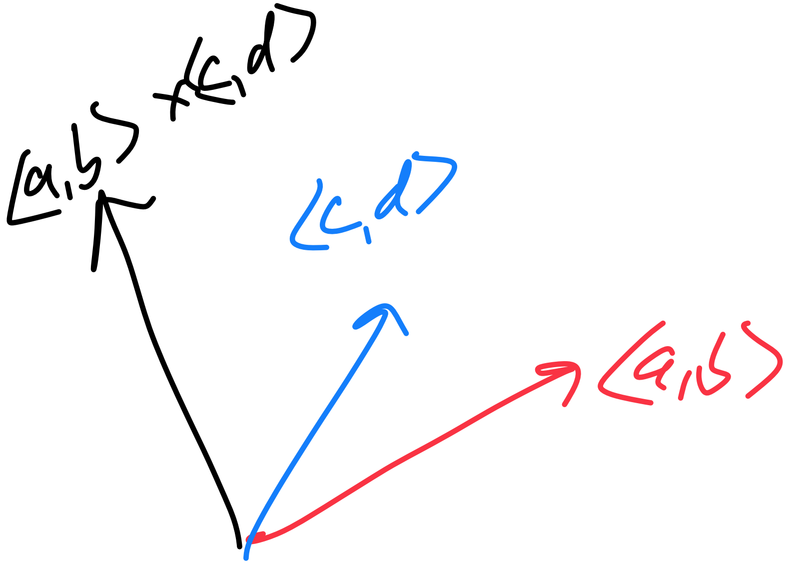
$$\text{Area}^2 = a^2d^2 - 2abcd + b^2c^2$$

$$\text{Area}^2 = (ad-bc)^2 \qquad \text{so Area is } ad-bc$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\langle a, b, 0\rangle \times \langle c, d, 0\rangle$$

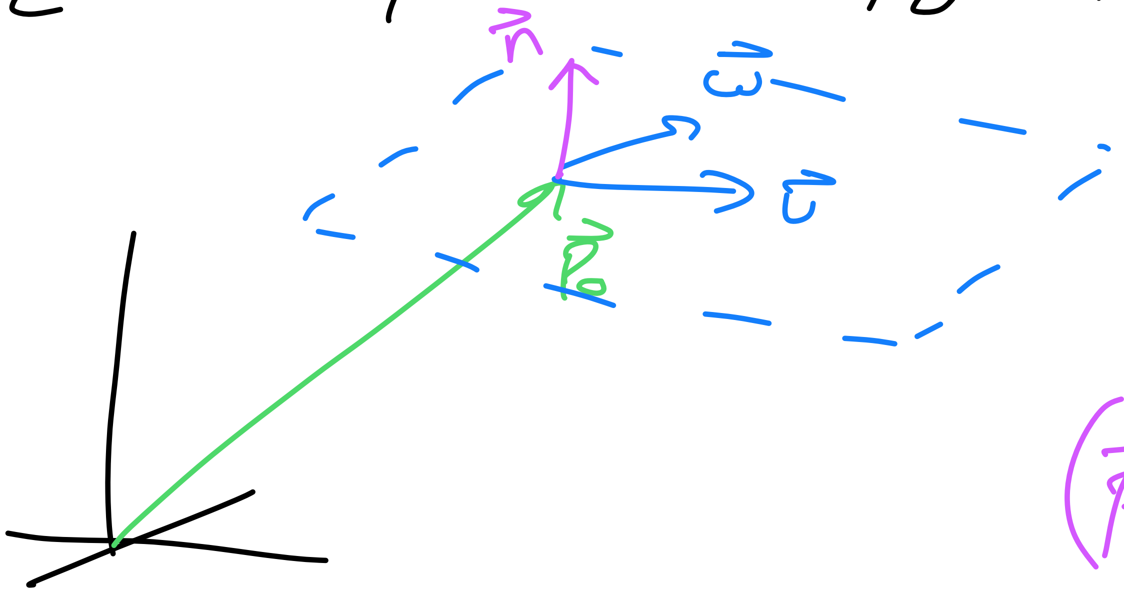
$$\left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ a & b & 0 & a & b \\ c & d & 0 & c & d \end{array}\right.$$



$$\vec{i}\, b0 + \vec{j}\, 0\cdot c + \underline{\vec{k}\, ad} - \underline{\vec{k}\, cb} - \vec{i}\, d0 - \vec{j}\, 0a$$

$$= (ad - bc)\,\vec{k}$$

Eq of a plane thru $\vec{P}_0$ and containing dirs $\vec{v}$ and $\vec{\omega}$



$$\vec{P} = \vec{P}_0 + t\vec{v} + s\vec{\omega}$$

$$(\vec{P} - \vec{P}_0) \cdot \vec{n} = 0$$

$$\vec{P} \cdot \vec{n} = \vec{P}_0 \cdot \vec{n}$$

$$\vec{n} = (a, b, c)$$
$$\vec{P} = (x, y, z)$$

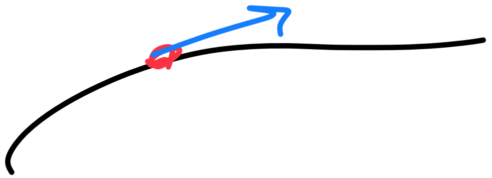
$$\vec{n} = \vec{v} \times \vec{\omega}$$

$$ax + by + cz = d = \vec{P}_0 \cdot \vec{n}$$

$$\vec{n} = \frac{\vec{v} \times \vec{\omega}}{\|\vec{v} \times \vec{\omega}\|} = \hat{n}$$

Projections

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$



7.7. **12.4: The Cross Product – Problems.** #0: Preliminary Question 12.4.6: When is the cross product $v \times w$ equal to zero? #1: Exercise 12.4.16: Calculate $(j - k) \times (j + k)$. #2: Exercise 12.4.30: What are the possible angles θ between two unit vectors e and f if $||e \times f|| = 1/2$?

7.9. **12.5: Planes in 3-Space – Problems.** #1: Exercise 12.5.13: Find a vector normal to the plane specified by $9x - 4y - 11z = 2$. #2: Exercise 12.5.18: Find the equation of the plane that passes through $(4, 1, 9)$ and is parallel to $x + y + z = 3$. #3: Exercise 12.5.48: Find the trace of the plane specified by $3x + 4z = -2$ in the xz coordinate plane. #4: Exercise 12.5.63: Find an equation of a plane making an angle of $\pi/2$ with the plane $3x + y - 4z = 2$.

Do all but one of the above.

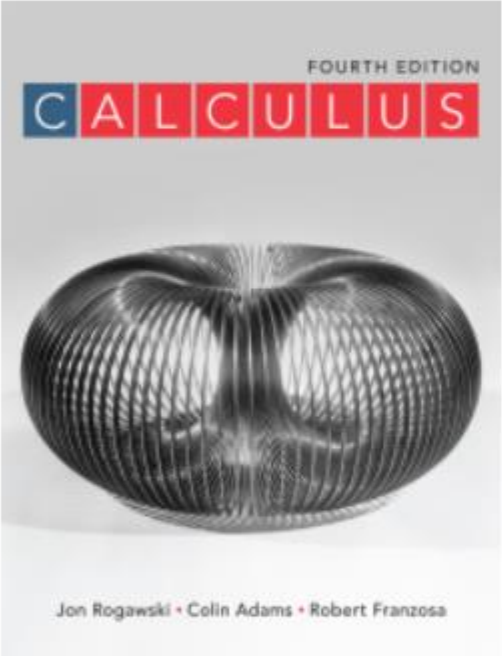
Math 150: Multivariable Calculus: Spring 2023:
Lecture 12: Level Sets, Special Coordinates: https://youtu.be/6QEQIMQf7g8

Plan for the day.

- Level Sets
- Coordinate Systems.

Note: all quoted text taken from the textbook for the class.

Calculus 4th Edition



Author(s)	Jon Rogawski; Colin Adams; Robert Franzosa
Publisher	W.H. Freeman & Company
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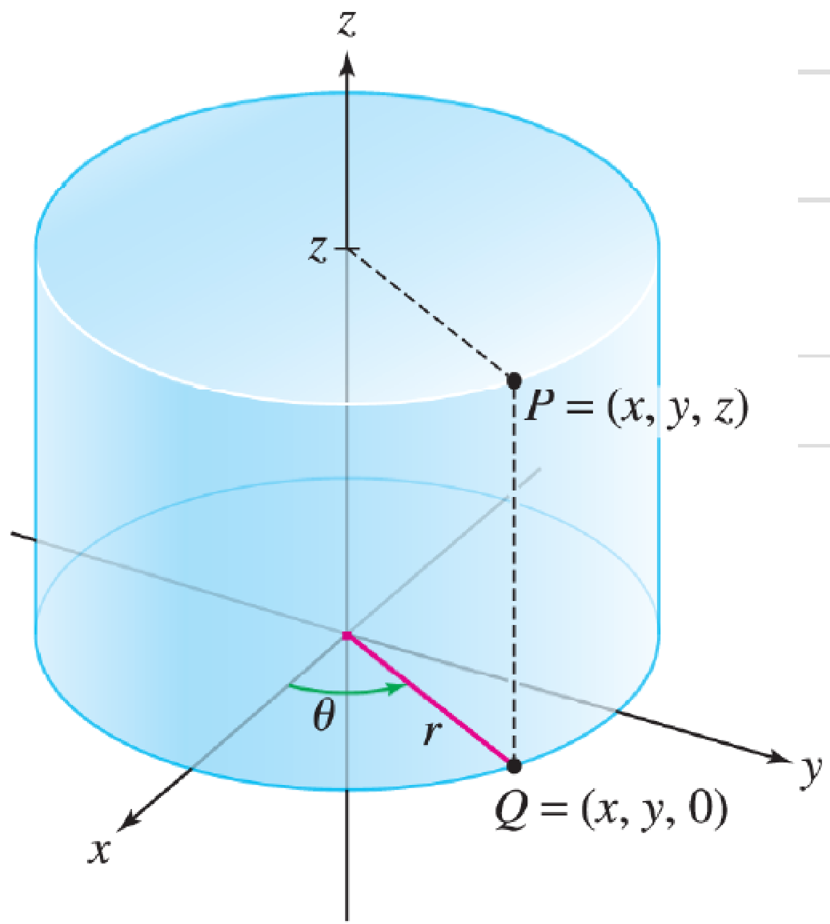
Edition	4th
Copyright	2019

3.7. **12.7: Cylindrical and Spherical Coordinates – Problems.** #1: Exercise 12.7.12: Describe $x^2 + y^2 + z^2 \leq 10$ in cylindrical coordinates. #2: Exercise 12.7.15: Describe $x^2 + y^2 \leq 9$, with $x \geq y$, in cylindrical coordinates. #3: Exercise 12.7.50: Describe $x^2 + y^2 + z^2 = 1$, with $z \geq 0$, in spherical coordinates. #4: Exercise 12.7.54: Describe $x^2 + y^2 = 3z^2$ in spherical coordinates.



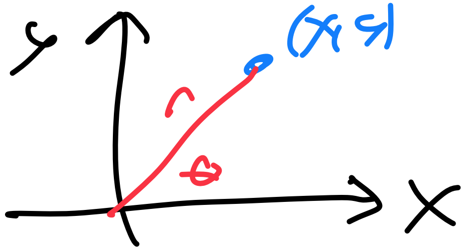
Rogawski et al., *Multivariable Calculus*, 4e, © 2019 W. H. Freeman and Company

Cylindrical to rectangular	Rectangular to cylindrical
$x = r\cos\theta$	$r = \sqrt{x^2 + y^2}$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = z$	$z = z$

Polar Coordinates

$x = r\cos\theta$ $\quad r = \sqrt{x^2+y^2}$

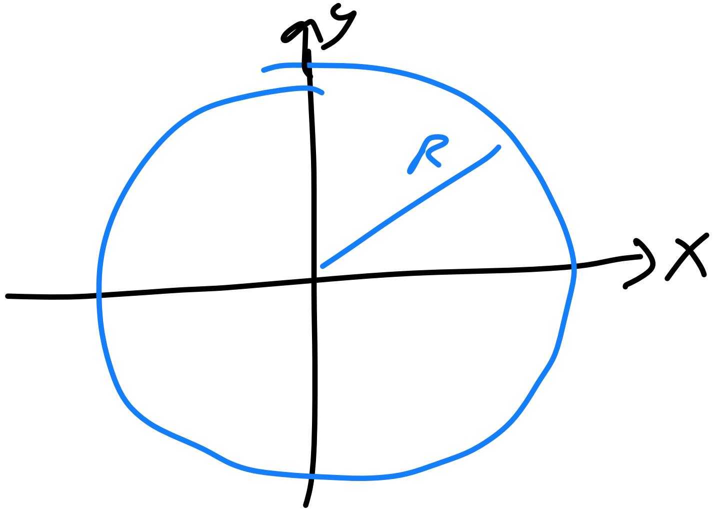
$y = r\sin\theta$ $\quad \tan\theta = y/x$



$r \geq 0$

$0 \leq \theta < 2\pi$

Circle of radius R



$x = r\cos\theta$

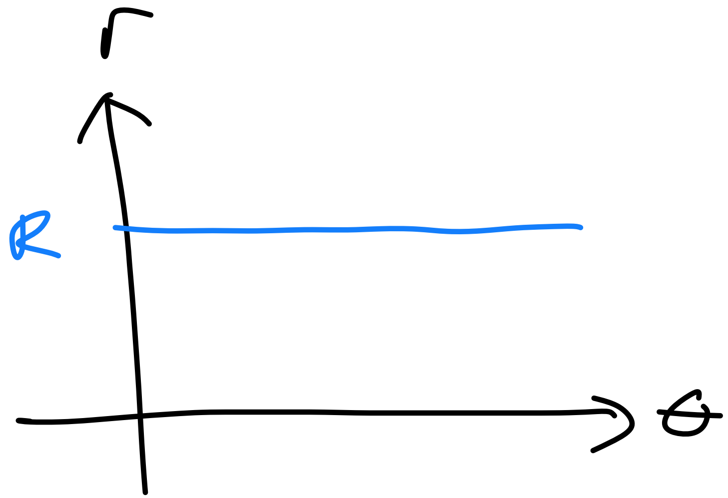
$y = r\sin\theta$

or

$r = \sqrt{x^2+y^2}$

$\tan\theta = y/x$

$\hookrightarrow \theta = \arctan\left(\frac{y}{x}\right)$



$\{(x,y) : x^2+y^2 = R^2\}$

boundary

$\{(r,\theta) : r = R\}$

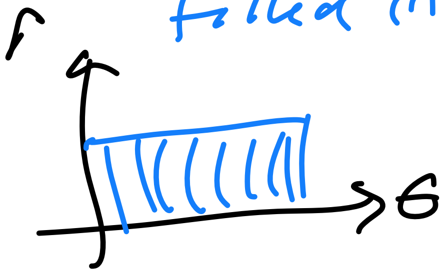
boundary

$\{(x,y) : x^2+y^2 \le R^2\}$

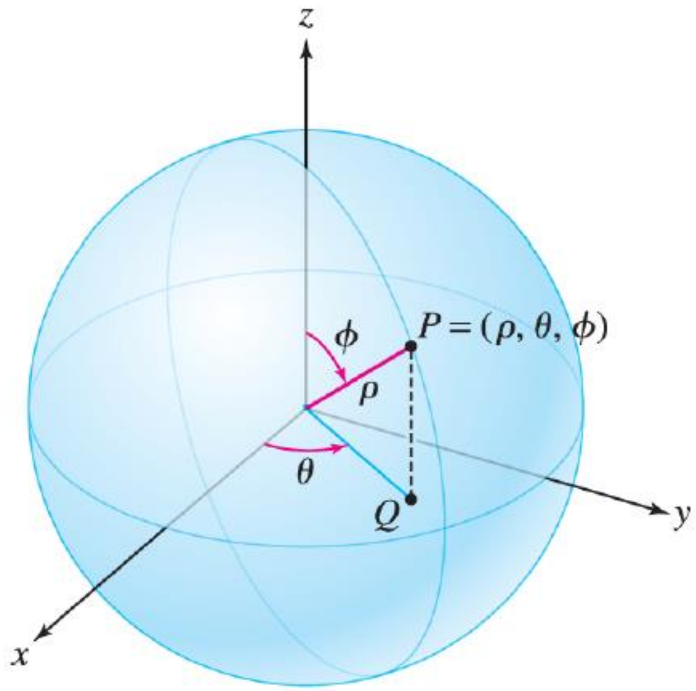
filled in

$\{(r,\theta) : r \le R\}$

filled in



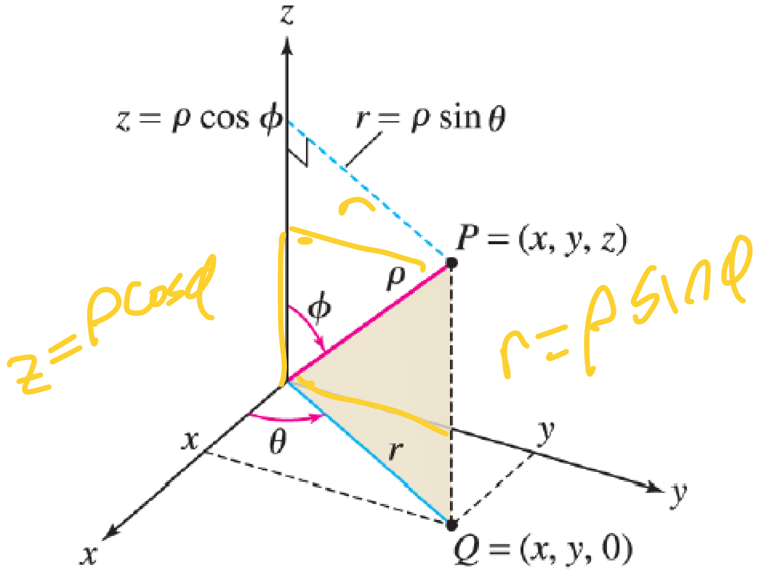
rectangle



Rogawski et al., *Multivariable Calculus*, 4e,

$$x = r\cos\theta = \rho\sin\phi\cos\theta, \qquad y = r\sin\theta = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi$$

Spherical to rectangular	**Rectangular to spherical**
$x = \rho\sin\phi\cos\theta = (\rho\sin\phi)\cos\theta = r\cos\theta$	$\rho = \sqrt{x^2 + y^2 + z^2}$
$y = \rho\sin\phi\sin\theta = (\rho\sin\phi)\sin\theta = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = \rho\cos\phi$	$\cos\phi = \dfrac{z}{\rho}$



Rogawski et al., *Multivariable Calculus*, 4e, © 2019 W. H. Freeman and Company

$\rho \geq 0$

$0 \leq \phi \leq \pi$

$0 \leq \theta < 2\pi$

If only depends on ρ:

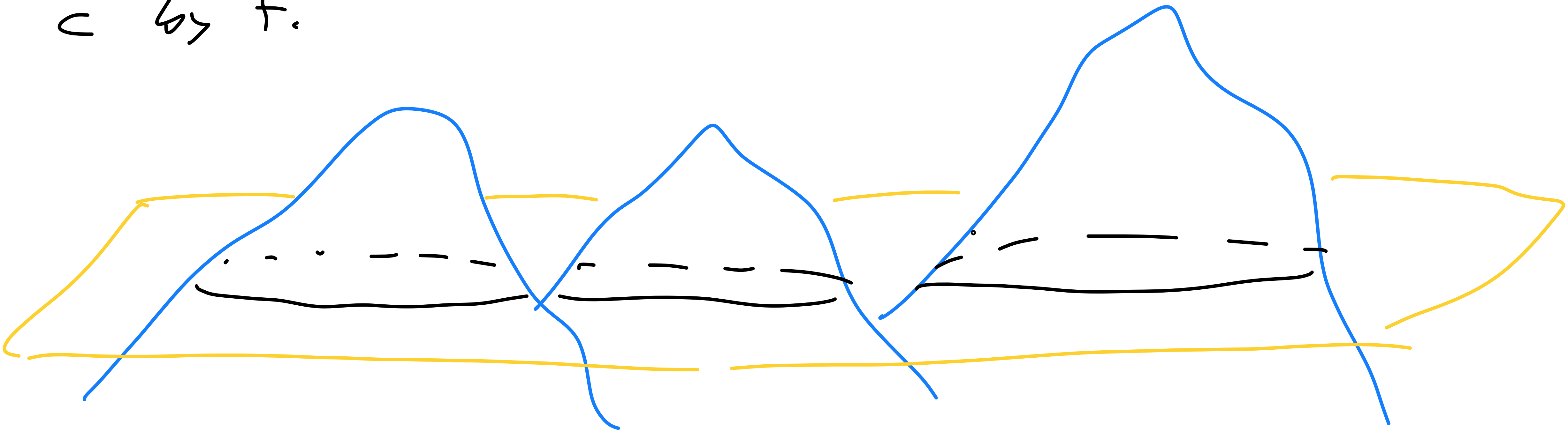
$$f(x,y,z) = f(\rho\sin\phi\cos\theta, \ldots, \ldots)$$

$$= g(\rho)$$

↑ new function

Level Sets

The level set of f of height c is all inputs sent to c by f.



Ex: $f(x,y,z) = x^2+y^2+z^2$

Find all (x,y,z) st $x^2+y^2+z^2 = c$

Case 1: $c < 0$

level set is empty

Case 2: $c = 0$

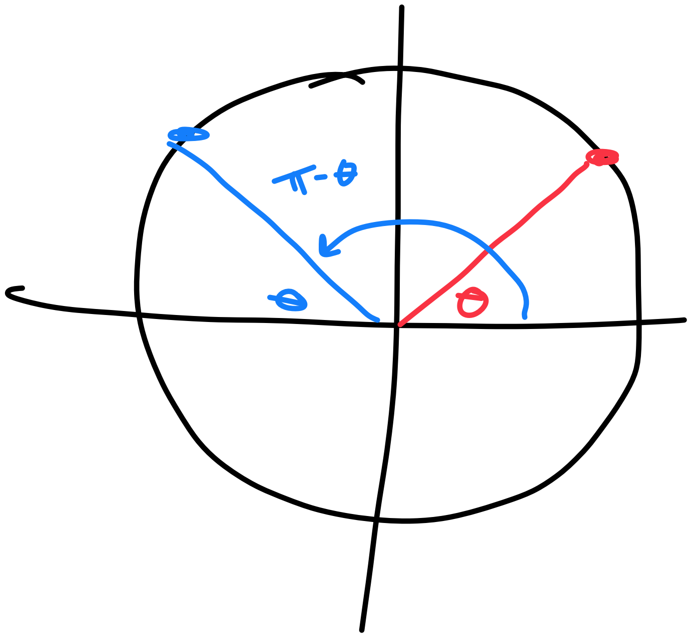
level set is the origin $(0,0,0)$

Case 3: $c > 0$

level set is a sphere of radius $\sqrt{c}$

Ex: $f(x,y) = \sin(x+y)$

Find level sets of height c



Case 1: $|c|>1$

level set is empty

Case 2: $|c|\le 1$

Solve $\sin(x+y)=c$

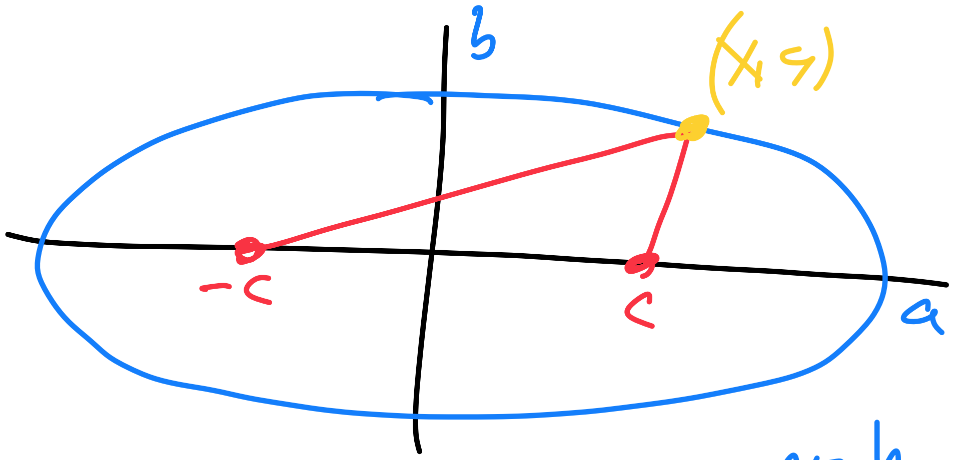
Fix x, let $y=\arcsin(c)-x$ Then $\sin(x+y)=c$

$\sin(\alpha)=\sin(\beta)$ Then $\beta=\alpha+2\pi n$

$\Rightarrow y=\arcsin(c)-x+2\pi n \quad n\in\mathbb{Z}$

and $y=\pi-(\arcsin(c)-x)+2\pi m \quad m\in\mathbb{Z}$

Ellipse



$a > b$

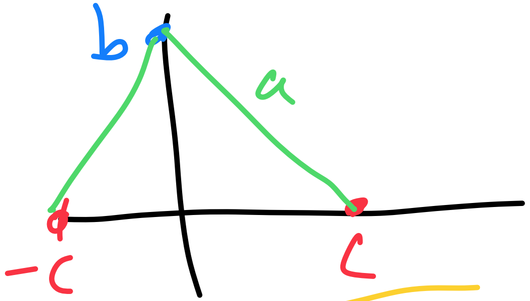
all (x,y) st

$$\|(x,y) - (-c,0)\| + \|(x,y) - (c,0)\| = \text{constant}$$

$(x,y) = (a,0)$

distances are $a-c$ and $a+c$

sum of distances is $(a-c)+(a+c)=2a$



$$a^2 = b^2 + c^2$$

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$$

& algebra

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$u = a\cos\theta$ $\qquad$ $v = b\sin\theta$

$$\underbrace{\left(\frac{a\cos\theta}{a}\right)^2 + \left(\frac{b\sin\theta}{b}\right)^2}_{\cos^2\theta + \sin^2\theta} = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$

YES!

Spherical:

$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

Solid sphere of radius R

$$x^2+y^2+z^2 \le R^2$$

In spherical coords

$$\rho \le R$$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

3.7. **12.7: Cylindrical and Spherical Coordinates – Problems.** #1: Exercise 12.7.12: Describe $x^2 + y^2 + z^2 \leq 10$ in cylindrical coordinates. #2: Exercise 12.7.15: Describe $x^2 + y^2 \leq 9$, with $x \geq y$, in cylindrical coordinates. #3: Exercise 12.7.50: Describe $x^2 + y^2 + z^2 = 1$, with $z \geq 0$, in spherical coordinates. #4: Exercise 12.7.54: Describe $x^2 + y^2 = 3z^2$ in spherical coordinates.

Math 150: Multivariable Calculus: Spring 2023:
Lecture 13: Mathematica: https://youtu.be/izUcZ0hwYeY

Plan for the day.

- Learning how to use Mathematica

Math 150: Multivariable Calculus: Spring 2023:
Lecture 14: Method of Least Squares: https://youtu.be/I2Z47ypMtBI

Slides: https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/MethodOfLeastSquaresLecture.pdf

Notes: https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/MethodLeastSquares.pdf

Math 150: Multivariable Calculus: Spring 2023:
Lecture 15: Chaos, Fractals, Newton’s Method: https://youtu.be/sRVXHXuMnJ4

Slides: https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/CToShiningC_HampshireCollege2022.pdf

Math 150: Multivariable Calculus: Spring 2023:
Lecture 16: In class exam.
Lecture 17: Review

Math 150: Multivariable Calculus: Spring 2023:
Lecture 18: Introduction to Differentiation

Plan for the day.

- Review writing up problems well.
- Review derivative in one-dimension.
- Discuss partial derivatives.
- Discuss tangent plane / hyperplanes.
- Discuss big theorems on differentiability.

$$\lim_{x\to\infty} \frac{2^n+3^n}{n^2+n^3}$$

as a_n does not go to zero, it diverges

$$\lim_{x\to\infty} \frac{n^2+n^3}{2^n+3^n}$$

as terms $\to 0$, has a chance to converge

$$\frac{n^2+n^3}{2^n+3^n} = \frac{n^3\left(1+1/n\right)}{3^n\left(1+\left(2/3\right)^n\right)}$$

Find b_n st $0 \le \frac{n^2+n^3}{2^n+3^n} \le b_n$ and $\sum b_n < \infty$

note $n^2+n^3 \le 2n^3$

$2^n+3^n \geq 2^n$ or $2^n+3^n \geq 3^n$

The above shows $\frac{n^2+n^3}{2^n+3^n} \le \frac{2n^3}{3^n}$, so by the comparison test if $\sum_{n=1}^{\infty} \frac{2n^3}{3^n}$ converges, so too does the original series.

Try ratio test on $b_n = 2 \cdot n^3 / 3^n$

$$\rho = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \frac{\cancel{2} \cdot (n+1)^3 / 3^{n+1}}{\cancel{2} \cdot n^3 / 3^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{1}{3}$$

$$= \frac{1}{3} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 = \frac{1}{3} \left(\lim_{n\to\infty} \frac{n+1}{n}\right)^3$$

$$= \frac{1}{3}$$

As $\rho < 1$, by ratio test it converges.

Compare x^m vs b^x

More generally: x^r vs b^x $\quad r > 0$

Claim: $x^r / b^x \longrightarrow 0$ as $x \to \infty$ for $b > 1$

Example: x^r vs e^x

wlog, let m be the smallest integer $\geq r$

If $x^m / e^x \longrightarrow 0$ so too does x^r / e^x

as $x^m \geq x^r$

$$\lim_{x\to\infty} \frac{x^n}{e^x} \underset{\text{L'Hopital}}{=} \lim_{x\to\infty} \frac{n x^{n-1}}{e^x} \underset{\text{L'H}}{=} \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x}$$

$$= \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0$$

Instead have $b^x = e^{x\log(b)} = e^{x\ln(b)}$

$$(b^x)' = e^{x\log b} \cdot (x\log b)' = b^x \cdot \ln b$$

Aside: $\lim_{n\to\infty} n^{1/n} = 1$

Proof: study $\log(n^{1/n}) = \frac{1}{n}\log(n) = \frac{\log(n)}{n}$

$$\lim_{n\to\infty} \frac{\log(n)}{n} \underset{L'H}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$$

as $\frac{\log(n)}{n} \to 0$, $e^{\log(n)/n} = n^{1/n} \to 1$

let $n = e^x$ as $n\to\infty$, $x\to\infty$

thus $\lim_{n\to\infty} \frac{\log(n)}{n} = \lim_{x\to\infty} \frac{\log(e^x)}{e^x} = \lim_{n\to\infty} \frac{x}{e^x} = 0$

(just did it)

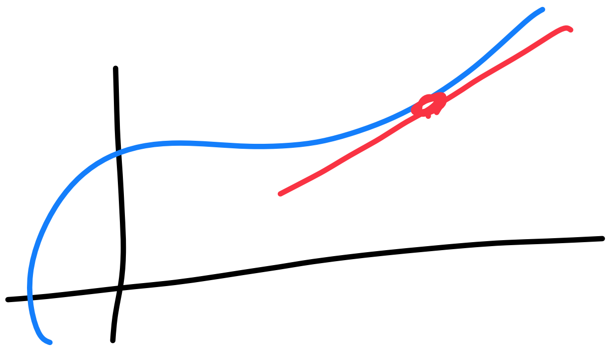
Deriv:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} = 0$$

looks like $f'(a)$

looks like $f'(a) \frac{x-a}{x-a} = f'(a)$



look at $f(x)$, subtract tangent line approx, show it is SO GOOD still go to zero when divide by $x-a$.

Partial Derivatives

$\frac{\partial f}{\partial x}$ means take deriv wrt x, all other variables fixed

$$f(x,y,z) = x^3 + \sin\left(y^3 \sin(y) \tan(z+y^2)\right)$$

$$\frac{\partial f}{\partial x} = 3x^2$$

$$\frac{\partial f}{\partial y} = \cos\left(y^3 \sin(y) \tan(z+y^2)\right) * \frac{\partial}{\partial y}\left[\left(y^3 \sin(y) \tan(z+y^2)\right)\right]$$

2 DIM

$f(x,y)$ is diff at (a,b) if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - \frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b)}{\|(x,y)-(a,b)\|} = 0$$

if $y=b$ always: subtracting tangent line in the x-dir

if $x=a$ always: similar in the y-dir (tangent line in the y-dir)

General: $f(x_1,\dots,x_n)$ is diff at $(a_1,\dots,a_n)$ if

$$\lim_{(x_1,\dots,x_n)\to(a_1,\dots,a_n)} \frac{f(x_1,\dots,x_n) - f(a_1,\dots,a_n) - (\nabla f)(a_1,\dots,a_n)\cdot(x_1-a_1,\dots,x_n-a_n)}{\|(x_1,\dots,x_n)-(a_1,\dots,a_n)\|}$$

is zero.

$\vec{x} = (x_1, \ldots, x_n)$ $\quad \vec{a} = (a_1, \ldots, a_n)$. Say f is differentiable if

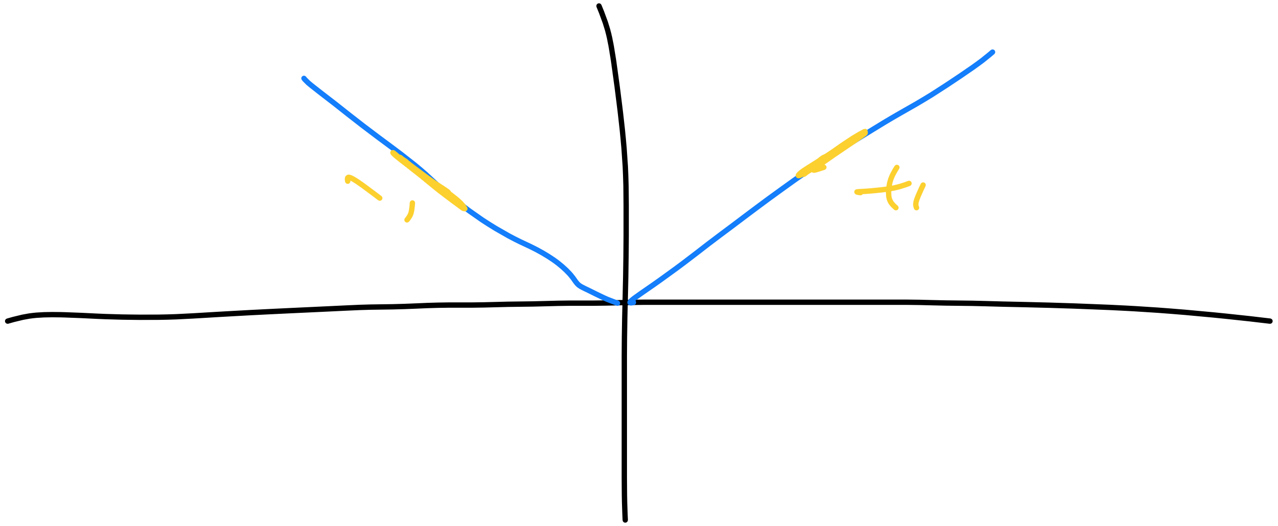
$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - f(\vec{a}) - (\nabla f)(\vec{a}) \cdot (\vec{x} - \vec{a})}{\| \vec{x} - \vec{a} \|} = 0$$

"gradient"

where $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) = \operatorname{grad}(f)$

so $(\nabla f)(\vec{a}) \cdot (\vec{x} - \vec{a}) = \frac{\partial f}{\partial x_1}(\vec{a})(x_1 - a_1) + \cdots + \frac{\partial f}{\partial x_n}(\vec{a})(x_n - a_n)$

In 1-dim consider $f(x) = |x|$



$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

not same

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{-x}{x} = -1$$

Not differentiable!

$\frac{\partial f}{\partial x} = f_x$ is the partial deriv of f wrt x

$\frac{\partial f}{\partial y} = f_y$ Similar with respect to y

What about f_{xy}? Is this

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right) \overset{?}{=} \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) ?$$

Might not be the same! $(f_x)_y$?

Bonus Review Session: 3-28-23

https://youtu.be/ceTXDndMUfg

Tests for Convergence

1) does $a_n \to 0$ as $n \to \infty$

If no, diverges: $\sum_{n=1}^{\infty} a_n$

If yes, <u>may</u> converge

ex. $\sum \frac{1}{n}$ diverges but $\sum \frac{1}{n^2}$ converges

p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges} & \text{if } p>1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$

2) Comparison test:

$0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges

$0 \le c_n \le a_n$ and $\sum c_n$ diverges then $\sum a_n$ diverges

Ex: $a_n = \dfrac{2^n + 3^n}{n^4 + 4^n} \le \dfrac{3^n + 3^n}{4^n} = \dfrac{2 \cdot 3^n}{4^n} = 2\left(\dfrac{3}{4}\right)^n$

Geometric Series, $r = 3/4$ so Converges

$\dfrac{2^n + 3^n}{n^4 + 4^n} \ge \dfrac{2^n + 2^n}{4^n + 4^n} = \dfrac{2 \cdot 2^n}{2 \cdot 4^n} = \left(\dfrac{2}{4}\right)^n$

not useful as lower bound

3) Integral test: Say a_n is non-increasing $f(n) = a_n$ and f is non-increasing

Then $\sum_{n=1}^{N} a_n \approx \int_1^N f(x)\,dx$

$$\sum_{n=10}^{\infty} \frac{1}{n \log n} \approx \int_{10}^{N} \frac{1}{x \log x}\,dx$$

$u = \log x$ $\quad x: 10 \to \infty$

$du = \frac{1}{x}dx$ $\quad u: \log(10) \to \infty$

$f(x) = \frac{1}{x \log x}$

(replace n with x)

$$= \int_{u=\log(10)}^{\infty} \frac{1}{u}\,du = \log(u)\Big|_{\log(10)}^{\infty} = \infty$$

diverges

Ex: try $\sum_{n=10}^{\infty} \frac{1}{n(\log n)^2}$: converges

4) Ratio: $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ if $\begin{cases} <1 & \text{Converges} \\ =1 & \text{no info} \\ >1 & \text{diverges} \end{cases}$

Ex: $a_n = n^3/3^n$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3/3^{n+1}}{n^3/3^n}$$

$$= \lim_{n\to\infty} \frac{3^n}{3^{n+1}} \frac{(n+1)^3}{n^3} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{3}\Big(\lim_{n\to\infty} \underbrace{\frac{n+1}{n}}_{1+\frac{1}{n}}\Big)^3 = \frac{1}{3}$$ Converge

5) Root test: $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}$ if $\begin{cases} <1 & \text{conv} \\ =1 & \text{no info} \\ >1 & \text{div} \end{cases}$

$\boxed{\text{note: } n^{1/n} \longrightarrow 1 \quad \text{as } n \to \infty}$

Study $\log(n^{1/n})$ show goes to 0

$$\log(n^{1/n}) = \frac{1}{n}\log(n) = \frac{\log(n)}{n}$$

$$\lim_{n\to\infty} \frac{\log(n)}{n} \underset{\text{L'Hopital}}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$$

Ex: $a_n = n^3/3^n$

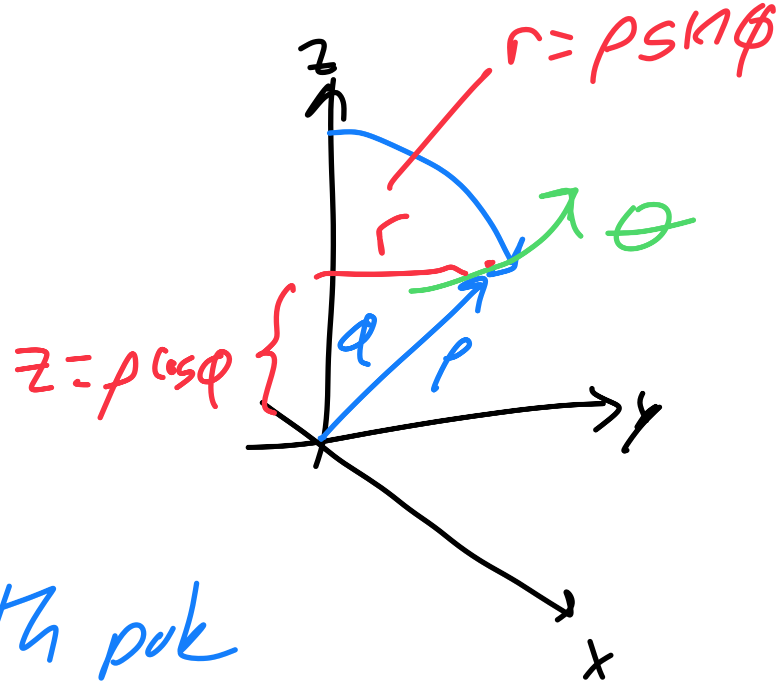
$$\rho = \lim_{n\to\infty}\left(\frac{n^3}{3^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^{3/n}}{3^{n/n}}$$

$$= \lim_{n\to\infty}\frac{(n^{1/n})^3}{3} = \frac{1}{3}\left(\lim_{n\to\infty} n^{1/n}\right)^3 = \frac{1}{3} < 1$$

Converges

Spherical Coordinates

$(x, y, z) \longleftrightarrow (\rho, \theta, \phi)$



$\rho = \sqrt{x^2 + y^2 + z^2}$

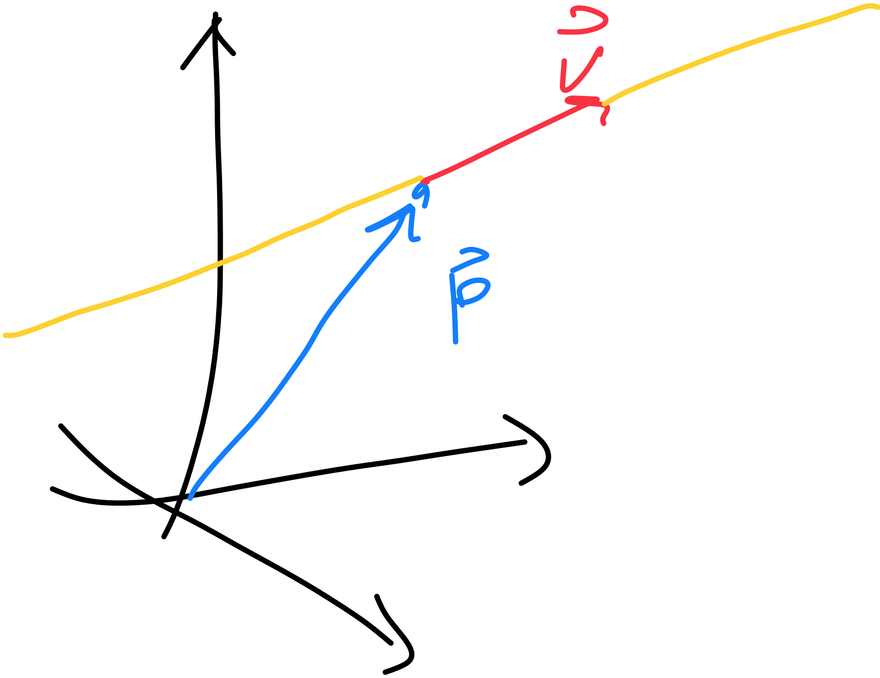
$0 \leq \phi \leq \pi$ north $\to$ south pole

$0 \leq \theta \leq 2\pi$ parallel to equator

polar coordinates with $r = \rho\sin\phi$ and θ

$x = r\cos\theta$
$y = r\sin\theta$
$\Longrightarrow$
$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$

Eqs of lines, planes, ...



line thru $\vec{p}$ in direction $\vec{v}$

$$\vec{P}(t) = \vec{p} + t\,\vec{v}$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} + t \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$t=0$: $\vec{P}(0) = \vec{p}$

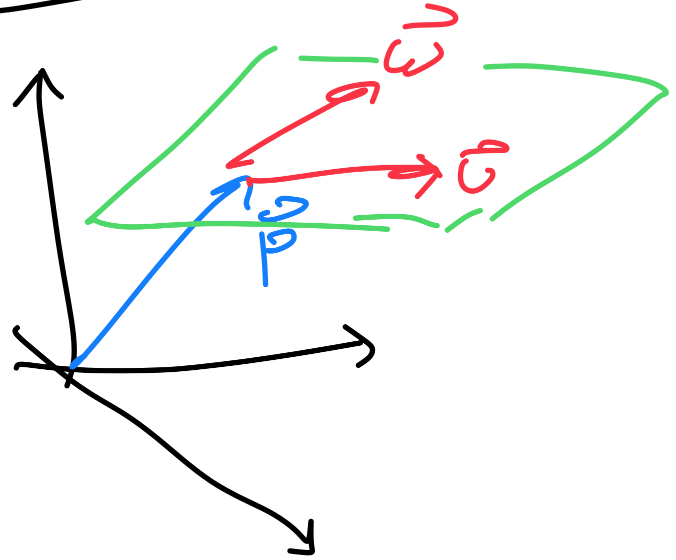
$t=1$: $\vec{P}(1) = \vec{p} + \vec{v}$

line thru $\vec{P}_1$ and $\vec{P}_2$

$\hookrightarrow \vec{v} = \vec{P}_2 - \vec{P}_1$

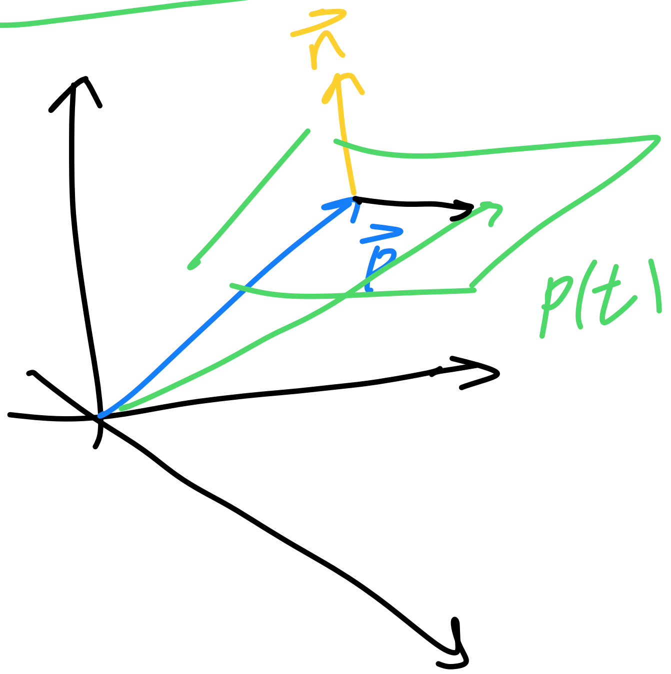
Eq is $\vec{P}(t) = \vec{P}_1 + t\vec{v}$

Eq of a Plane



$\vec{P}(t,s) = \vec{P} + t\vec{v} + s\vec{w}$

Normal Form



$\vec{p}(t) - \vec{p} \perp \vec{n}$

$\vec{n} \cdot (\vec{p} - \vec{p}) = 0$

$\vec{n} \cdot \vec{p} = \vec{n} \cdot \vec{p}$

$n = (a, b, c) \quad \vec{p} = (x, y, z)$

$ax + by + cz = d$

where $d = \vec{n} \cdot \vec{p}$

$\vec{v}, \vec{w}$ are in the plane
and not parallel
then $\vec{v} \times \vec{w}$ and $\vec{n}$
are in same direction

Math 150: Multivariable Calculus: Spring 2023:
Lecture 19: Level Sets, Limits, Partial Differentiation: https://youtu.be/CF1y6yZDvao

Plan for the day: 14.1 – 14.3

- Review Level Sets, Domain and Range
- Review Limits
- Review Partial Derivatives

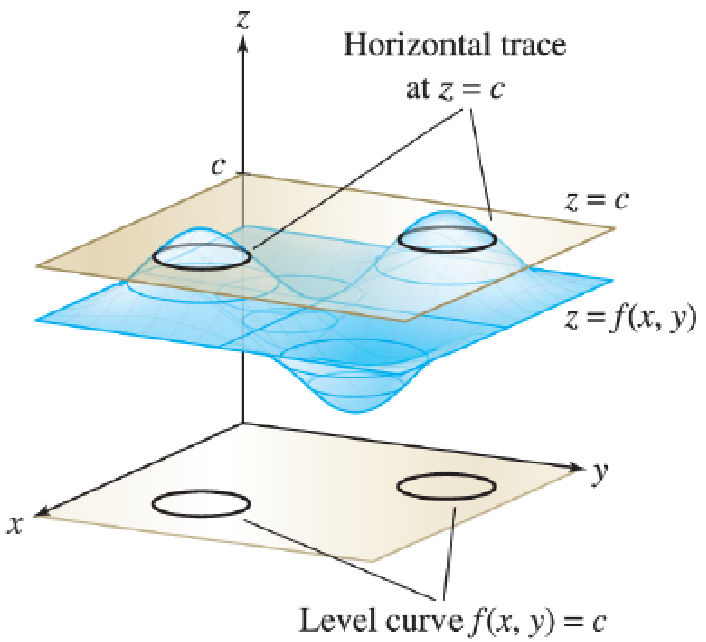
Homework due at the start of class 20:

5.1. **14.1: Functions of Two or More Variables – Problems.** #1: Exercise 14.1.18: Describe the domain and range of $g(r,s) = \cos^{-1}(rs)$. #2: Exercise 14.1.21: Matching functions with their graphs, see book. #3: Exercise 14.1.22: Matching functions with their contour maps, see book.

5.2. **14.2: Limits and Continuity in Several Variables – Problems.** #1: Exercise 14.2.5: Using continuity, evaluate $\lim_{(x,y)\to(\pi/4,0)} \tan x \cos y$. #2: Exercise 14.2.32: Evaluate $\lim_{(x,y)\to(0,0)} xy/(\sqrt{x^2+y^2})$. #3: Exercise 14.2.40: Evaluate $\lim_{(x,y)\to(0,0)}(x+y+2)e^{-1/(x^2+y^2)}$.

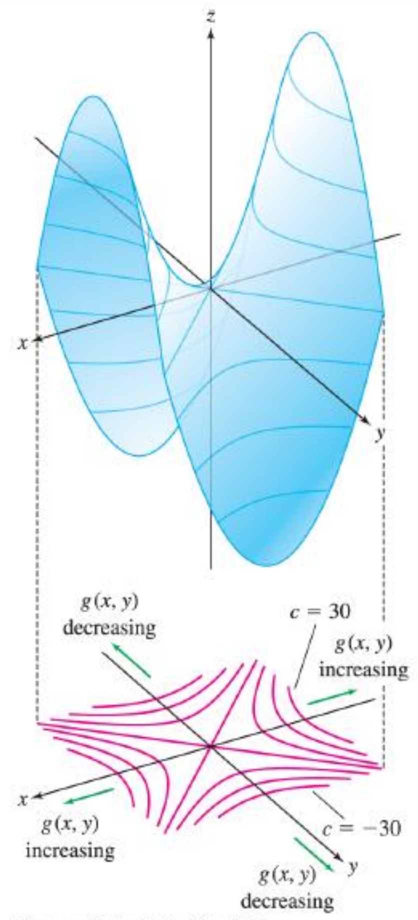
5.3. **14.3: Partial Derivatives – Problems.** #1: Exercise 14.3.20: Compute the first-order partial derivatives of $z = x/(x-y)$. #2: Exercise 14.3.23: Compute the first-order partial derivatives of $z = (\sin x)(\cos y)$. #3: Exercise 14.3.35: Compute the first-order partial derivatives of $U = e^{-rt}/r$. #4: Exercise 14.3.58: Compute the derivative $g_{xy}(-3,2)$ of $g(x,y) = xe^{-xy}$. #5: Exercise 14.3.69: Find a function such that $\partial f/\partial x = 2xy$ and $\partial f/\partial y = x^2$.

- **Level curve:** The curve $f(x, y) = c$ in the xy-plane



Rogawski et al., *Multivariable Calculus*, 4e, © 2019 W. H. Freeman and Company

DF FIGURE 9 The level curve consists of all points (x, y) where the function takes on the value c.



Rogawski et al., *Multivariable Calculus*, 4e, © 2019 W. H. Freeman and Company

FIGURE 12 $g(x, y) = x^2 - 3y^2$. Contour interval $m = 10$.

$(x-\sqrt{3}y)(x+\sqrt{3}y)$

Thus, the level curve corresponding to c consists of all points (x, y) in the domain of f in the xy-plane where the function takes the value c. Each level curve is the projection onto the xy-plane of the horizontal trace on the graph that lies above it.

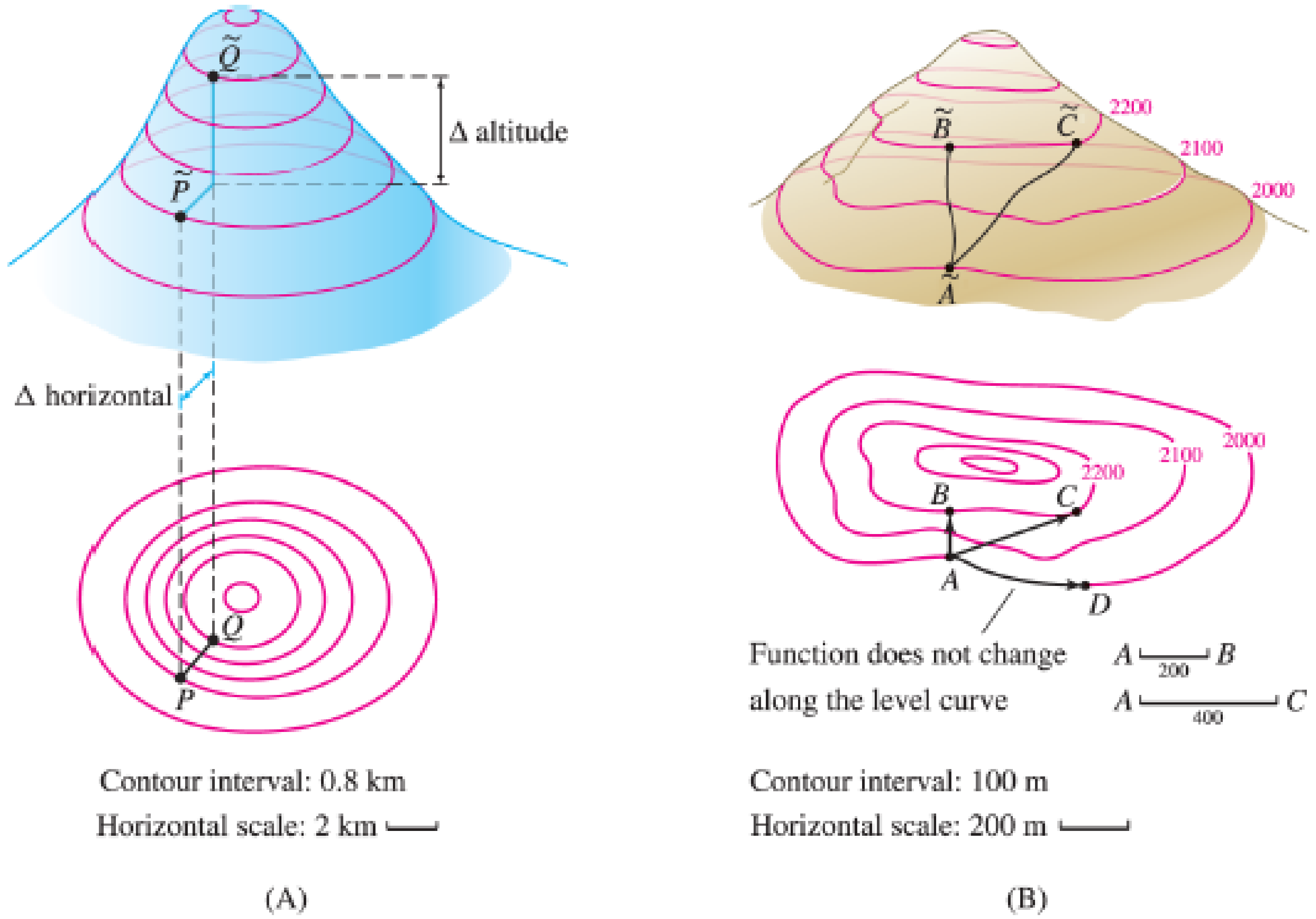

$\tilde{Q}$
Δ altitude
$\tilde{P}$
Δ horizontal
Q
P
Contour interval: 0.8 km
Horizontal scale: 2 km
(A)
2200
2100
2000
$\tilde{B}$
$\tilde{C}$
$\tilde{A}$
2000
2100
2200
B
C
A
D
Function does not change
along the level curve
A 200 B
A 400 C
Contour interval: 100 m
Horizontal scale: 200 m
(B)

FIGURE 14

Contour Example: f(x,y) = cos(x^2 + y^2)

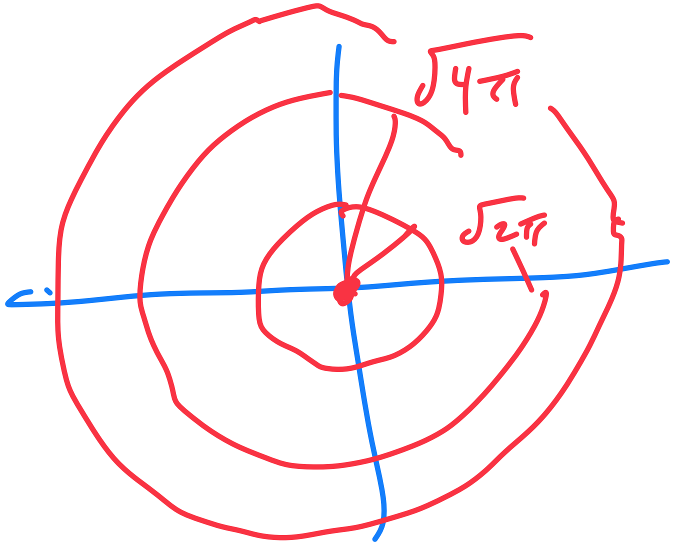
Solve for (x,y) such that $\cos(x^2+y^2)=c$ for a fixed c.

Empty if $|c|>1$

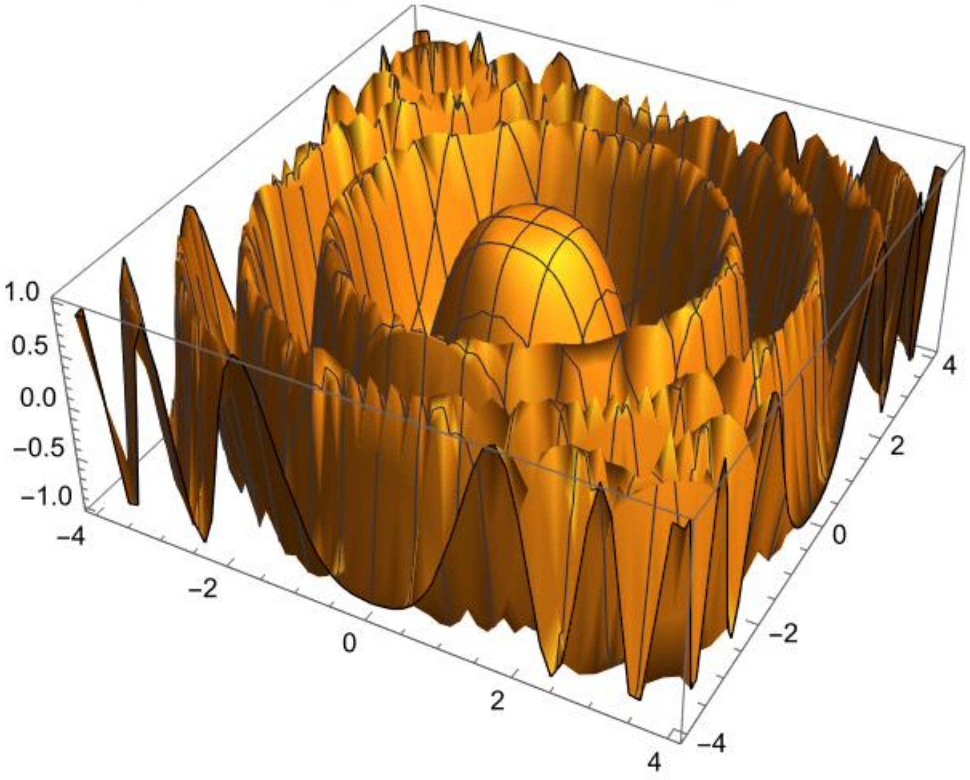
Level sets union of circles if $|c|\le 1$

Ex: $c=1$: Then $x^2+y^2=0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$

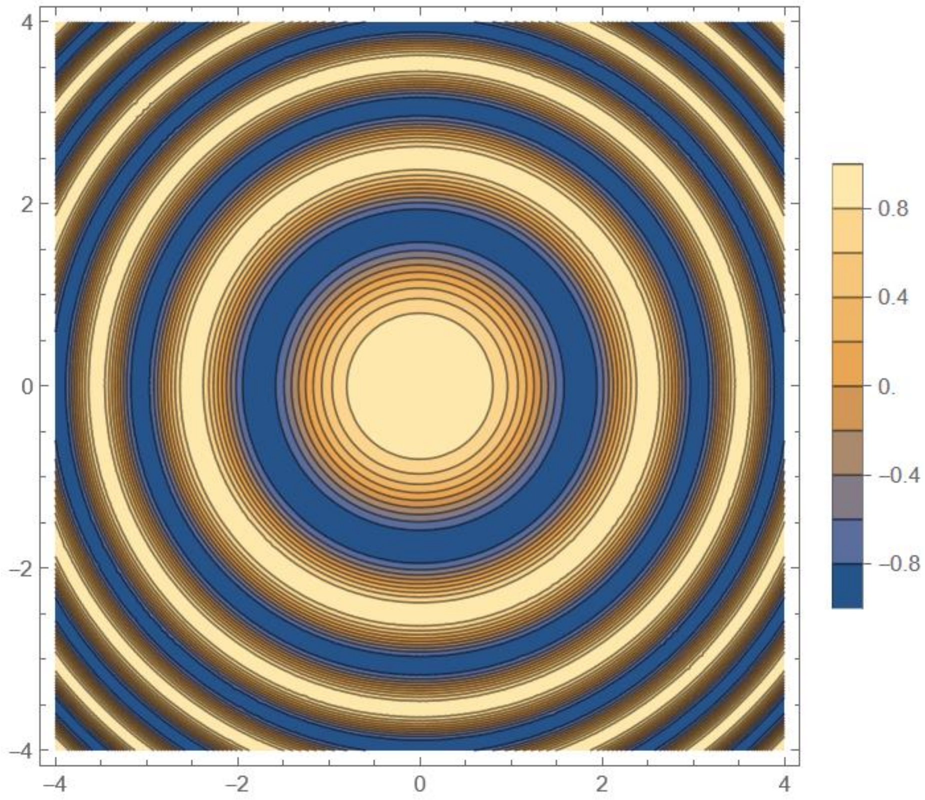
only positive signs matter



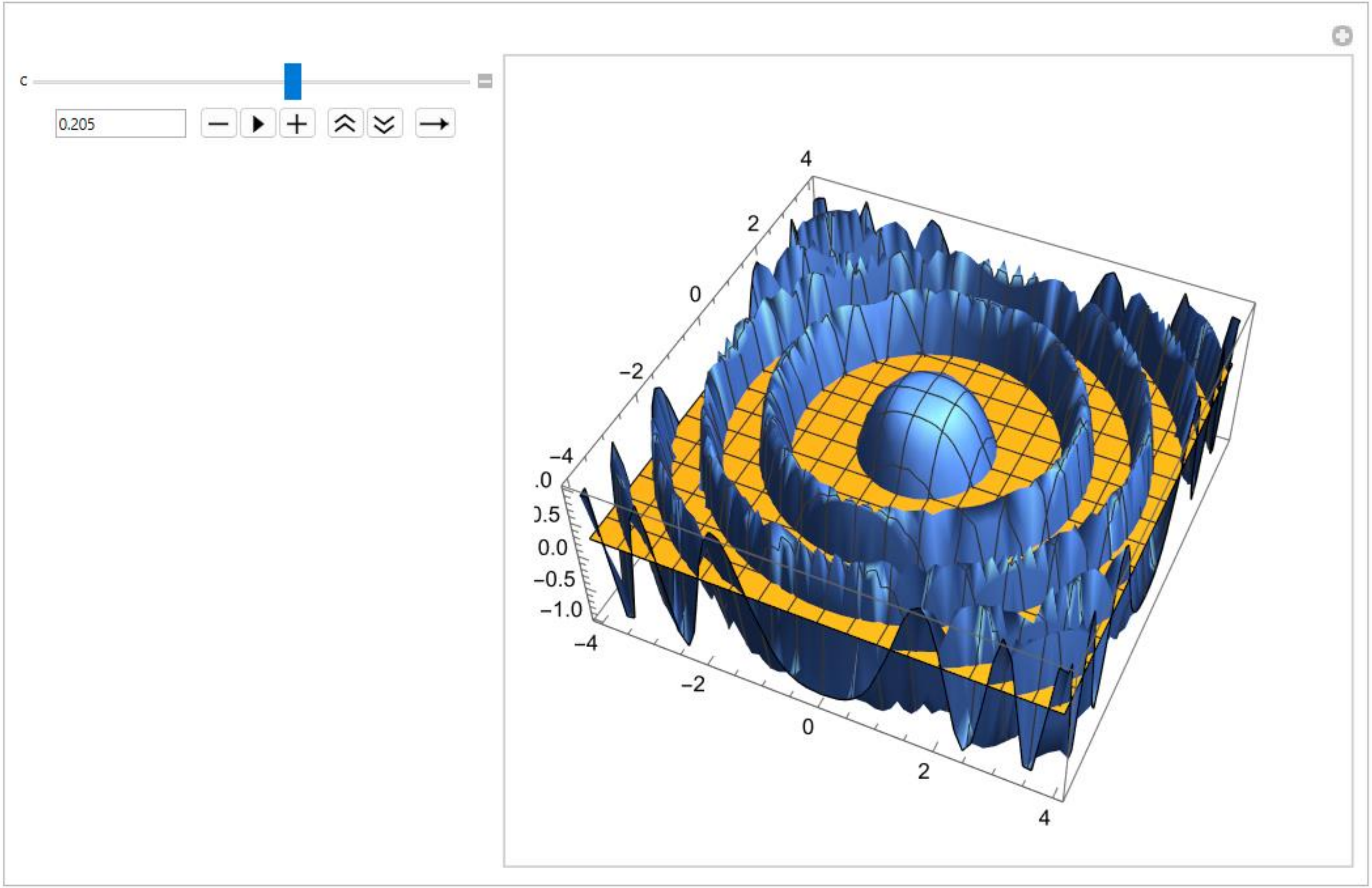
```
Plot3D[Cos[x^2 + y^2], {x, -4, 4}, {y, -4, 4}]
```




```
ContourPlot[Cos[x^2 + y^2], {x, -4, 4}, {y, -4, 4}, PlotLegends → Automatic]
```



```
Manipulate[Plot3D[{c, Cos[x^2 + y^2]}, {x, -4, 4}, {y, -4, 4}], {c, -1, 1}]
```



DEFINITION

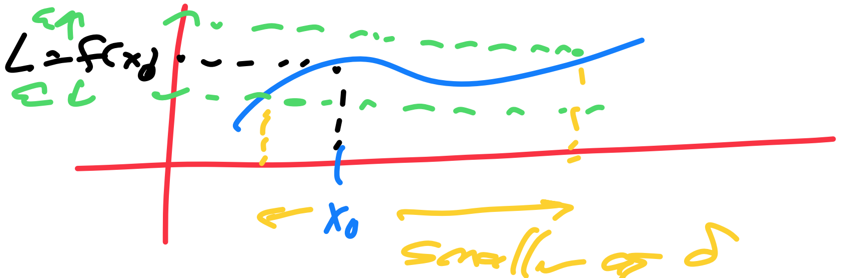
Limit

Assume that $f(x, y)$ is defined near $P = (a, b)$. Then

$$\lim_{(x,y)\to P} f(x, y) = L$$

if, for any $\epsilon > 0$, there exists $\delta > 0$ such that if (x, y) satisfies

$$0 < d((x, y), (a, b)) < \delta, \quad \text{then} \quad |f(x, y) - L| < \epsilon$$



THEOREM 1

Limit Laws

Assume that $\lim_{(x,y)\to P} f(x, y)$ and $\lim_{(x,y)\to P} g(x, y)$ exist.

i. **Sum Law:**
$$\lim_{(x,y)\to P} (f(x, y) + g(x, y)) = \lim_{(x,y)\to P} f(x, y) + \lim_{(x,y)\to P} g(x, y)$$

ii. **Constant Multiple Law:** For any number k,
$$\lim_{(x,y)\to P} kf(x, y) = k \lim_{(x,y)\to P} f(x, y)$$

iii. **Product Law:**
$$\lim_{(x,y)\to P} f(x, y)\, g(x, y) = \left(\lim_{(x,y)\to P} f(x, y)\right)\left(\lim_{(x,y)\to P} g(x, y)\right)$$

iv. **Quotient Law:** If $\lim_{(x,y)\to P} g(x, y) \neq 0$, then
$$\lim_{(x,y)\to P} \frac{f(x, y)}{g(x, y)} = \frac{\lim_{(x,y)\to P} f(x, y)}{\lim_{(x,y)\to P} g(x, y)}$$

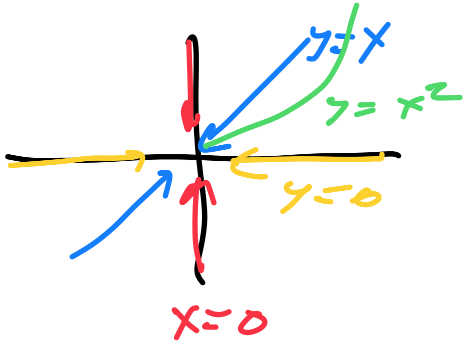
Continuous if the limit equals the value of the function at the point.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+e^{xy}-y}{xy+2} = \frac{1}{2}$$ replace x with 0, y with 0

Methods to find limits:

- Direct substitution.
- Polar transformation.

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$$

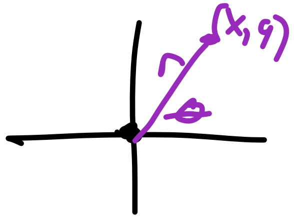


$$\lim_{\substack{x\to 0\\(y=0)}} \frac{x^4}{x^2} = \lim_{x\to 0} x^2 = 0$$

$$\lim_{\substack{y\to 0\\(x=0)}} \frac{y^4}{y^2} = 0$$

$$\lim_{\substack{x\to 0\\y=x}} \frac{2x^4}{2x^2} =$$

$$\lim_{\substack{x\to 0\\(y=x^2)}} \frac{x^4+x^8}{x^2+x^4} = 0$$



$x=r\cos\theta$, $y=r\sin\theta$ $\quad (x,y)\to(0,0)$ have $r\to 0$, θ free

$$\lim_{\substack{r\to 0\\ \theta \text{ free}}} \frac{r^4\cos^4\theta + r^4\sin^4\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{\substack{r\to 0\\ \theta \text{ free}}} r^2(\cos^4\theta+\sin^4\theta) = 0$$

$$0 \le r^2(\cos^4\theta + \sin^4\theta) \le 2r^2$$

Squeeze Theorem $\quad r\to 0 \longrightarrow 0 \longleftarrow r\to 0$

The **partial derivatives** are the rates of change with respect to each variable separately. A function $f(x, y)$ of two variables has two partial derivatives, denoted f_x and f_y, defined by the following limits (if they exist):

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}, \qquad f_y(a, b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

Thus, f_x is the derivative of $f(x, b)$ as a function of x alone, and f_y is the derivative of $f(a, y)$ as a function of y alone. We refer to f_x as **the partial derivative of f with respect to x** or **the x-derivative of f.** We refer to f_y similarly. The Leibniz notation for partial derivatives is

$$\frac{\partial f}{\partial x} = f_x, \qquad \frac{\partial f}{\partial y} = f_y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b), \qquad \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a, b)$$

Higher Order Partial Derivatives

The higher order partial derivatives are the derivatives of derivatives. The *second-order* partial derivatives of f are the partial derivatives of f_x and f_y. We write f_{xx} for the x-derivative of f_x and f_{yy} for the y-derivative of f_y:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

We also have the *mixed partials*:

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$(f_x)_y$ $\qquad$ $(f_y)_x$

The process can be continued. For example, f_{xyx} is the x-derivative of f_{xy}, and f_{xyy} is the y-derivative of f_{xy} (perform the differentiation in the order of the subscripts from left to right). The Leibniz notation for higher order partial derivatives is

$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}, \qquad f_{yx} = \frac{\partial^2 f}{\partial x \partial y}, \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

THEOREM 1

Clairaut's Theorem: Equality of Mixed Partials

If f_{xy} and f_{yx} both exist and are continuous on a disk $D,$ then $f_{xy}(a,b) = f_{yx}(a,b)$ for all $(a,b) \in D.$

Therefore, on $D,$

$$\boxed{\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}}$$

The Heat Equation

Show that $u\left(x,t\right)=\dfrac{1}{2\sqrt{\pi t}}e^{-\left(x^2/4t\right)}$, defined for $t>0,$ satisfies the heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

2

Math 150: Multivariable Calculus: Spring 2023:

Lecture 20: Tangent Planes, Approximation, Directional Derivatives:

https://youtu.be/sndIgR0iTxI

Plan for the day: 14.4 – 14.5

- Tangent Planes and Differentiability
- Approximation
- Directional Derivatives

Homework due at the start of class 22 (not 21 – class 21 is on applications):

5.4. **14.4: Differentiability, Tangent Planes, and Linear Approximation – Problems.** #1: Exercise 14.4.5: Find an equation of the tangent plane at $(4, 1)$ of $f(x, y) = x^2 + y^{-2}$. #2: Exercise 14.4.14: Find the points on the graph of $f(x, y) = (x + 1)y^2$ at which the tangent plane is horizontal. #3: Exercise 14.4.23: Estimate $f(2.1, 3.8)$ assuming that $f(2, 4) = 5$, $f_x(2, 4) = 0.3$, and $f_y(2, 4) = -0.2$.

5.5. **14.5: The Gradient and Directional Derivatives – Problems.** #1: Exercise 14.5.24: Calculate the directional derivative of $\sin(x - y)$ at $P = (\pi/2, \pi/6)$ in the direction of $v = \langle 1, 1\rangle$. #2: Exercise 14.5.35: Determine the direction in which $f(x, y, z) = xy/z$ has maximum rate of increase from $P = (1, -1, 3)$, and give the rate of change in that direction. #3: Exercise 14.5.41: Find a vector normal to the surface $x^2 + y^2 - z^2 = 6$ at $P = (3, 1, 2)$. #4: Exercise 14.5.55: Find a function $f(x, y, z)$ such that $\nabla f = \langle z, 2y, x\rangle$.

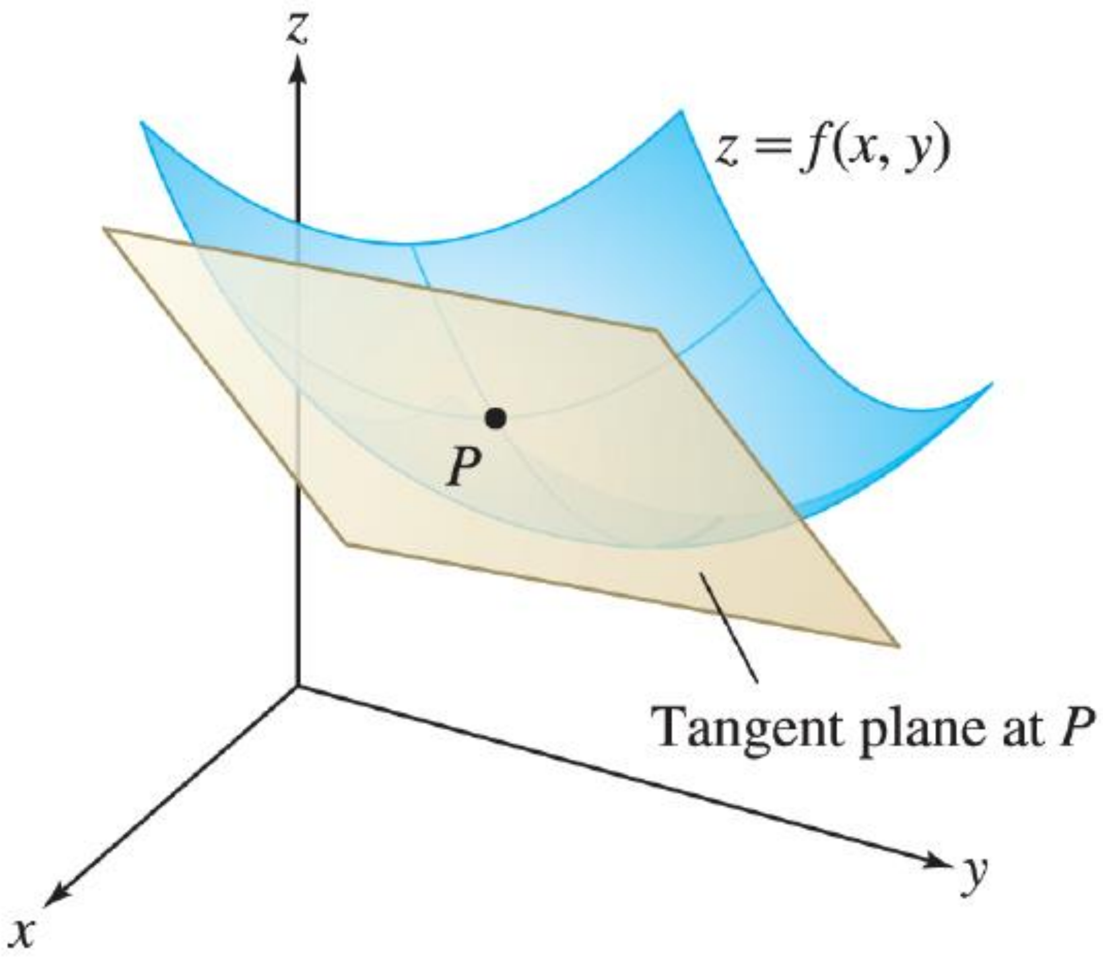
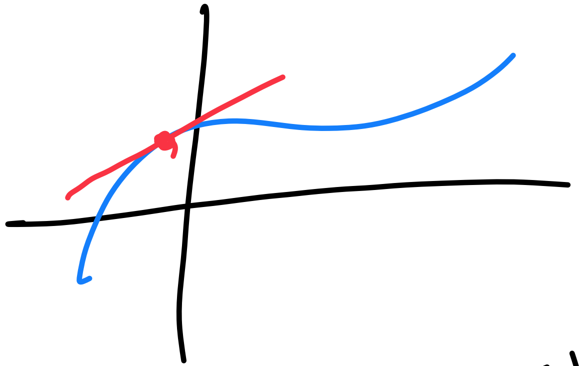


FIGURE 1 Tangent plane to the graph of $f(x, y)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

$$\lim_{x \to a} \left| \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} \right| = 0$$



$$\lim_{\vec{x} \to \vec{a}} \frac{\left| f(\vec{x}) - f(\vec{a}) - (\nabla f|(\vec{a})) \cdot (\vec{x} - \vec{a}) \right|}{\|\vec{x} - \vec{a}\|} = 0$$

REMINDER

A plane through the point $P = (x_0, y_0, z_0)$ *with normal vector* $\mathbf{n} = \langle A, B, C \rangle$ *has equation* $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

If $f(x, y)$ is differentiable at (a, b), then the **tangent plane** to the graph at $(a, b, f(a, b))$ is the plane with equation $z = L(x, y)$. Explicitly, the equation of the tangent plane is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (1)$$

$$f(a + \Delta x, b + \Delta y) \approx f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y \quad (3)$$

$(x, y) = (a + \Delta a, b + \Delta b)$ assume know
$f(a,b)$, $f_x(a,b)$, $f_y(a,b)$

Let f(x,y) = sqrt(x^2 + 3y). Approximate f(4.1, 2.9).

$(a,b) = (4,3) \quad f(4,3) = 5 \quad \Delta a = .1 \quad \Delta b = -.1$

$$f_x(x,y) = \frac{\partial}{\partial x}(x^2+3y)^{1/2} = \frac{1}{2}(x^2+3y)^{-1/2}\frac{\partial}{\partial x}(x^2+3y) = \frac{x}{\sqrt{x^2+3y}}$$

$$f_x(4,3) = \frac{4}{5} = \frac{8}{10} = .8$$

$$f_y(x,y) = \frac{\partial}{\partial y}(x^2+3y)^{1/2} = \frac{1}{2}(x^2+3y)^{-1/2}\frac{\partial}{\partial y}(x^2+3y) = \frac{3}{2\sqrt{x^2+3y}}$$

$$f_y(4,3) = \frac{3}{2.5} = \frac{3}{10} = .3$$

$$f(4.1, 2.9) \approx f(4,3) + f_x(4,3)(.1) + f_y(4,3)(-.1)$$
$$= 5 + (.8)(.1) + (.3)(-.1) = 5.05$$

How far is $(4.1, 2.9)$ from $(4,3)$? Is $\sqrt{(4.1-4)^2 + (2.9-3)^2}$

or $\sqrt{2}/10 \approx .141$

$f(4.1, 2.9) \approx 5.0574\ldots$

notice error is much smaller than how far we moved.

The gradient of a function of n variables is the vector

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$$

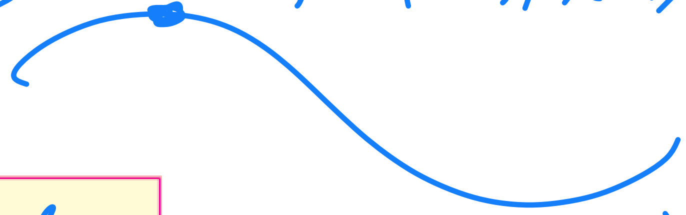
$u = x - y$ $\quad$ $x = \frac{u+v}{2}$

$v = x + y$ $\quad$ $y = \frac{-u+v}{2}$

A function f that is defined along a path $\mathbf{r}(t)$ results in a composition $f(\mathbf{r}(t))$. The Chain Rule for Paths is used to find the derivative of these composite functions.

$r(t) = (x(t), y(t), z(t))$

If f and $\mathbf{r}(t)$ are differentiable, then



$r(t)$

curve

$$\boxed{\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f_{\mathbf{r}(t)} \cdot \mathbf{r}'(t) = \nabla f_{r(t)} \cdot \frac{dr}{dt}}$$

$t \longrightarrow r(t) = (x(t), y(t), z(t)) \longrightarrow f(r(t))$

$= f(x(t), y(t), z(t))$

$f: R^3 \to R$

$r: R \to R^3$

so $f \circ r: R \to R$

$g(t) = f(r(t))$, $g: R \to R$

$$g(t) = f(x(t), y(t), z(t))$$

$$g'(t) = \lim_{h\to 0} \frac{f(x(t+h), y(t+h), z(t+h)) - f(x(t), y(t), z(t))}{h}$$

$$\frac{f(x(t+h), y(t+h), z(t+h)) - f(x(t), y(t+h), z(t+h)) + f(x(t), y(t+h), z(t+h)) \dots}{h}$$

want $f(x(t)+h)$

$$\frac{f(x(t+h), y(t+h), z(t+h)) - f(x(t), y(t+h), z(t+h))}{\cancel{h}\,(x(t+h) - x(t))} \quad \frac{x(t+h) - x(t)}{h}$$

"$a = x(t)$"

"x" $=$ $x(t+h)$"

"$x \to a$" $\leq$ "$h \to 0$"

$$f_x(r(t)) \qquad x'(t)$$

defn of the deriv of x

The directional derivative of f at $P = (a, b)$ in the direction of a unit vector $\mathbf{u} = \langle h, k \rangle$ is the limit (assuming it exists)

$$D_{\mathbf{u}} f(P) = D_{\mathbf{u}} f(a, b) = \lim_{t \to 0} \frac{f(a + th, b + tk) - f(a, b)}{t}$$

If f is differentiable at P and $\mathbf{u}$ is a unit vector, then the directional derivative in the direction of $\mathbf{u}$ is given by

$$D_{\mathbf{u}} f(P) = \nabla f_P \cdot \mathbf{u}$$

2

$r(t) = \vec{P} + t\vec{u} = (x(t), y(t))$ $\quad r(0) = \vec{P}$

$r'(t) = \vec{u} = (h, k) = (x'(t), y'(t))$

$g(t) = f(r(t))$

$g'(t) = \nabla f_{r(t)} \cdot r'(t)$ so for us it's $\nabla f_{\vec{P}} \cdot \vec{u}$

$D_{\vec{u}} f$ is $\nabla f_{\vec{p}} \cdot \vec{u}$

Take $\vec{u} = (1,0,0) \Longrightarrow \frac{\partial f}{\partial x}$

$\vec{u} = (0,1,0) \Longrightarrow \frac{\partial f}{\partial y}$

ALWAYS TAKE

$\|\vec{u}\| = 1$

Math 150: Multivariable Calculus: Spring 2023:
Lecture 21: Application (Trafalgar), Review: https://youtu.be/gxCZOZZx9KQ

Plan for the day: 14.4 – 14.5

- Application: Battle of Trafalgar
- Review: Differentiation Rules

Homework due at the start of class 22:

5.4. **14.4: Differentiability, Tangent Planes, and Linear Approximation – Problems.** #1: Exercise 14.4.5: Find an equation of the tangent plane at $(4, 1)$ of $f(x, y) = x^2 + y^{-2}$. #2: Exercise 14.4.14: Find the points on the graph of $f(x, y) = (x + 1)y^2$ at which the tangent plane is horizontal. #3: Exercise 14.4.23: Estimate $f(2.1, 3.8)$ assuming that $f(2, 4) = 5$, $f_x(2, 4) = 0.3$, and $f_y(2, 4) = -0.2$.

5.5. **14.5: The Gradient and Directional Derivatives – Problems.** #1: Exercise 14.5.24: Calculate the directional derivative of $\sin(x - y)$ at $P = (\pi/2, \pi/6)$ in the direction of $v = \langle 1, 1\rangle$. #2: Exercise 14.5.35: Determine the direction in which $f(x, y, z) = xy/z$ has maximum rate of increase from $P = (1, -1, 3)$, and give the rate of change in that direction. #3: Exercise 14.5.41: Find a vector normal to the surface $x^2 + y^2 - z^2 = 6$ at $P = (3, 1, 2)$. #4: Exercise 14.5.55: Find a function $f(x, y, z)$ such that $\nabla f = \langle z, 2y, x\rangle$.

Differential Equations: I: First Order

Lots of differential equations can study.

Consider $f'(x) = af(x)$ with initial condition $f(0) = C$.

Special case: $a = 1$ solution $f(x) = Ce^x$....

Solution: $f(x) = Ce^{ax}$ ($f(0) = C$ yields unique soln).

Check: $f(x) = Ce^{ax}$ then $f'(x) = aCe^{ax} = af(x)$.

Differential Equations: II: Second Order

What about $f''(x) \; = \; af'(x) + bf(x)$?

Similar to our difference equations! Try exponential!

$f(x) = e^{\rho x}$ ($e^{\rho x} = (e^{\rho})^x$ like r^n from before) yields

$$\rho^2 e^{\rho x} \; = \; a\rho e^{\rho x} + be^{\rho x}.$$

Yields characteristic equation

$$\rho^2 - a\rho - b \; = \; 0 \quad \text{with roots } \rho_1, \rho_2,$$

general solution (if $\rho_1 \neq \rho_2$)

$$f(x) \; = \; \alpha e^{\rho_1 x} + \beta e^{\rho_2 x}.$$

In general have several variables and/or related quantities.

Consider a system involving $f(x)$ and $g(x)$:

$$\begin{aligned} f'(x) &= af(x) + bg(x) \\ g'(x) &= cf(x) + dg(x). \end{aligned}$$

How do we solve? Think back to similar examples.

Differential Equations: III: System: Solution

$$
\begin{aligned}
f'(x) &= af(x) + bg(x) \\
g'(x) &= cf(x) + dg(x).
\end{aligned}
$$

In linear algebra solved for one variable in terms of others.

$g(x) \;=\; \frac{1}{b}f'(x) - \frac{a}{b}f(x)$, substitute:

$$
\begin{aligned}
\left[\frac{1}{b}f'(x) - \frac{a}{b}f(x)\right]' &= cf(x) + d\left[\frac{1}{b}f'(x) - \frac{a}{b}f(x)\right] \\
f''(x) &= (a+d)f'(x) + (cb - ad)f(x),
\end{aligned}
$$

reducing to previously solved problem!

Differential Equations: III: Matrix Formulation for System

$$V'(x) = AV(x), \quad V(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Formally looks like $f'(x) = af(x)$, guess solution is $V(x) = e^{Ax}V(0)$, where

$$e^{Ax} = I + Ax + \frac{1}{2!}A^2x^2 + \frac{1}{3!}A^3x^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^kx^k.$$

Can justify term-by-term differentiation of series for e^{Ax}, see importance of matrix exponential.

Mentioned Baker-Campbell-Hausdorf formula; in general product of matrices is hard but $(e^{Ax})' = Ae^{Ax} = e^{Ax}A$.

Application: Battle of Trafalgar

Modified from *Mathematics in Warfare* by F. W. Lancaseter.

Battle of Trafalgar

Wikipedia: “The battle was the most decisive naval victory of the war. Twenty-seven British ships of the line led by Admiral Lord Nelson aboard HMS Victory defeated thirty-three French and Spanish ships of the line under French Admiral Pierre-Charles Villeneuve off the southwest coast of Spain, just west of Cape Trafalgar, in Caños de Meca.

The Square Law: I

Forces $r(t)$ and $b(t)$, effective fighting values N and M:

$$\begin{aligned} b'(t) &= -Nr(t) \\ r'(t) &= -Mb(t). \end{aligned}$$

Can solve using techniques from before: what do you expect solution to look like?

If take derivatives again find

$$b''(t) = -Nr'(t) = NMb(t), \quad \text{yields}$$

$$b(t) = \beta_1 e^{\sqrt{NM}t} + \beta_2 e^{-\sqrt{NM}t}, \quad r(t) = \alpha_1 e^{\sqrt{NM}t} + \alpha_2 e^{-\sqrt{NM}t}.$$

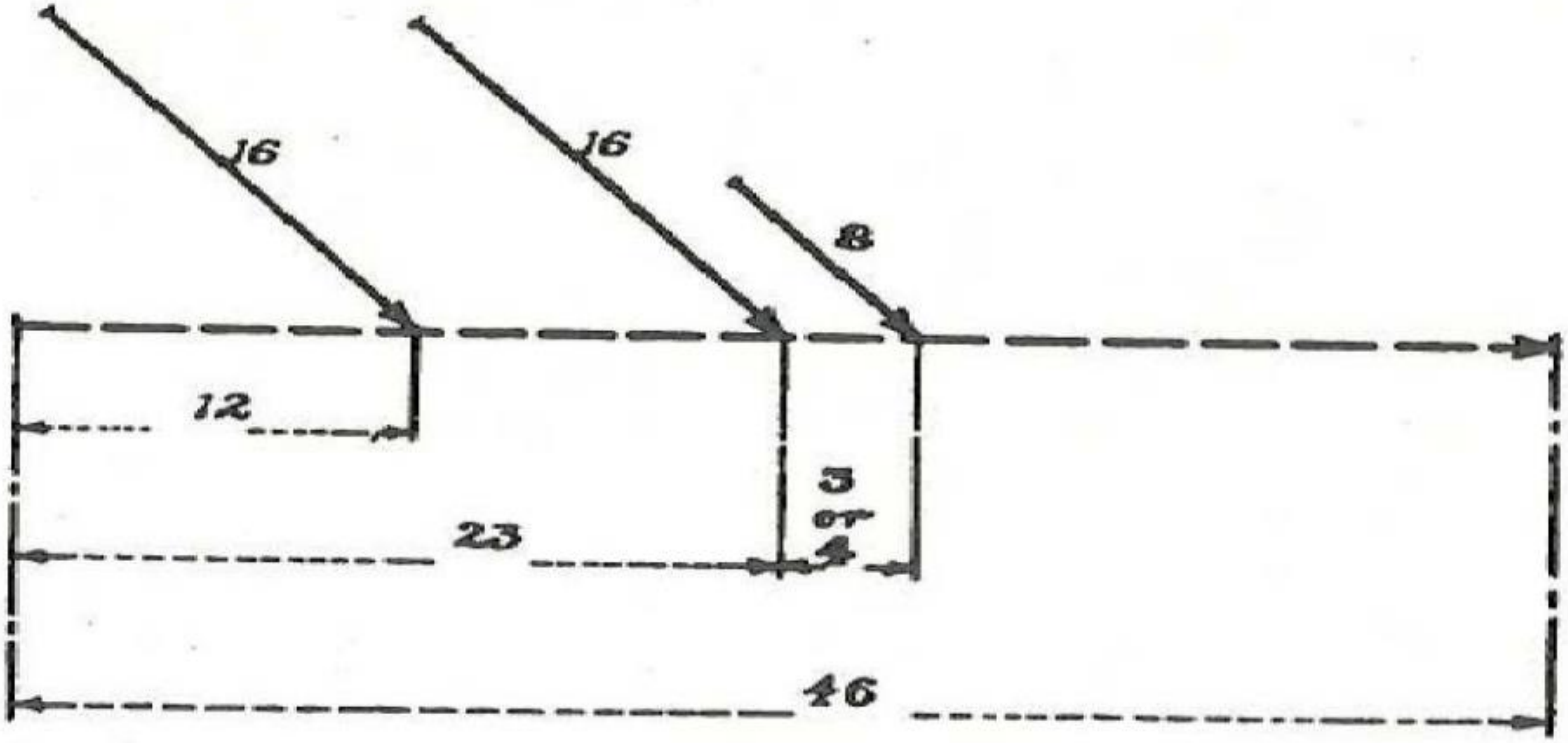
$b'(t)/b(t) = r'(t)/r(t)$ yields $Nr(t)^2 = Mb(t)^2$ (square law).

Trafalgar

Nelson outnumbered – how could he win?

Analysis of Nelson's Plan: I

Nelson assumed for the purpose of framing his plan of attack that his own force would consist of forty sail of the line, against forty-six of the com-



Analysis of Nelson's Plan: II

If for the purpose of comparison we suppose the total forces had engaged under the conditions described by Villeneuve as "the usage of former days," we have:—

Strength of combined fleet, 46^2	....	$= 2116$
" British " 40^2	....	$= 1600$
Balance in favour of enemy	....	516

Analysis of Nelson's Plan: III

Dealing with the position arithmetically, we have:—

Strength of British (in arbitrary n^2 units),

$$32^2 + 8^2 = 1088$$

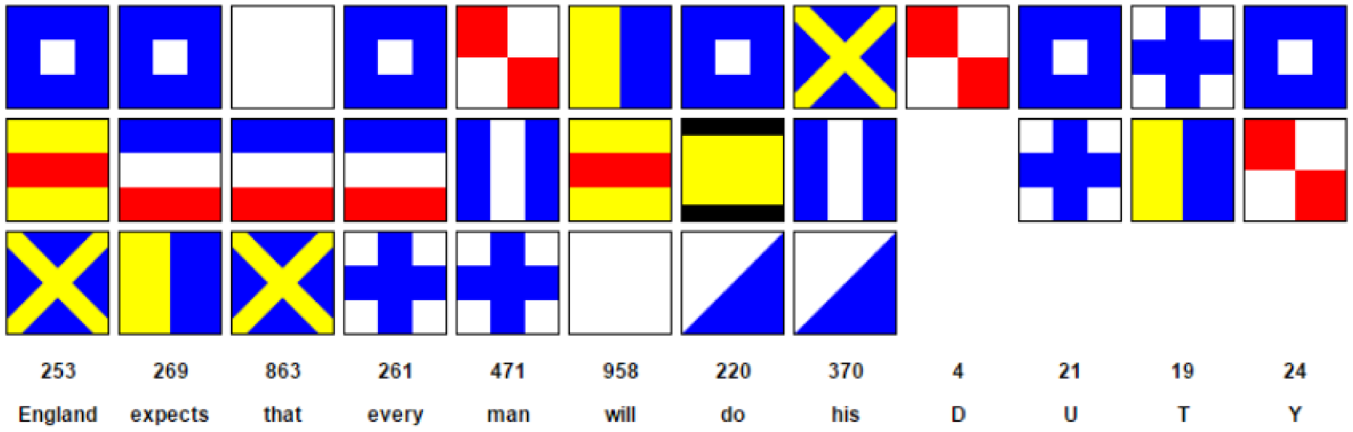
And combined fleet,

$$23^2 + 23^2 = 1058$$

British advantage 30

Wikipedia: "The battle was the most decisive naval victory of the war. Twenty-seven British ships of the line led by Admiral Lord Nelson aboard HMS Victory defeated thirty-three French and Spanish ships of the line under French Admiral Pierre-Charles Villeneuve off the southwest coast of Spain, just west of Cape Trafalgar, in Caños de Meca. The Franco-Spanish fleet lost twenty-two ships, without a single British vessel being lost."

AfterMATH of Battle of Trafalgar



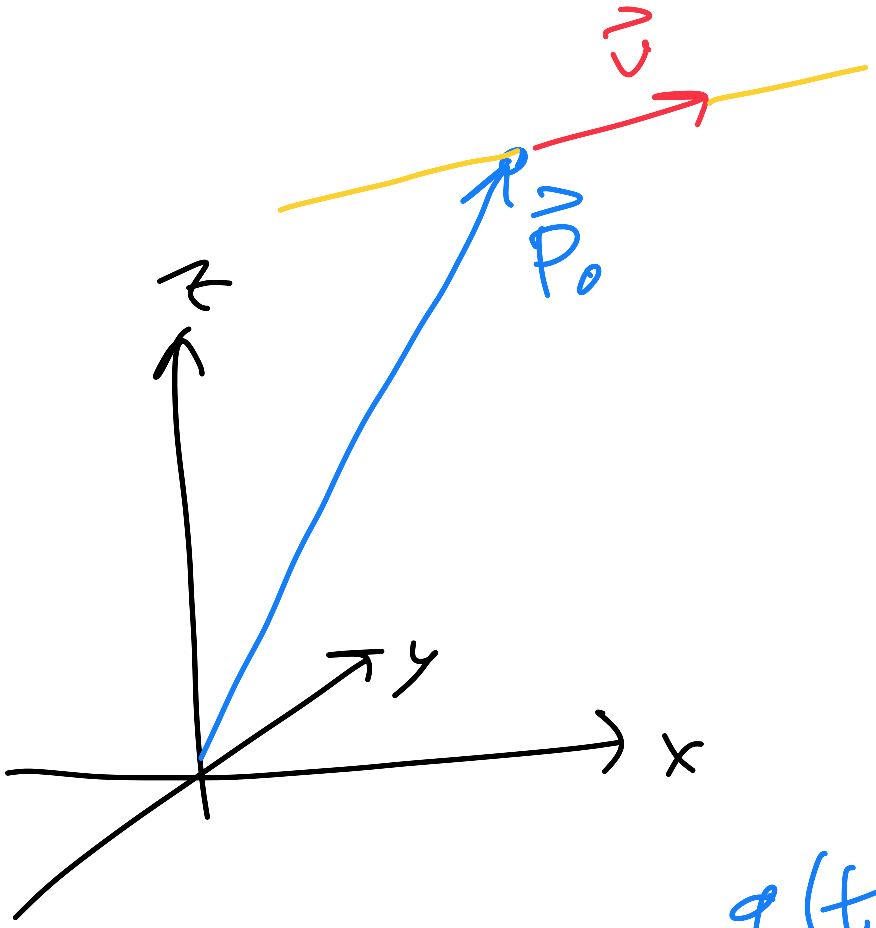
British: 0 of 27 ships, 1,666 dead or wounded.
Franco-Spanish: 22 of 33 ships, 13,781 captured, dead or wounded.

Biggest issue is deterministic.

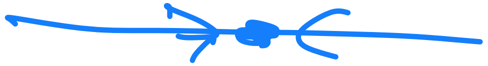
Make fighting effectiveness random variables!

Leads to stochastic differential equations.

http://en.wikipedia.org/wiki/Stochastic_differential_equation.

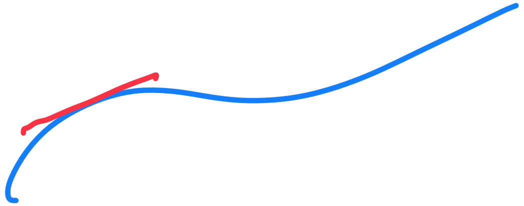


$$\vec{P}(t) = \vec{P}_0 + t\,\vec{v} = r(t) = c(t)$$



$$f(x, y, z)$$

$$g(t) := (f \circ r)(t) = f(r(t))$$
$$= f(x(t), y(t), z(t))$$



$$g(t) = f(r(t))$$

$$g'(t) = f_x(r(t))\,x'(t) + f_y(r(t))\,y'(t) + f_z(r(t))\,z'(t)$$

$$\downarrow$$

$$\frac{\partial f}{\partial x}(r(t))\,\frac{dx}{dt}(t)$$

Math 150: Multivariable Calculus: Spring 2023:

Lecture 22: Chain Rule, Optimization / 2nd Derivative Test:

https://youtu.be/kNqwNfczw74

Plan for the day: 14.6 – 14.7

- Chain Rule
- Optimization / Second Derivative Test

Homework due at the start of class 23: Extra credit if you do 14.6.31 – we will not do implicit differentiation in the class (we will do more applications instead).

5.6. **14.6: Multivariable Calculus Chain Rules – Problems.** #1: Exercise 14.6.8: Use the Chain Rule to calculate $\partial f/\partial u$ for $f(x,y) = x^2 + y^2$, $x = e^{u+v}$, $y = u + v$. #2: Exercise 14.6.12: Use the Chain Rule to evaluate $\partial f/\partial s$ at $(r,s) = (1,0)$, where $f(x,y) = \ln(xy)$, $x = 3r + 2s$, and $y = 5r + 3s$. #3: Exercise 14.6.31: Use implicit differentiation to calculate $\partial z/\partial y$ for $e^{xy} + \sin(xz) + y = 0$.

5.7. **14.7: Optimization in Several Variables – Problems.** #1: Exercise 14.7.12: Find the critical points of $f(x,y) = x^3 + y^4 - 6x - 2y^2$, then apply the Second Derivative Test. #2: Exercise 14.7.17: Find the critical points of $f(x,y) = \sin(x+y) - \cos x$, then apply the Second Derivative Test. #3: Exercise 14.7.24: Show that $f(x,y) = x^2$ has infinitely many critical points (as a function of two variables) and that the Second Derivative Test fails for all of them. What is the minimum value of f? Does $f(x,y)$ have an local maxima?

Chain Rule for Paths

$g(t) = f(r(t))$

$g'(t) = \frac{dg}{dt}$

If f and $\mathbf{r}(t)$ are differentiable, then

$$\boxed{\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)}$$

In the cases of two and three variables, this chain rule states:

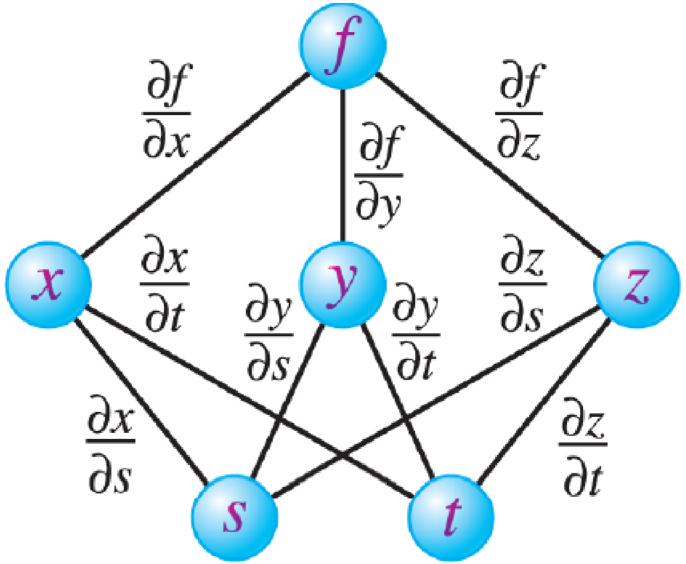
$$\frac{d}{dt} f(\mathbf{r}(t)) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(t), y'(t) \rangle = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

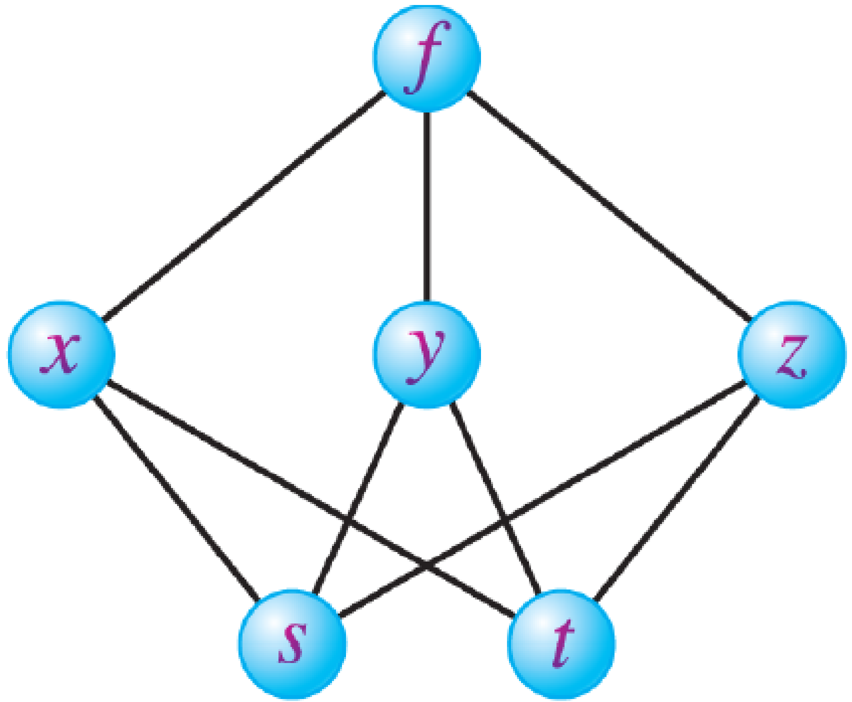
$$\frac{d}{dt} f(\mathbf{r}(t)) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \langle x'(t), y'(t), z'(t) \rangle = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

The Chain Rule expresses the derivatives of f with respect to the independent variables. For example, the partial derivatives of $f(x(s,t), y(s,t), z(s,t))$ are

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$





Rogawski et al., *Multivariable Calculus*, 4e, © 2019 W. H. Freeman and Company

FIGURE 1 Keeping track of the relationships between the variables.

THEOREM 2

General Version of the Chain Rule

Let $f(x_1, \ldots, x_n)$ be a differentiable function of n variables. Suppose that each of the variables $x_1, \ldots, x_n$ is a differentiable function of m independent variables $t_1, \ldots, t_m$. Then, for $k = 1, \ldots, m$,

$$\frac{\partial f}{\partial t_k} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_k} \tag{4}$$

$x = r\cos\theta \qquad x(r,\theta) = r\cos\theta \qquad \frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta \qquad y(r,\theta) = r\sin\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

$f(r,\theta) = f(x(r,\theta), y(r,\theta))$

$f(x,y) = x^3 + xy - y^2$

Find $\frac{\partial}{\partial \theta} f(x(r,\theta), y(r,\theta))$

option 1: $f(x(r,\theta), y(r,\theta)) = r^3\cos^3\theta + r^2\overbrace{\cos\theta\sin\theta}^{\frac{1}{2}\sin 2\theta} - r^2\sin^2\theta$

$\frac{\partial}{\partial \theta}\cos^3\theta = -3\cos^2\theta\sin\theta$

$\neq \cos 3\theta \quad (\theta = 0)$

$\frac{\partial}{\partial \theta}(\sin^2\theta) = \sin 2\theta$

$\frac{\partial}{\partial \theta}\sin^2\theta = 2\sin\theta\cos\theta = \sin 2\theta$

$\frac{\partial f}{\partial x} = 3x^2 + y$

$\frac{\partial f}{\partial x}(x(r,\theta), y(r,\theta)) = 3r^2\cos^2\theta + r\sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta \implies \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} = -(3r^2\cos^2\theta + r\sin\theta)$

$* r\sin\theta$

Local Extreme Values

A function $f(x,y)$ has a **local extremum** at $P=(a,b)$ if there exists an open disk $D(P,r)$ such that

- **Local maximum:** $f(x,y) \leq f(a,b)$ for all $(x,y) \in D(P,r)$
- **Local minimum:** $f(x,y) \geq f(a,b)$ for all $(x,y) \in D(P,r)$

local max at (1, 2)

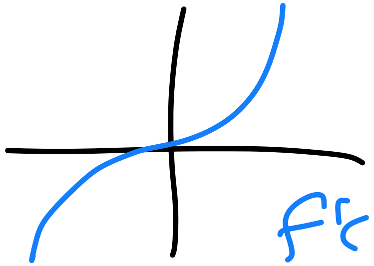
$f(x,y) = -(x-1)^2 - (y-2)^2$

Critical Point

A point $P=(a,b)$ in the domain of $f(x,y)$ is called a **critical point** if:

- $f_x(a,b)=0$ or $f_x(a,b)$ does not exist, and
- $f_y(a,b)=0$ or $f_y(a,b)$ does not exist.

$f(x) = x^3$ has no max or min at $x=0$



$f'(x) = 3x^2$, $f'(0)=0$

$f''(x) = 6x$, $f''(0)=0$

As in the one-variable case, there is a Second Derivative Test determining the type of a critical point (a, b) of a function $f(x, y)$ in two variables. This test relies on the sign of the **discriminant** $D = D(a, b)$, defined as follows:

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - f_{xy}^2(a, b)$$

$$\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

Assume all derivs continuous

so $f_{xy} = f_{yx}$

Second Derivative Test for $f(x,y)$

$D = f_{xx} f_{yy} - f_{xy}^2$

Let $P=(a,b)$ be a critical point of $f(x,y)$. Assume that f_{xx}, f_{yy}, f_{xy} are continuous near P. Then

$D>0 \qquad f_{xx}(a,b)>0, \qquad f(a,b)$

---

i. If and then is a local minimum.
ii. If $D>0$ and $f_{xx}(a,b)<0,$ then $f(a,b)$ is a local maximum.
iii. If $D<0,$ then f has a saddle point at (a,b).
iv. If $D=0,$ the test is inconclusive.

If $D>0$, then $f_{xx}(a,b)$ and $f_{yy}(a,b)$ must have the same sign, so the sign of $f_{yy}(a,b)$ also determines whether $f(a,b)$ is a local minimum or a local maximum in the $D>0$ case.

Assume critical point (wlog) at $x=0$

Wlog assume $f(0)=0$

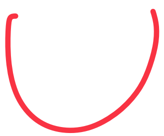
Taylor Series: $f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots$

So $f(x) = f''(0)\frac{x^2}{2!} + \dots$

if x is small, $|x^3| \lll |x|^2$

So $f(x) \approx f''(0)\, x^2/2!$

$A(x) = x^2$



$A'(x) = 2x$

$A''(x) = 2$

$A''(0) = 2 > 0$

$B(x) = -x^2$

$B'(x) = -2x$

$B''(x) = -2$

$B''(0) = -2 < 0$

Multivariable Taylor

Take $(x,y) = (0,0)$ and $f(0,0) = 0$ (wlog)

Gradient is $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ and $(\nabla f)(0,0) = (0,0)$ at critical point

$$f(x,y) = f(0,0) + (\nabla f)(0,0) \cdot (x,y) + \underbrace{\frac{1}{2}(x \;\; y)(Hf)(0,0)\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{Gives a number}}$$

$$(Hf)(0,0) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}_{(0,0)}$$

Hessian

$$\frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$f_{xx} f_{yy} - f_{xy}^2$

determinant!

$$= \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} f_{xx}\, x + f_{xy}\, y \\ f_{xy}\, x + f_{yy}\, y \end{pmatrix}$$

$$= \frac{1}{2}\left[f_{xx} \cdot x^2 + f_{xy}\, xy + f_{xy}\, xy + f_{yy}\, y^2 \right]$$

$$= \frac{f_{xx}}{2} x^2 + f_{xy} \cdot xy + \frac{f_{yy}}{2} y^2$$

Two derivs wrt x, get f_{xx}

Two " " y, get f_{xy}

One deriv wrt x, one wrt y, get f_{xy}

Math 150: Multivariable Calculus: Spring 2023:
Lecture 23: Lagrange Multipliers: https://youtu.be/omW5MRL_zVw

Plan for the day: 14.8

- Lagrange Multipliers

Homework due at the start of class 24:

5.8. **14.8: Lagrange Multipliers: Optimizing with a Constraint – Problems.** #1: Exercise 14.8.10: Find the minimum and maximum vales of $f(x, y) = x^2y^4$, subject to the constraint $x^2 + 2y^2 = 6$. #2: Exercise 14.8.15: Find the minimum and maximum vales of $f(x, y) = xy + xz$, subject to the constraint $x^2 + y^2 + z^2 = 4$. #3: Exercise 14.8.18: Find the rectangular box of maximum volume if the sum of the lengths of the edges is 300 cm.

Optimization in Several Variables

(1) Interior: look for $\nabla f = \vec{0} = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$

(2) boundary: Surface $g(x_1, \dots, x_n) = c$

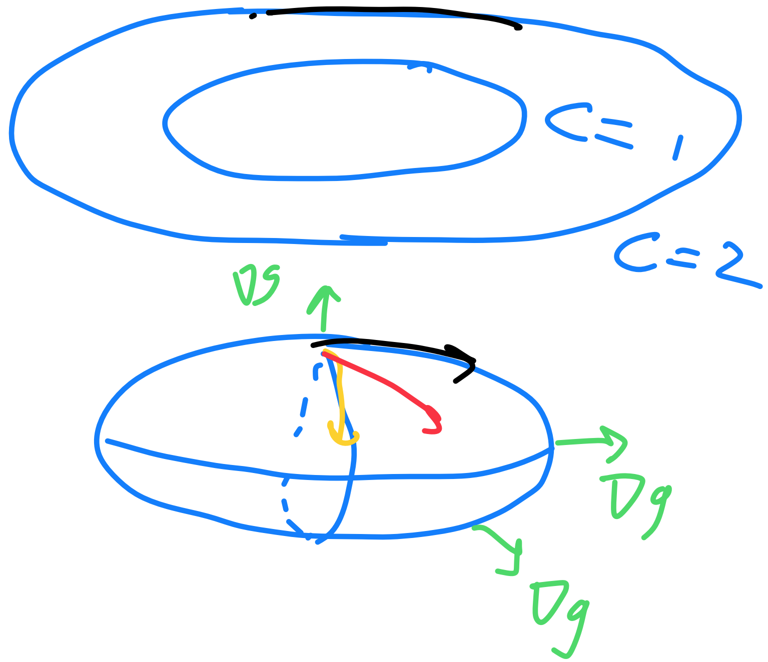
Then must have a λ st $\nabla f = \lambda \nabla g$

at a candidate

$$\left.\begin{aligned} \partial f/\partial x_1 &= \lambda\, \partial g/\partial x_1 \\ &\vdots \\ \partial f/\partial x_n &= \lambda\, \partial g/\partial x_n \\ g(x_1, \dots, x_n) &= c \end{aligned}\right\} \begin{array}{l} n+1 \text{ eqs} \\ n+1 \text{ variables} \end{array}$$

$g(x_1, \ldots, x_n) = c$ Level set of g of height c

Ex: $g(x,y) = x^2 + 4y^2 = c > 0$



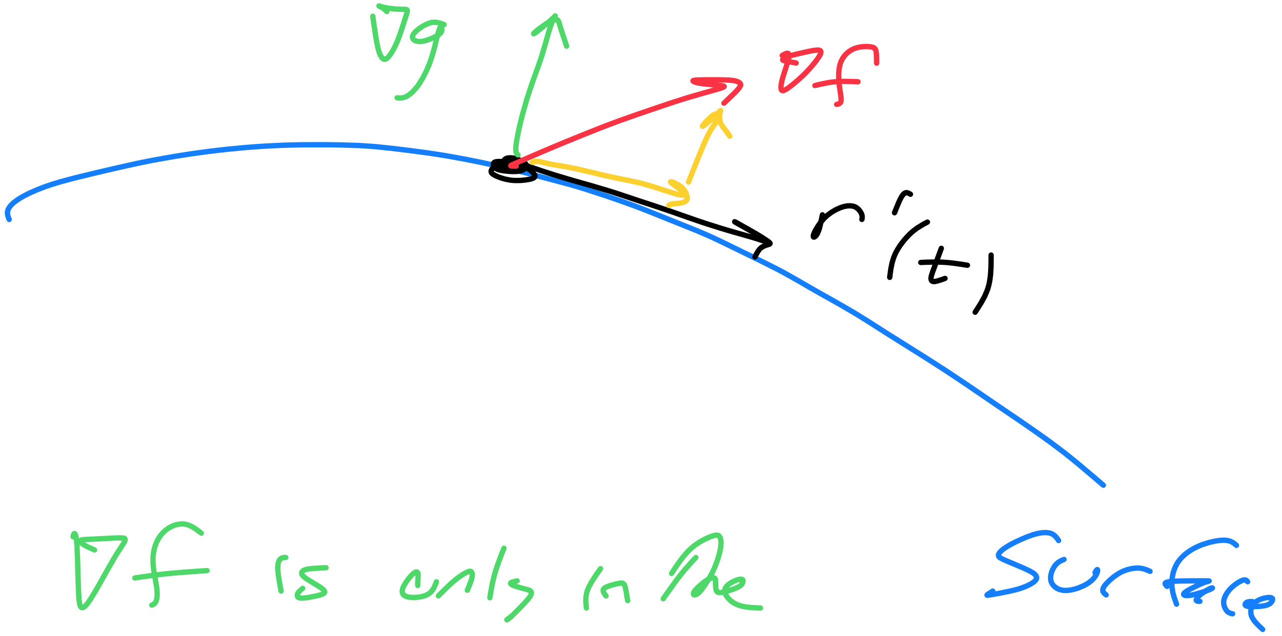
$g(x,y,z) = x^2 + 4y^2 + 9z^2 = c$

Curve $r(t)$ on the ellipsoid of height c

$A(t) = g(r(t)) = c$

$A'(t) = (\nabla g)(r(t)) \cdot r'(t) = 0$

at every t, $(\nabla g)(r(t))$ is $\perp$ to $r'(t)$, the tangent to the curve $r(t)$ at t, ∇g is normal dir



If move on surface, change is a directional derivative

$$D_{r'(t)} f = (\nabla f)(r(t)) \cdot r'(t)$$

(recall $(\nabla g)(r(t)) \cdot r'(t) = 0$)

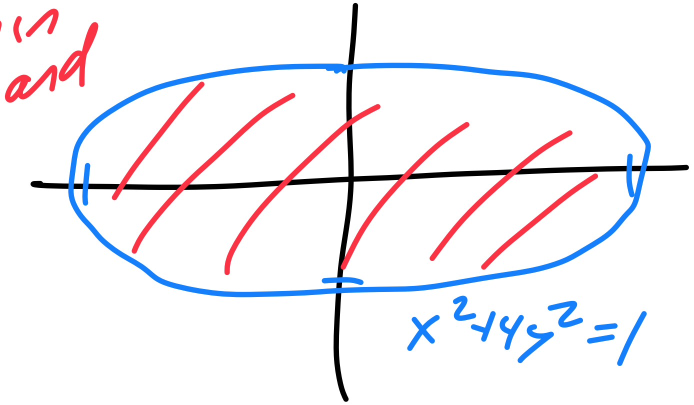
If ∇f is only in the dir of ∇g, then all dir deriv of f as stay on the surface are zero, so have candidates for max/min.

If ∇f has something $\perp$ to ∇g, move in that dir for max $\nearrow$ and opposite dir for max $\downarrow$.

Ex: $f(x,y) = 4x^4 + 2y^4$

$g(x,y) = x^2 + 4y^2 = 1$

Find max/min of f on and inside $g(x,y) \leq 1$



Interior: Need $\nabla f = \vec{0}$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 16x^3, 8y^3 \rangle$$

If $(0,0)$ need $(x,y) = (0,0)$: MIN

Boundary $\nabla f = \lambda \nabla g$ and $g(x,y)=1$, $\nabla g = \langle 2x, 8y \rangle$

$f_x = \lambda g_x$ $\qquad 16x^3 = \lambda 2x$

$f_y = \lambda g_y$ $\qquad 8y^3 = \lambda 8y$

$g(x,y) = 1$ $\qquad x^2 + 4y^2 = 1$

$16x^3 = \lambda 2x$

$8y^3 = \lambda 8y$

$x^2 + 4y^2 = 1$

Case 1: $x=0$

Get $4y^2 = 1$

so $y = \pm 1/2$

Case 2: $y=0$

$x^2 = 1$

so $x = \pm 1$

Case 3: $x, y \neq 0$

Divide 1st by 2nd equation: $\frac{16x^3}{8y^3} = \frac{\lambda 2x}{\lambda 8y}$ or $2\left(\frac{x}{y}\right)^3 = \frac{1}{4}\left(\frac{x}{y}\right)$

As $x/y \neq 0$ get $(x/y)^2 = \frac{1}{8}$ or $x^2 = \frac{1}{8}y^2$ so $x = \pm \frac{\sqrt{2}}{4} y$

Now use $x^2 + 4y^2 = 1$ so $\frac{1}{8}y^2 + 4y^2 = 1$ or $\frac{33}{8}y^2 = 1$

$y^2 = 33/8$ so $y = \pm\sqrt{\frac{33}{8}}$ and $x = \pm \frac{\sqrt{66}}{4\sqrt{8}}$

Candidates: $(\pm 1, 0)$, $(0, \pm\frac{1}{2})$, $\left(\pm\frac{\sqrt{66}}{4\sqrt{8}}, \pm\sqrt{\frac{33}{8}}\right)$ have 8 points

Check all! Really only 3....

$g(x,y) = x^2 + 4y^2 = 1 \qquad f(x,y) = 4x^4 + 2y^4$

Reduce to 1-dim

$x = 2\cos\theta \qquad y = \sin\theta$

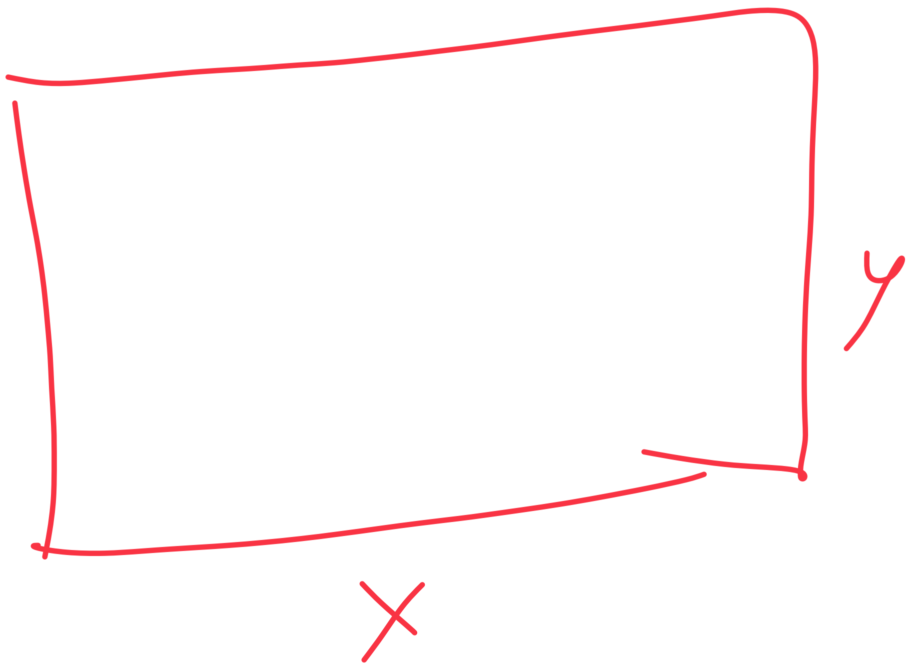
$x^2 + 4y^2 = 4\cos^2\theta + 4\sin^2\theta = 1$

$f(x(\theta), y(\theta)) = 4(2\cos\theta)^4 + 2(\sin\theta)^4 =: f(\theta)$

Find $f'(\theta) = 0$

Solve using Calc I!

Farmer Brown



100 meters of fence

Max rectangular area

$P = 2x + 2y = 100$

$A = xy$

$x + y = 50$ so $y = 50 - x$ so $\text{Area}(x) = x(50 - x)$

Theory: $50x - x^2$

Simplify, Simplify

boundary $x = 0, 50$: min

critical points: $50 - 2x = 0$

Get $x = 25$ square

Math 150: Multivariable Calculus: Spring 2023:
Lecture 24: Lagrange Multipliers II, Derivatives: https://youtu.be/-QEyiSaZQZo

Plan for the day:

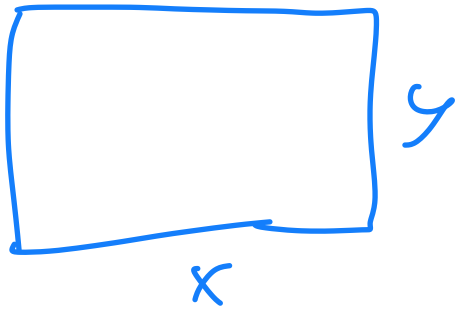
- Lagrange Multipliers
- Rules for Derivatives

Monday: Class 25: Sabermetrics lecture, prospectives visiting.

Midterm II: Class 26: Wednesday (can show up at 8am if wish)

Farmer Brown

100 meters of fence, maximize rectangular area



$2x+2y=100$ $\qquad$ $y=50-x$

$\max xy$ $\qquad$ $\max x(50-x) = 50x - x^2$

$g(x,y) := x+y = 50$ $\qquad$ $f(x,y) = xy$

$\nabla f = \lambda \nabla g$ $\qquad$ $y = \lambda 1$

$g(x,y) = 50$ $\qquad$ $x = \lambda 1$ $\qquad$ $x = y$

Note: $\nabla f = \langle y, x \rangle$ $\qquad$ $x+y=50$

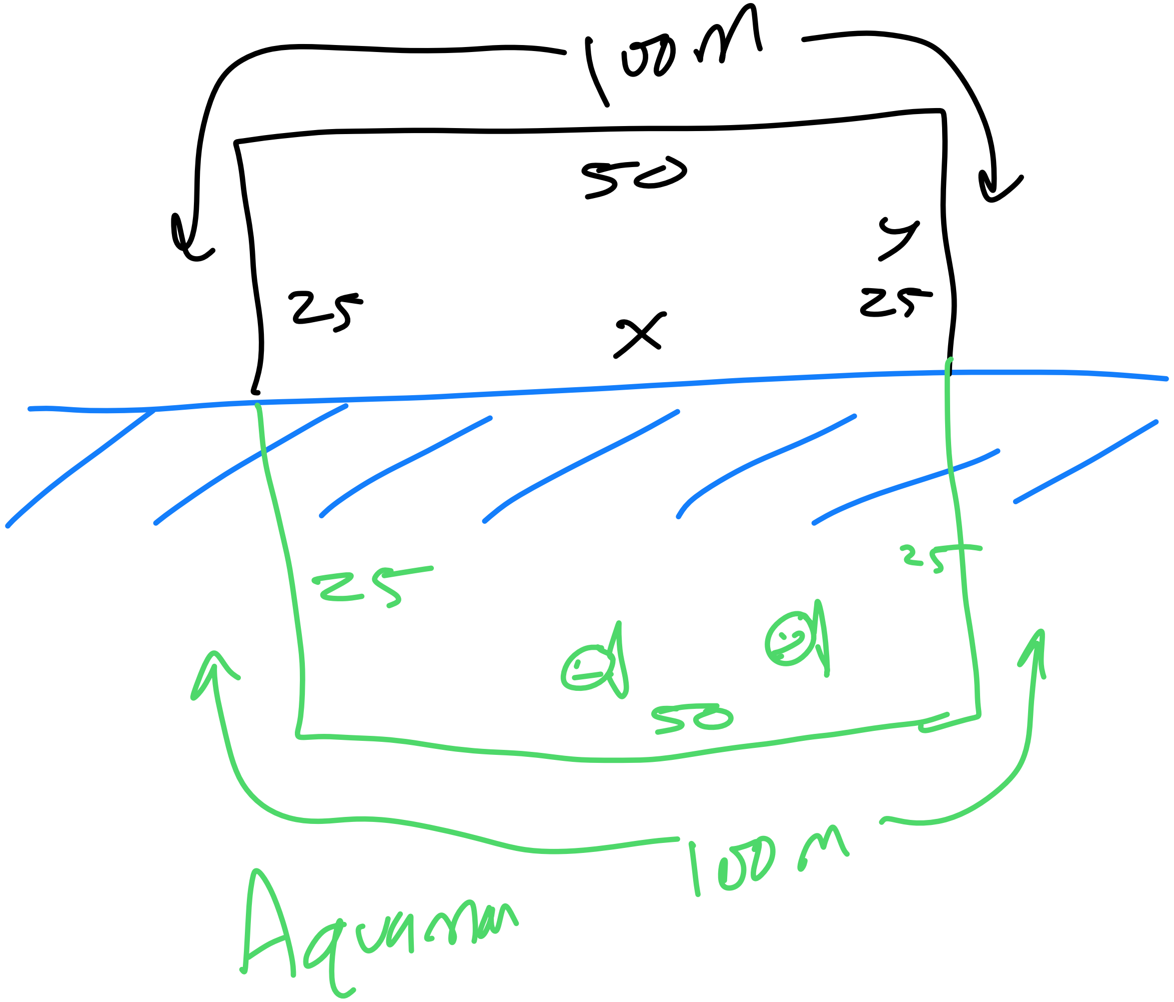
$\nabla g = \langle 1, 1 \rangle$

$2x = 50$

$x = 25$

$y = 25$

Farmer Tim



100 meters of fencing

only need 3 sides

Maximum area?

$g(x,y) := x + 2y = 100$

$f(x,y) = xy$

$\nabla g = \langle 1, 2 \rangle$

$\nabla f = \langle y, x \rangle$

$\nabla f = \lambda \nabla g$

$g(x,y) = 100$

$y = \lambda \cdot 1$

$x = \lambda \cdot 2$

$x + 2y = 100$

$2\lambda + 2\lambda = 100$

so $\lambda = 25$

thus $x = 50$

$y = 25$

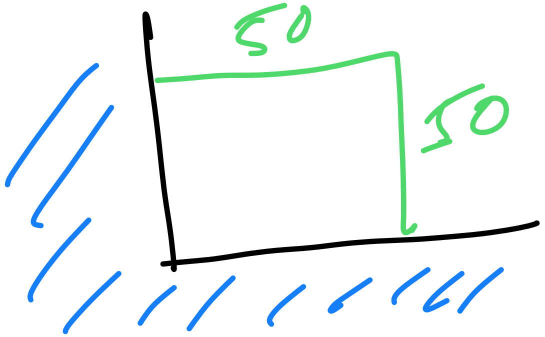
Same as Farmer Brown with 200 m of fence, so must be 50×50

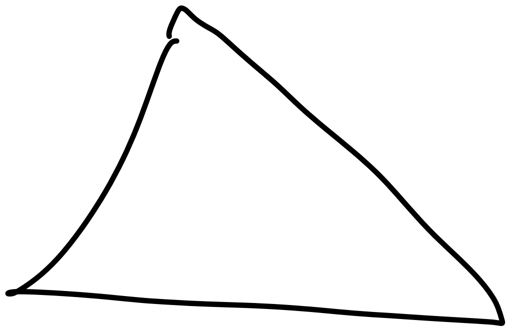
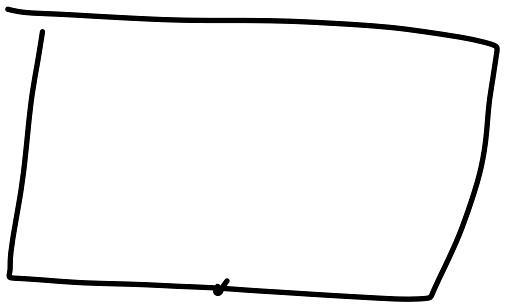
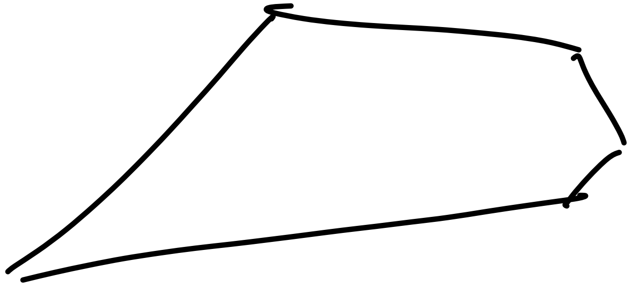
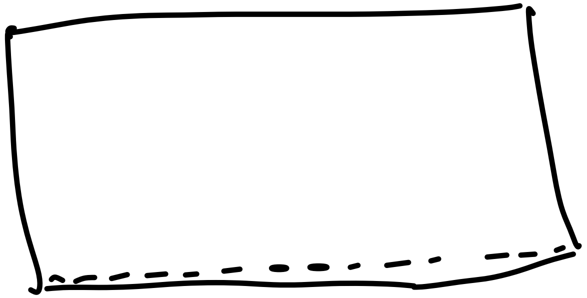
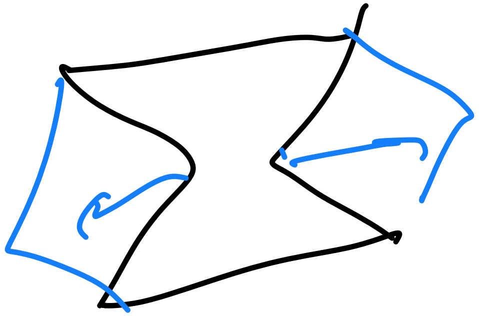
1, 2, ..., 9

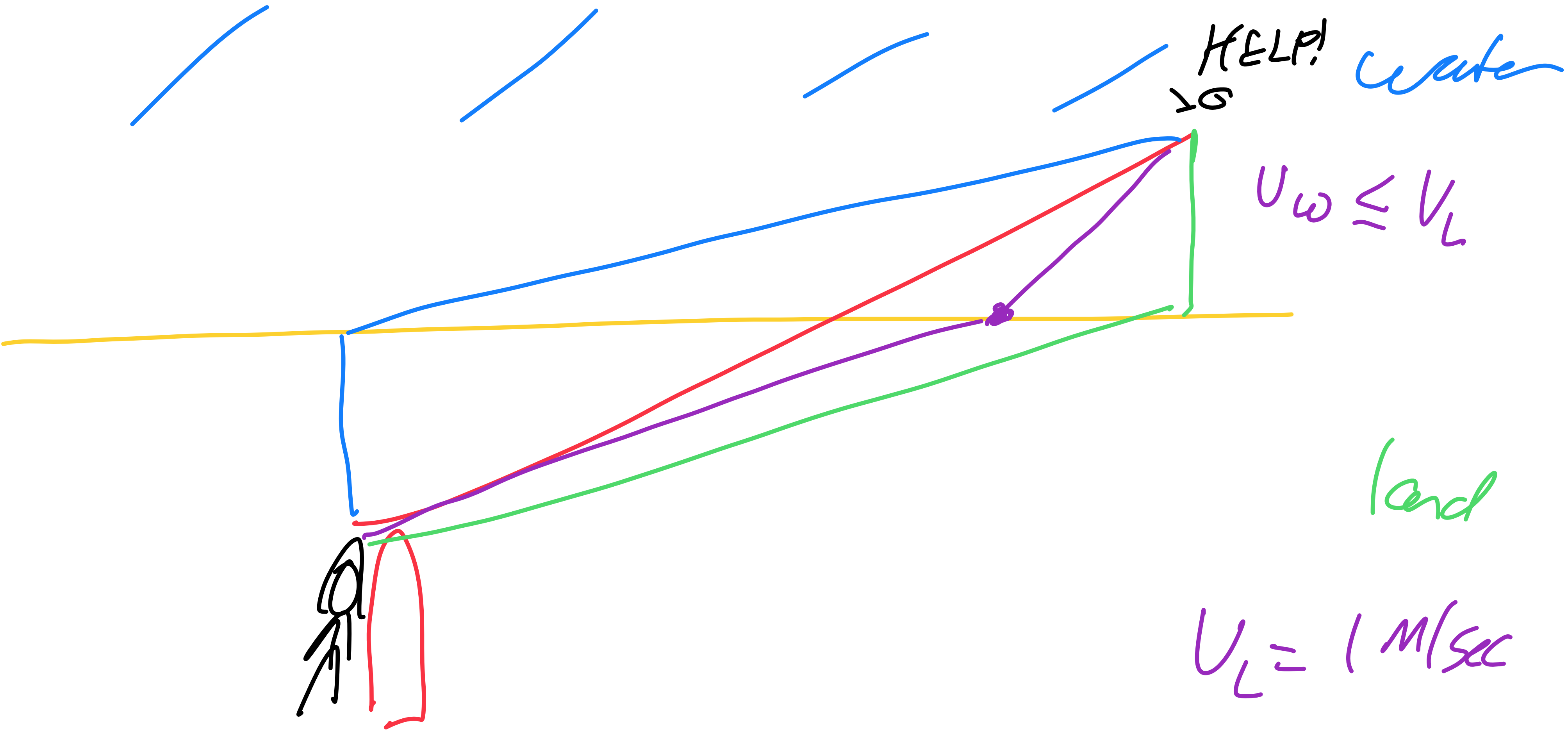
$$\frac{\boxed{3}}{\boxed{8}\,\boxed{7}} + \frac{\boxed{9}}{\boxed{2}\,\boxed{4}} + \frac{\boxed{1}}{\boxed{5}\,\boxed{6}} \stackrel{?}{=} 1$$

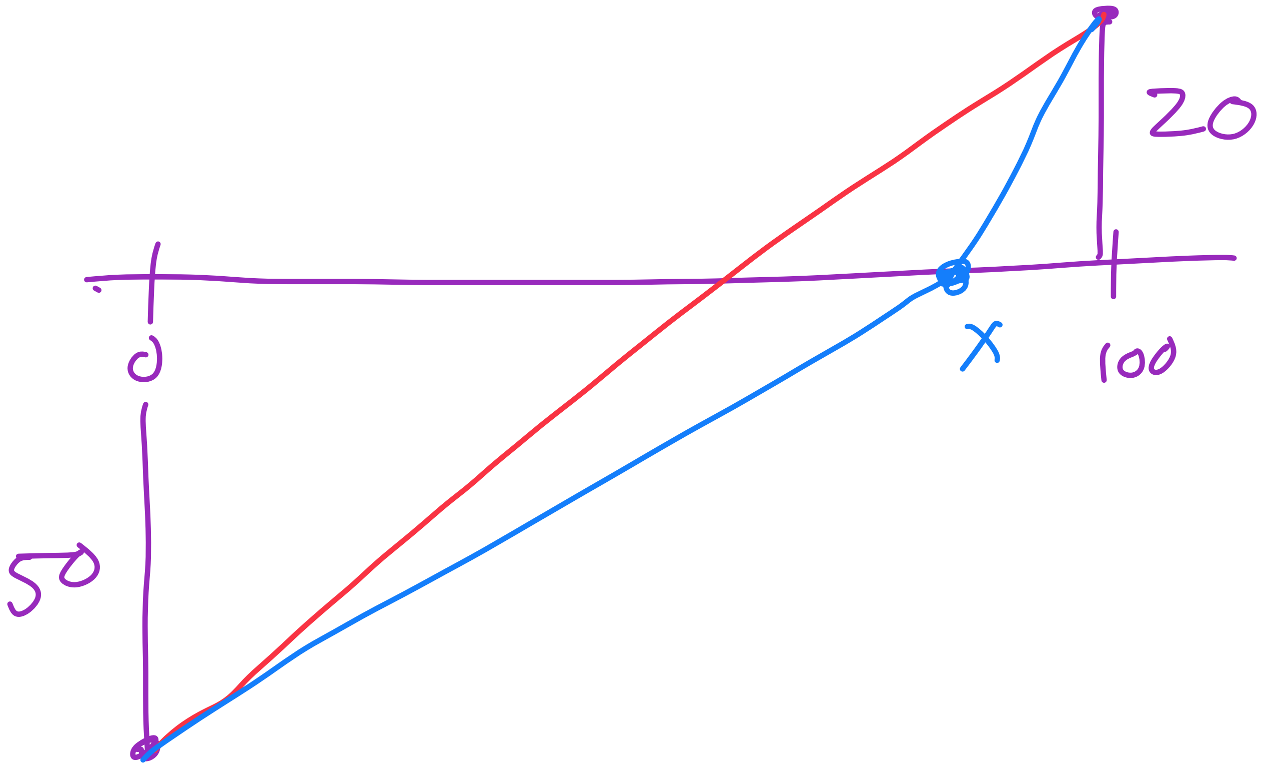
possibilities: $9! \approx 360{,}000$ or $330{,}000$

Reduce by $3! = 6$




HELP!
water
$U_w \leq V_L$
land
$U_L = 1$ m/sec



$0 \le x \le 100$

time on land

= land dist / land speed

$= \sqrt{x^2 + 50^2} / 1$

time in water

= water dist / water speed

$= \sqrt{(100-x)^2 + 20^2} / V_w$

$$Time(x) = \sqrt{x^2+50^2} + \sqrt{(100-x)^2+20^2}/V_w$$

Solve $Time'(x) = 0$, Compare with $Time(0)$, $Time(100)$

Math 150: Multivariable Calculus: Spring 2023:
Lecture 25: Sabermetrics; Lecture 26: Midterm II
Lecture 27: Fundamental Theorem of Calculus: https://youtu.be/IQj0lHPx3-4

Plan for the day:

- Need inputs (IVT, MVT)
- Proof of Fundamental Theorem of Calculus in 1 Variable

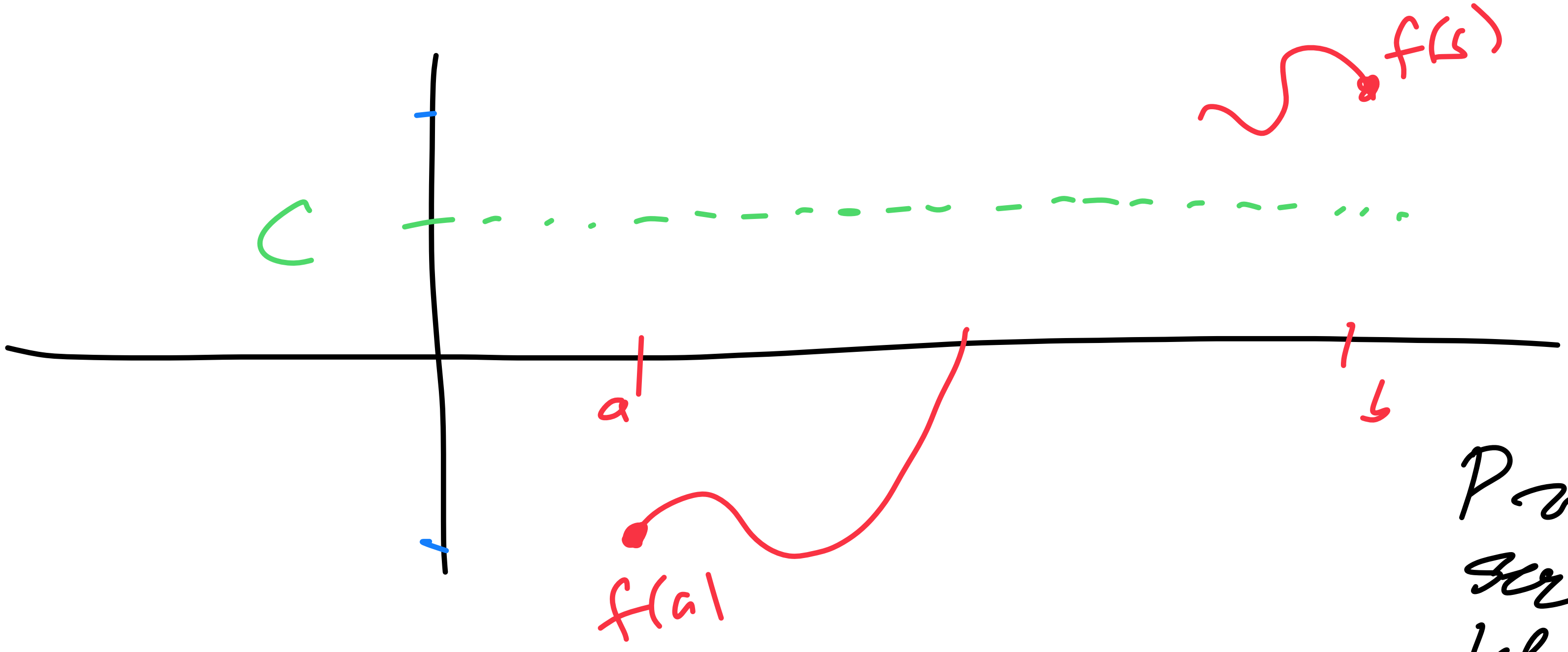
Monday: Class 25: Sabermetrics lecture, prospectives visiting.
Slides:
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/PythagWLTalk_DeveloperCloud85_2017.pdf
Video: https://youtu.be/reUdQ0NPbPY
Paper:
https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/MillerEtAl_Pythagoras.pdf

Midterm II: Class 26: Wednesday (can show up at 8am if wish)

Intermediate Value Thm (IVT)

If f is cont on $[a,b]$ then if C is b/w $f(a)$ and $f(b)$

There is a $c \in [a,b]$ with $f(c) = C$



Proof: look at a seq of midpoints so left is $\leq C$, right is $\geq C$, keep subdividing/converge to common value

The Mean Value Thm (MVT)

If f is cont and diff on $[a,b]$ there is a $c \in [a,b]$

such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ or $f(b)-f(a) = f'(c)(b-a)$

$f(b) = f(a) + f'(c)(b-a)$

i.e., at some time the instantaneous speed = average speed.

Proof: ave speed is 70.

Case 1: always travel < 70: Contradiction

Case 2: always travel > 70: Contradiction

Case 3: either at some time 70 or at some point < 70 and another point > 70

$\hookrightarrow$ by IVT hit 70 at some time

Bounded

Say f is bounded by B if $|f(x)| \leq B$

Ex: $f(x) = x^2 + 3 + 3x + e^{x\cos(2x^3)} - 1701$

Say $x \in [-1, 10]$

$$|f(x)| \leq |x|^2 + 3 + 3|x| + e^{|x\cos(2x^3)|} + 1701$$

$$\leq 100 + 3 + 30 + e^{10} + 1701$$

$$\leq 10^{1000}$$

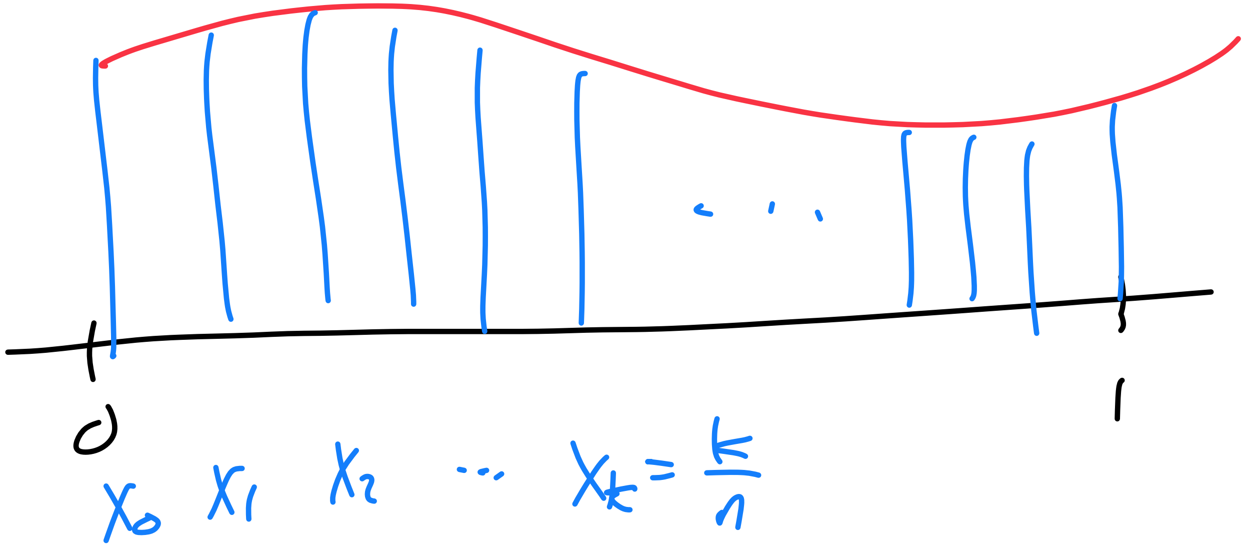
Fund Thm of Calc

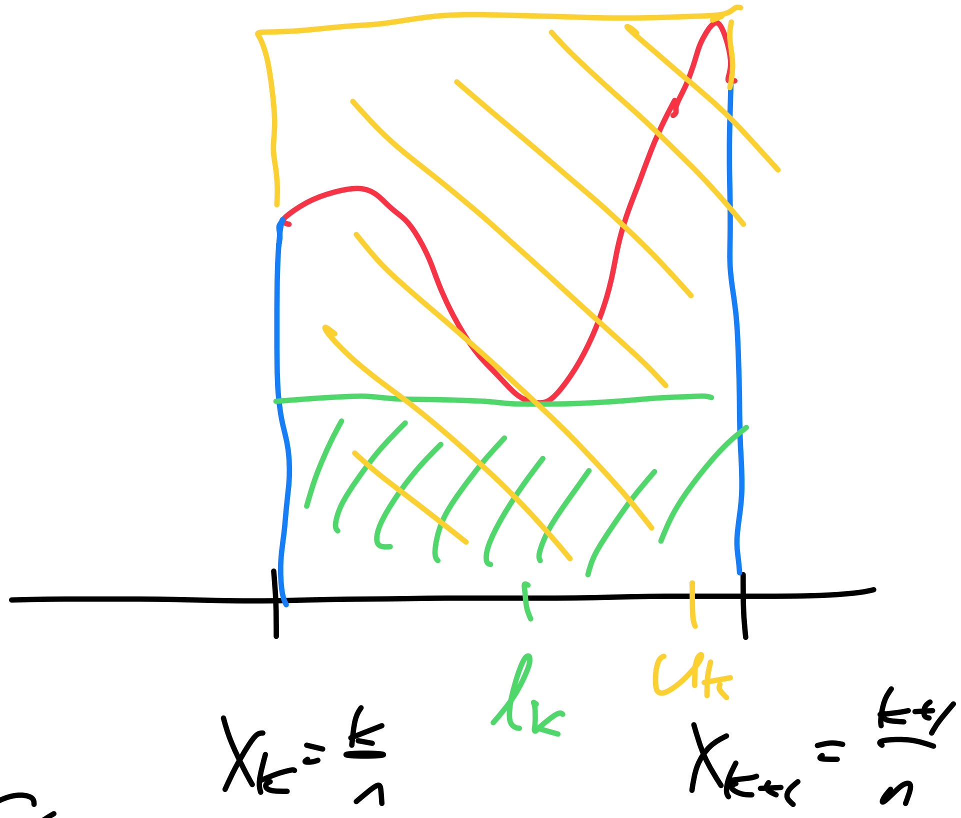
Let f be a cont and diff function on a finite interval $[a,b]$ with $|f'(x)| \leq B$ for some B. Let $F' = f$.

Then the area under the curve $y = f(x)$ from $x=a$ to $x=b$ is $F(b) - F(a)$, and denote this by $\underbrace{\int_a^b f(x)\,dx}_{\text{area}}$.

$$\text{Area} = \int_a^b f(x)\,dx = F(b) - F(a)$$

Rieman Sums





on $[x_k, x_{k+1}]$ have $f(l_k) \le f(x) \le f(u_k)$

$$f(l_k)\,\frac{1}{n} \le \int_{x_k}^{x_{k+1}} f(x)\,dx \le f(u_k)\,\frac{1}{n}$$

area under f
in this interval

Sum over all pieces:

$$\underbrace{L(n)}_{\text{lower sum with } n \text{ pieces}} := \sum_{k=0}^{n-1} f(l_k)\frac{1}{n} \leq \text{Area} \leq \sum_{k=0}^{n-1} f(u_k)\frac{1}{n} =: \underbrace{U(n)}_{\text{upper sum}}$$

Show $L(n)$ and $U(n)$ converge to a common value

↳ if yes, has to be the area

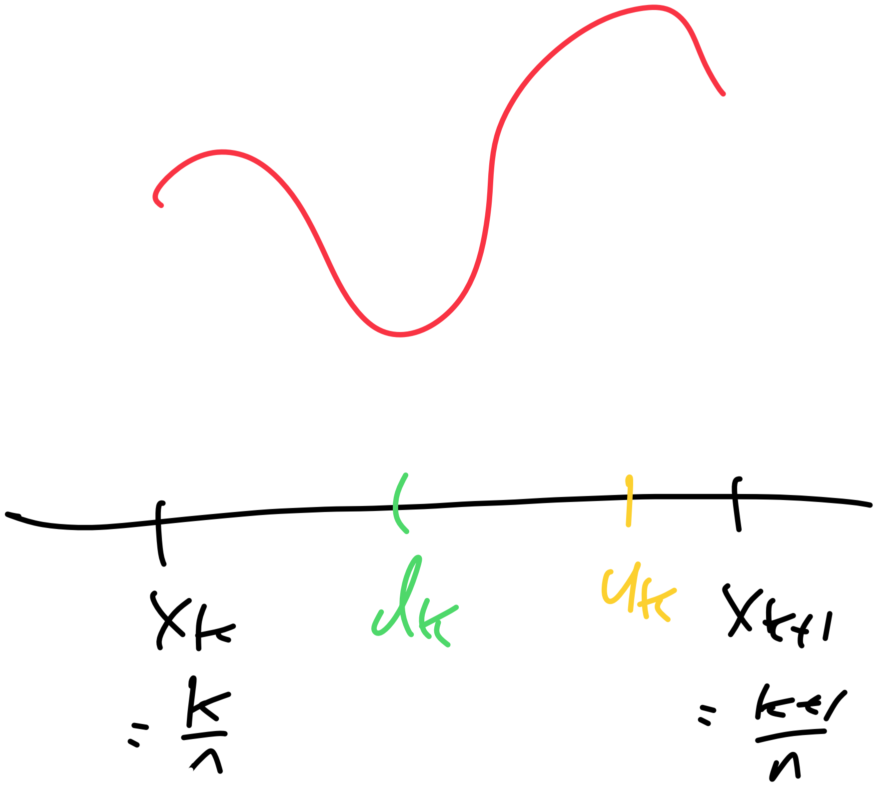
Study $U(n) - L(n) = \sum_{k=0}^{n-1} \underbrace{\left[f(u_k) - f(l_k)\right]}_{\text{show this goes to } 0} \frac{1}{n}$

Study $f(u_k) - f(l_k) = f'(c_k)\,(u_k - l_k)$ by MVT

$$|f(u_k) - f(l_k)| = \underbrace{|f'(c_k)|}_{\text{by Assumption this is at most } B} \; \underbrace{|u_k - l_k|}_{\text{at most } \frac{1}{n}}$$

by Assumption this is at most B

at most $\frac{1}{n}$ as each in interval from $x_k = \frac{k}{n}$ to $x_{k+1} = \frac{k+1}{n}$



Substitute:

$$U(n) - L(n) \le \sum_{k=0}^{n-1} \left(B \cdot \frac{1}{n}\right) \frac{1}{n}$$

$$= \frac{B}{n^2} \sum_{k=0}^{n-1} 1 = \frac{Bn}{n^2} = \frac{B}{n} \xrightarrow[n\to\infty]{} 0$$

$\blacksquare$

Consider:

$$\begin{array}{r}
\cancel{14} - 1 \\
+ \cancel{17} - \cancel{14} \\
+ \cancel{110} - \cancel{17} \\
+ \cancel{1701} - \cancel{110} \\
+ 24601 - \cancel{1701} \\
\hline
24600
\end{array}$$

telescoping sum

Consider:

$$\sum_{k=1}^{n} \left[F(x_k) - F(x_{k-1}) \right]$$

$$\Rightarrow \quad \begin{array}{l}
F(\cancel{x_1}) - F(x_0) \\
+ F(\cancel{x_2}) - F(\cancel{x_1}) \\
+ F(\cancel{x_3}) - F(\cancel{x_2}) \\
\vdots \\
+ F(x_n) - F(\cancel{x_{n-1}}) \\
\hline
F(x_n) - F(x_0)
\end{array}$$

Know: $$\sum_{k=1}^{n} \left[\underbrace{F(x_k) - F(x_{k-1})} \right] = F(x_n) - F(x_0)$$

MVT applied to F $$F(x_k) - F(x_{k-1}) = F'(m_k)(x_k - x_{k-1})$$

with m_k in $[x_{k-1}, x_k]$

so $$F(x_k) - F(x_{k-1}) = f(m_k)\,\frac{1}{n}$$

on the interval $[x_{k-1}, x_k]$ have $f(l_{k-1}) \le f(m_k) \le f(u_{k-1})$

Get $$L(n) \le \sum_{k=1}^{n} f(m_k)\,\frac{1}{n} \le U(n)$$

$$\underbrace{L(n) \leq \sum_{k=1}^{n} f(m_k)\frac{1}{n} \leq U(n)}$$

$$L(n) \leq \underbrace{F(x_n) - F(x_0)} \leq U(n)$$

$n \to \infty$ $\qquad$ $n \to \infty$

$x_n = \frac{n}{n} = 1$

$x_0 = \frac{0}{n} = 0$

AREA

Thus Area, denoted $\int_a^b f(x)\,dx$, is $F(1) - F(0)$ $\blacksquare$

Math 150: Multivariable Calculus: Spring 2023:
Lecture 28: Integration in Several Variables: https://youtu.be/gLEfgNRcKmA

Plan for the day:

- Integration in Several Variables
- Switching orders of integration (generalizing $f_{xy} = f_{yx}$).

11.1. **15.1: Integration in Two Variables – Problems.**

15. Evaluate the integral

$$\int\int_{\mathcal{R}} x^3 dA, \quad = \int_{y=0}^{5}\left[\int_{x=-4}^{4} x^3 dx\right] dy$$

or

$$= \int_{x=-4}^{4}\left[\int_{y=0}^{5} x^3 dy\right] dx$$

where $\mathcal{R} = [-4, 4] \times [0, 5]$.

27. Evaluate

$$\int_0^1 \left[\int_0^2 \left(x + 4y^3\right) dx\right] dy.$$

31. Evaluate

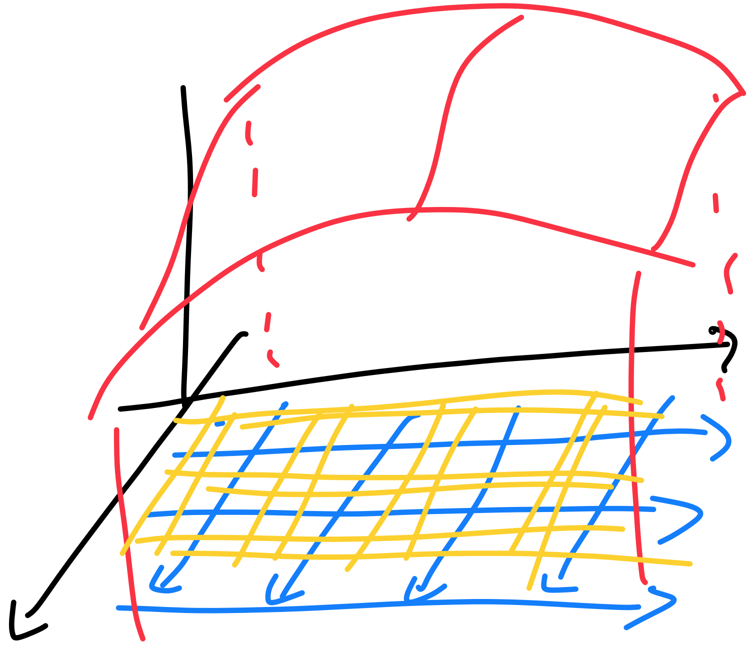
$$\int_1^2 \left[\int_2^4 e^{3x-y} dy\right] dx.$$

41. Evaluate

$$\int\int_{\mathcal{R}} e^x \sin y dA,$$

where $\mathcal{R} = [0, 2] \times [0, \frac{\pi}{4}]$.

Basics of Integrals in Several Variables



Find volume under the surface

- Rectangle in (x,y) plane:

$$\left.\begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix}\right\} \text{all finite}$$

- Riemann sums

look at max/min in each rectangle

sum, take limit, show converge to a common value, denote this by

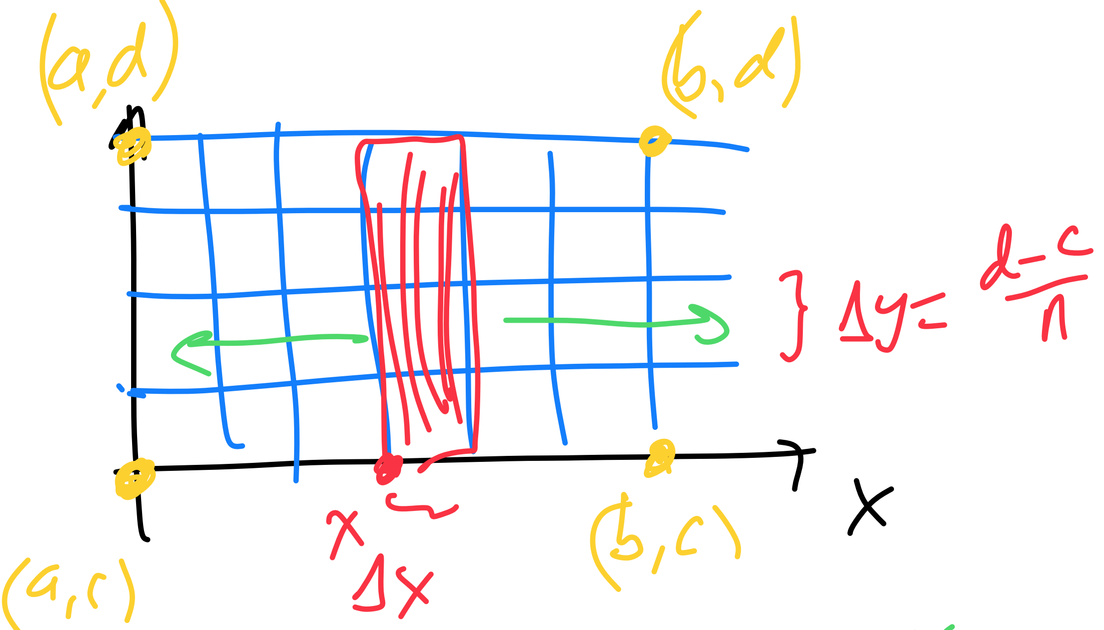
$$\iint_R f(x,y)\,dA$$

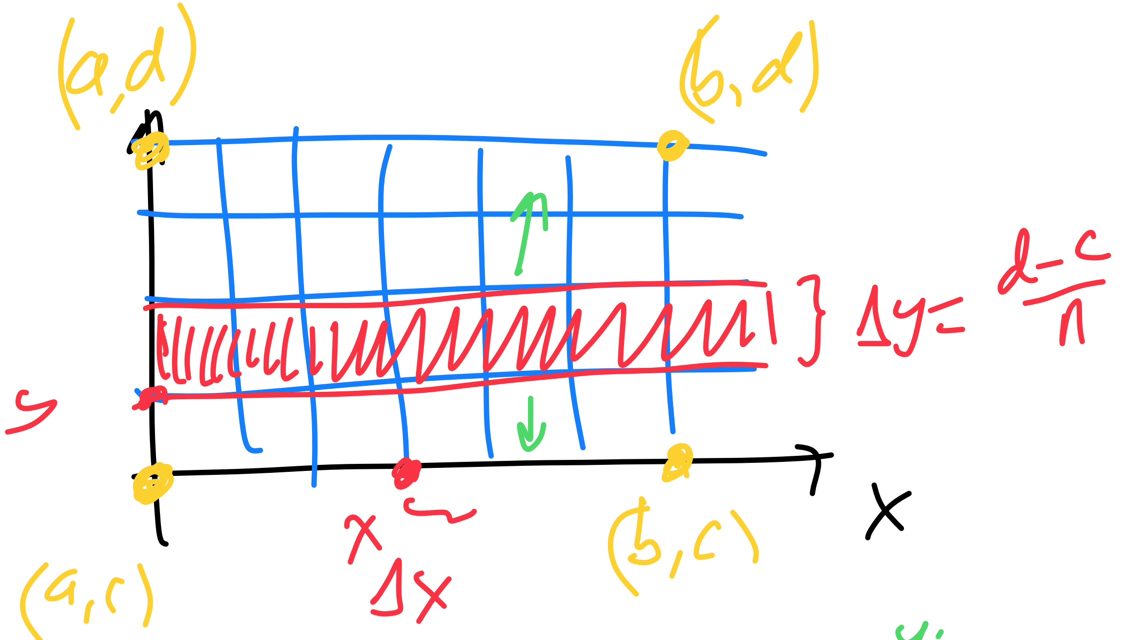
$$R = [a,b] \times [c,d]$$

$$= \{(x,y) : x \in [a,b],\ y \in [c,d]\}$$

Hope:

$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\left[\int_{y=c}^{d} f(x,y)\,dy\right]dx = \int_{y=c}^{d}\left[\int_{x=a}^{b} f(x,y)\,dx\right]dy$$





$\Delta x = \frac{b-a}{n}$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\int_{y=c}^{d} f(x_i,y)\,dy\,\Delta x$$

x lives between x and $x+\Delta x$

MVT

$f(x+\Delta x, y) = f(x,y) + \frac{\partial f}{\partial x}(x,y)\,\Delta x$

$\Delta x = \frac{b-a}{n}$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\int_{x=a}^{b} f(x,y_i)\,dx\,\Delta y$$

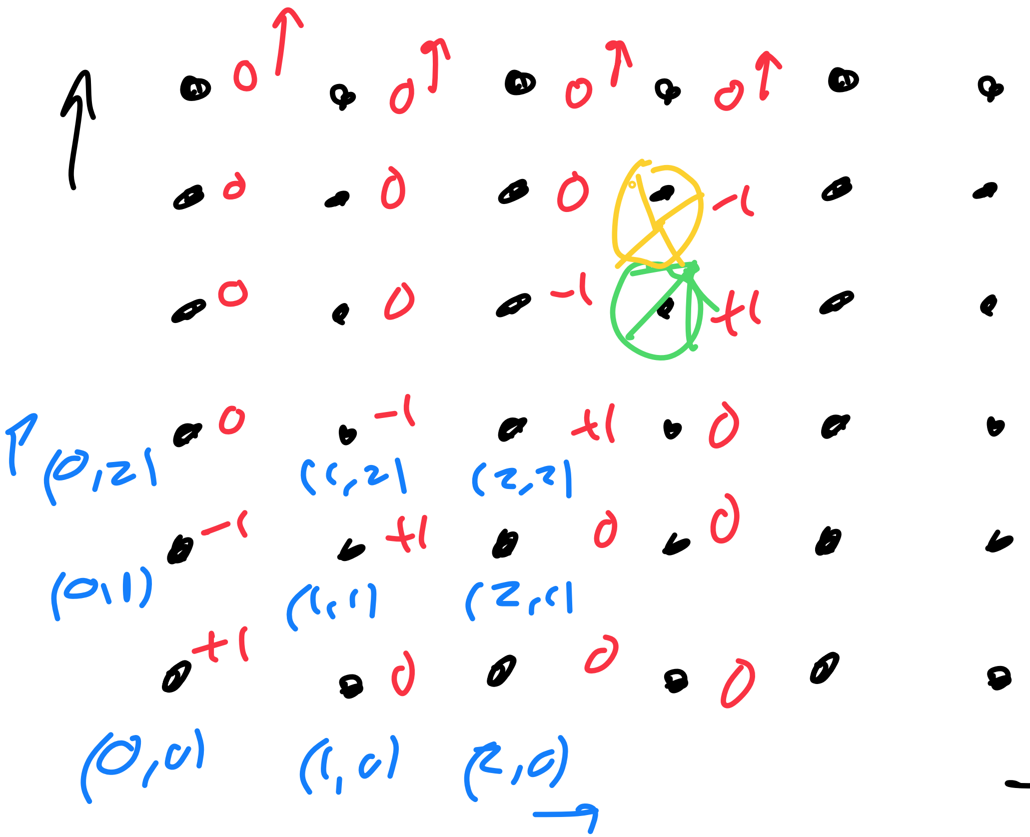
Rules of Integration

- $$\int \cdots \int_R c\, f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$
  $$= c \int \cdots \int_R f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

- $$\int \cdots \int_R \left[f(x_1, \ldots, x_n) + g(x_1, \ldots, x_n) \right] dx_1 \cdots dx_n$$
  $$= \int \cdots \int_R f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n + \int \cdots \int_R g(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

Why Order Can Matter

a_{mn}



a_{mn} : m is row
n is column

row fixed — column fixed

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{mn}\right) \quad \text{vs} \quad \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{mn}\right)$$

$$= \sum_{m=0}^{\infty} \begin{cases} 1 & \text{if } m=0 \\ 0 & \text{otherwise} \end{cases} = 1 \neq 0 = \sum_{n=0}^{\infty} 0$$

Issue: $\sum_m \sum_n |a_{mn}| = \infty$

Fubini's Thm

Assume f is a cont fn on a finite rectangle $[a,b]\times[c,d]=:R$.

Then $$\iint_R f\,dA = \int_{x=a}^{b}\left[\int_{y=c}^{d} f(x,y)\,dy\right]dx = \int_{y=c}^{d}\left[\int_{x=a}^{b} f(x,y)\,dx\right]dy$$

More generally ok if $\iint_R |f|\,dA < \infty$

Harmonic Series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$

Alternating Harmonic

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$ converges to $\ln(2)$ (I think)

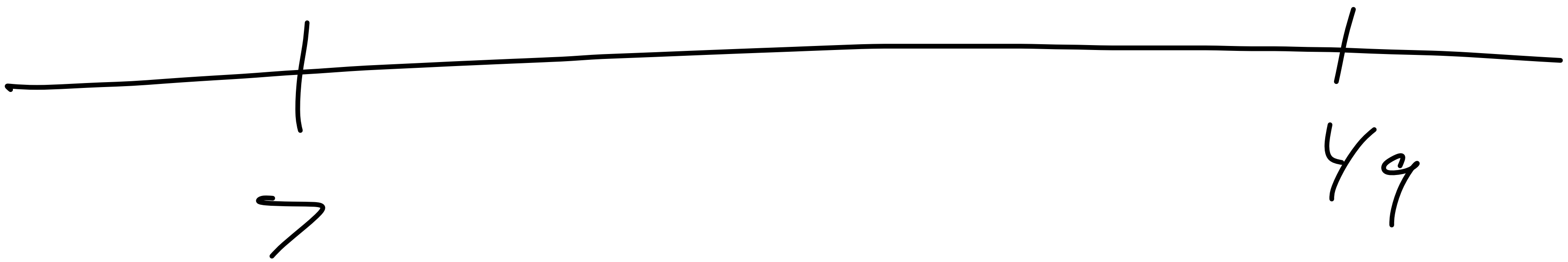
Reorder Alternating Harmonic

$1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$ $\quad -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, -\frac{1}{10}, \ldots$

$\frac{1}{n_1+2} + \frac{1}{n_1+4} + \cdots + \frac{1}{n_3}$

$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \cdots - \frac{1}{n_2}$

$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{n_1}$

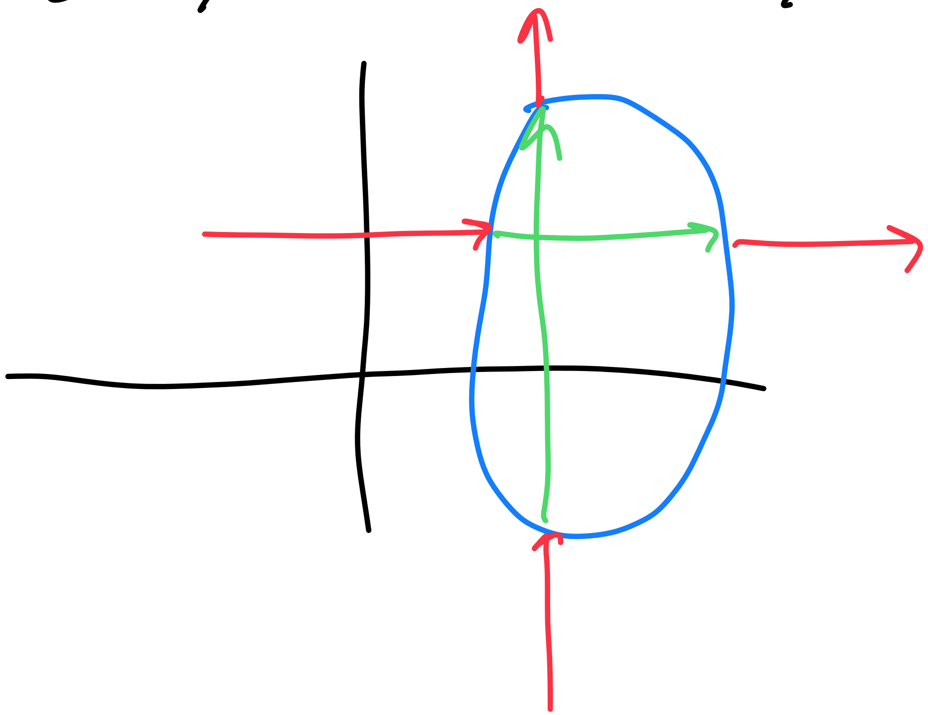


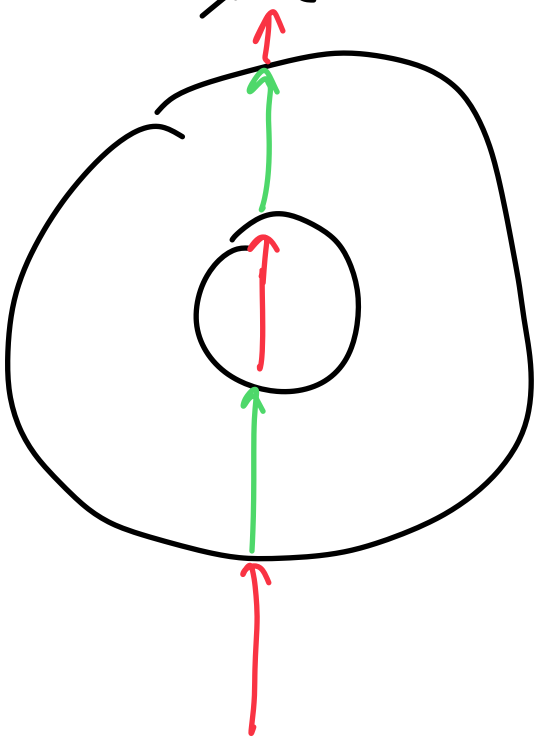
Simple Regions:

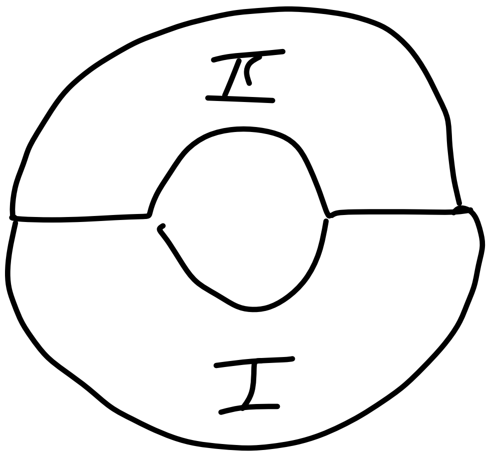
x-simple if for each x enter once, leave once

y-simple " " " y " ", leave once

Simple if x-simple and y-simple







Math 150: Multivariable Calculus: Spring 2023:

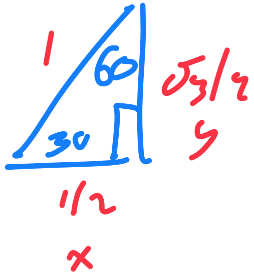
Lecture 29: Monte Carlo Integration: https://youtu.be/QHgSQDNQQTU

Plan for the day:

- Erf Function
- Central Limit Theorem
- Monte Carlo Integration

$(\sin x)' = \cos x$, $(\cos x)' = -\sin x$ in radians

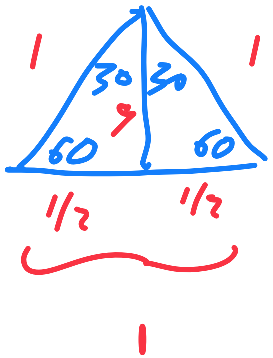
Compute by Taylor Series or identities from nice angles



why?

$1^2 = x^2 + y^2$

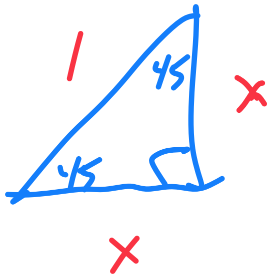
?



$1^2 = \left(\frac{1}{2}\right)^2 + y^2$

$\sqrt{\frac{3}{4}} = y$

or $y = \frac{\sqrt{3}}{2}$



$1^2 = x^2 + x^2$

So $2x^2 = 1$

so $x = \sqrt{2}/2$

$$e^x = \sum_{n=0}^{\infty} x^n/n! = 1 + x + x^2/2! + x^3/3! + \cdots$$

$$e^{ix} = \cos x + i \sin x$$

$$\cos x = 1 - x^2/2! + x^4/4! - \cdots$$

$$\sin x = x - x^3/3! + x^5/5! - \cdots$$

Probability: $f(x)$ is a density if

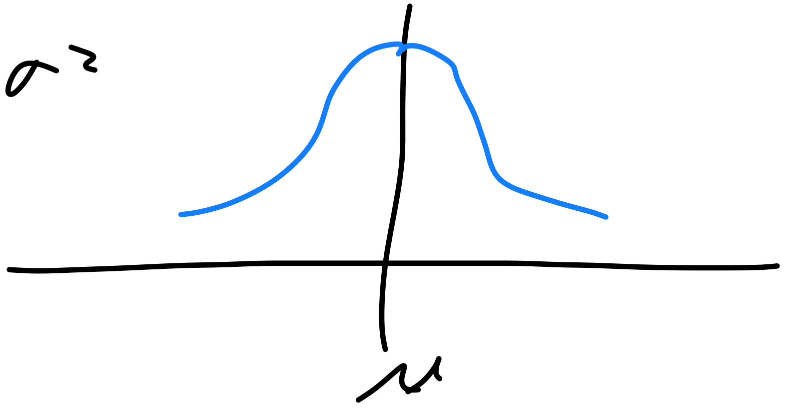
- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Cumulative Distribution function is the prob at most x:

$$\int_{-\infty}^{x} f(t)\,dt = F(x)$$

Normal mean μ standard deviation σ is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$



$\mu = 0$ $\sigma = 1$

Find prob at most x:

$$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

When $x = 0$ integral is $1/2$

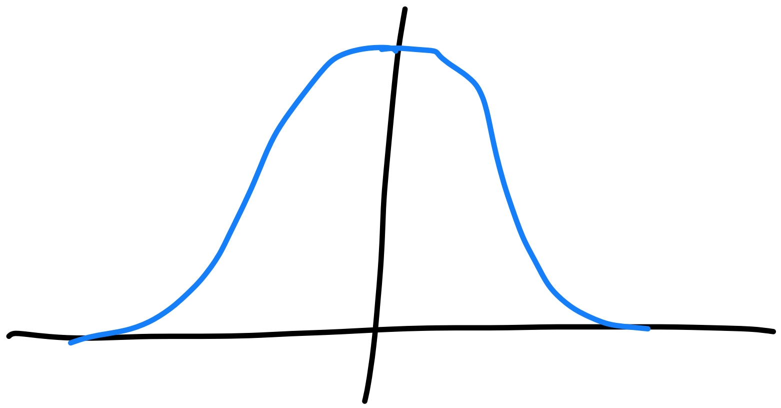
when $x = \infty$ integral is 1

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \sum_{n=0}^{\infty} \frac{(-t^2/2)^n}{n!}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^n} \int_{-\infty}^{x} t^{2n}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^n} \left.\frac{t^{2n+1}}{2n+1}\right|_{-\infty}^{x}$$

$$= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^n} \left[\frac{x^{2n+1}}{2n+1} - \frac{(-\infty)^{2n+1}}{2n+1}\right]$$



BAD!

Want $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt$ for $x > 0$

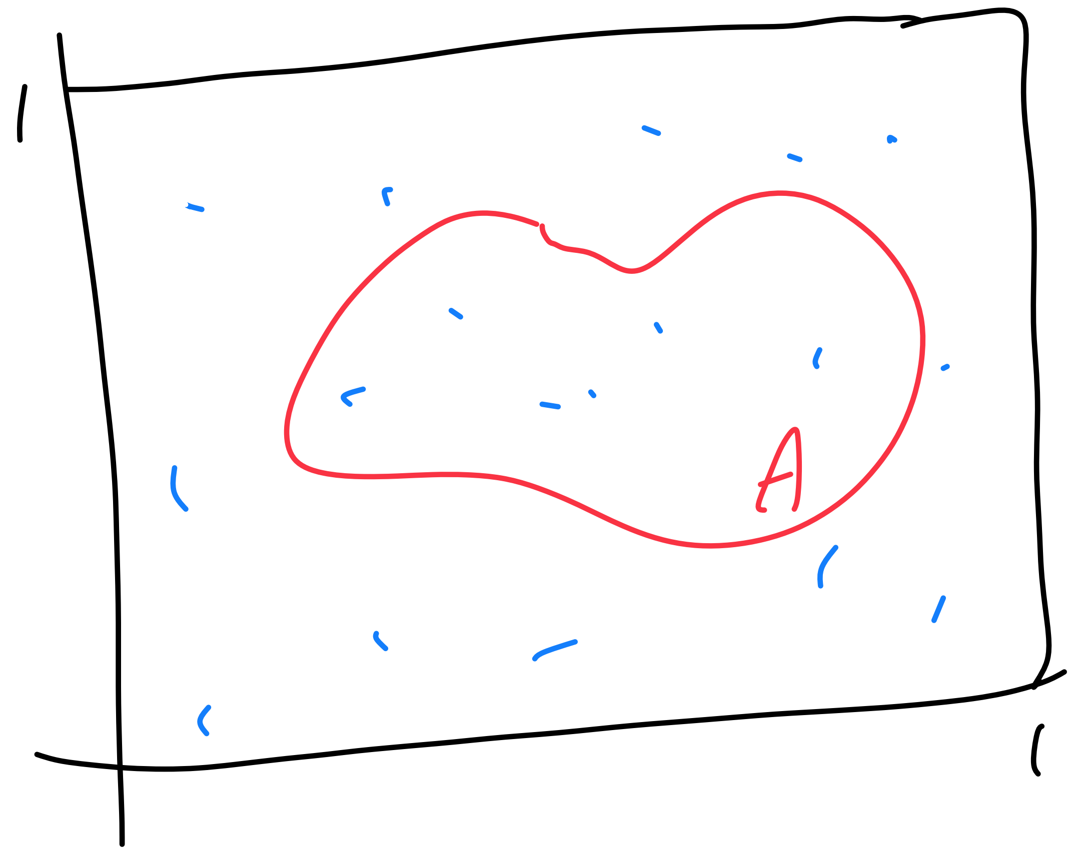
Write as $\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-t^2/2} \, dt} + \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-t^2/2} \, dt$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, 2^n} \left. \frac{t^{2n+1}}{2n+1} \right|_0^x$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, 2^n} \frac{x^{2n+1}}{2n+1}$$

"related to erf function"

Monte Carlo Integration



Throw N darts

Estimate:

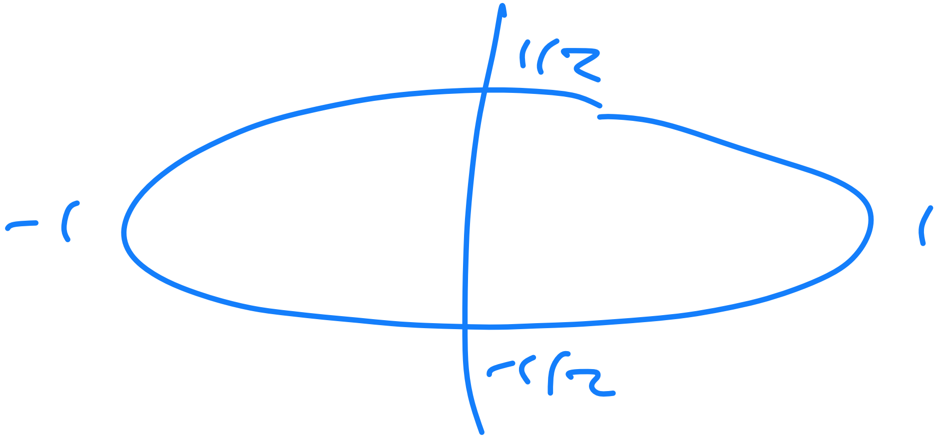
$$\frac{\text{Area}(A)}{\text{Area}(\text{Rectangle})} \approx \frac{\#\text{ hits (inside } A)}{\#\text{ tosses}}$$

error is on the order of

$$\frac{1}{\sqrt{N}}$$

for the area of A

(Central Limit Theorem)



area $\approx 1.52 \approx \pi \frac{1}{2}$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$a=1, b=\frac{1}{2}$

Guess: $\pi\left(\frac{a+b}{2}\right)^2$ $\qquad \pi \frac{9}{16}$

$\pi a b$ $\qquad \pi \frac{8}{16}$

if $a=b=r$

Circle

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

area πr^2

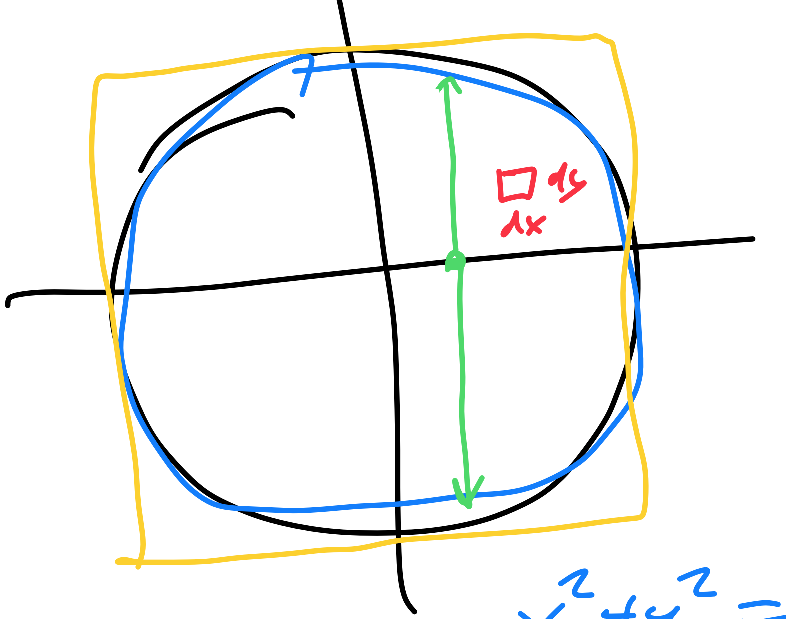
Math 150: Multivariable Calculus: Spring 2023:
Lecture 30: Integration over Simple Regions: https://youtu.be/qOZZxLiFIP0

Plan for the day:

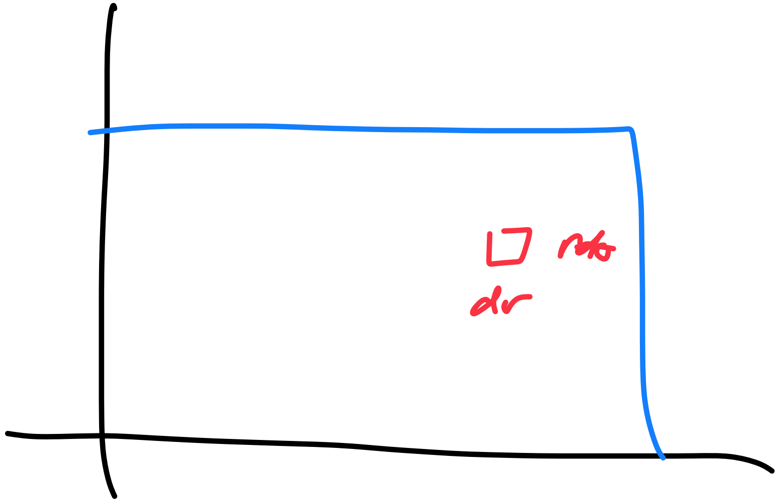
- Integration over Simple Regions:

6.2. **15.2: Double Integrals over More General Regions – Problems.**

13. Calculate the double integral of $f(x, y) = x + y$ over the domain $\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 4, \quad y \geq 0\}$ (this is a semicircle of radius 2).
17. Calculate the double integral of $f(x, y) = x^3 y$ over the domain $\mathcal{D} = \{(x, y) : 0 \leq x \leq 5, \quad x \leq y \leq 2x + 3\}$.
45. Find the volume of the region bounded by $z = 40 - 10y$, $z = 0$, $y = 0$, $y = 4 - x^2$.



(r, θ)



$$x^2+y^2=R^2 \Rightarrow y=\pm\sqrt{R^2-x^2}$$

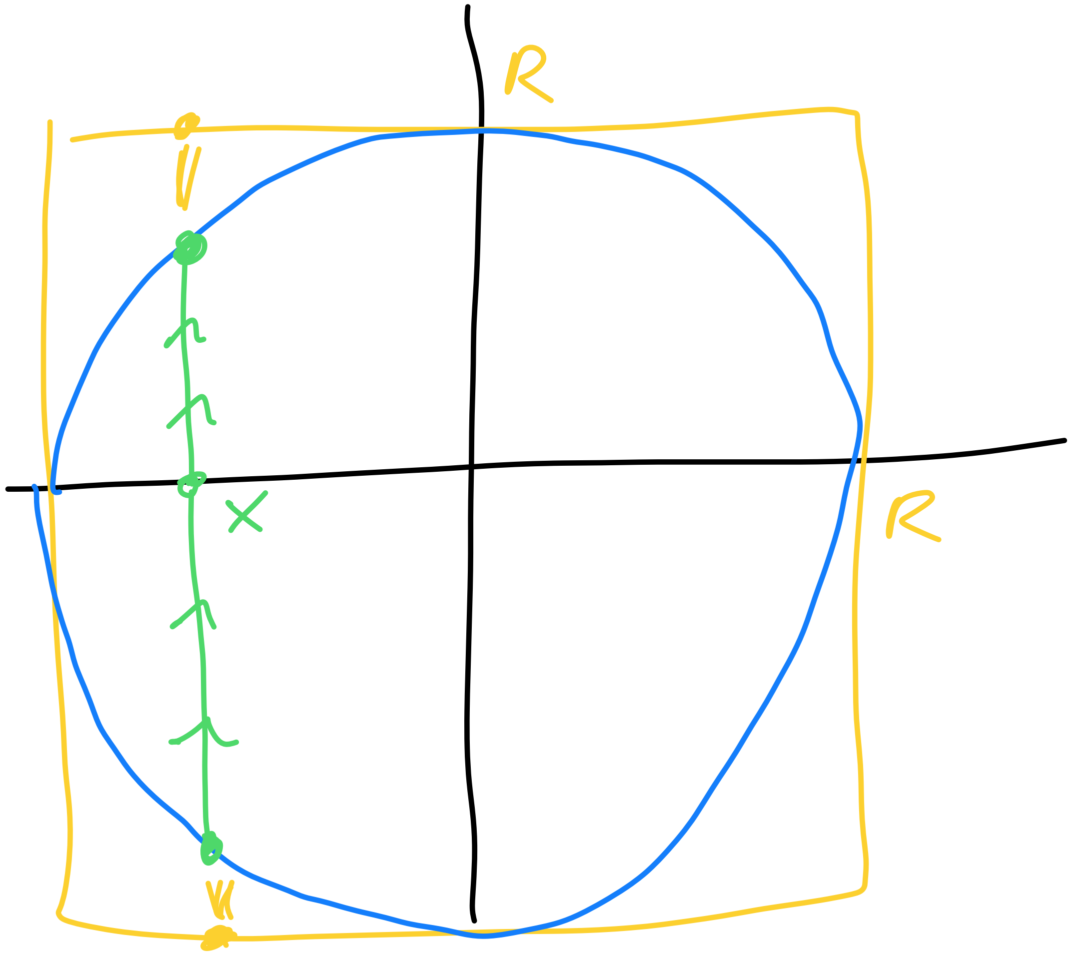
$$\int_{x=-R}^{R}\left[\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} f(x,y)\,dy\right]dx$$

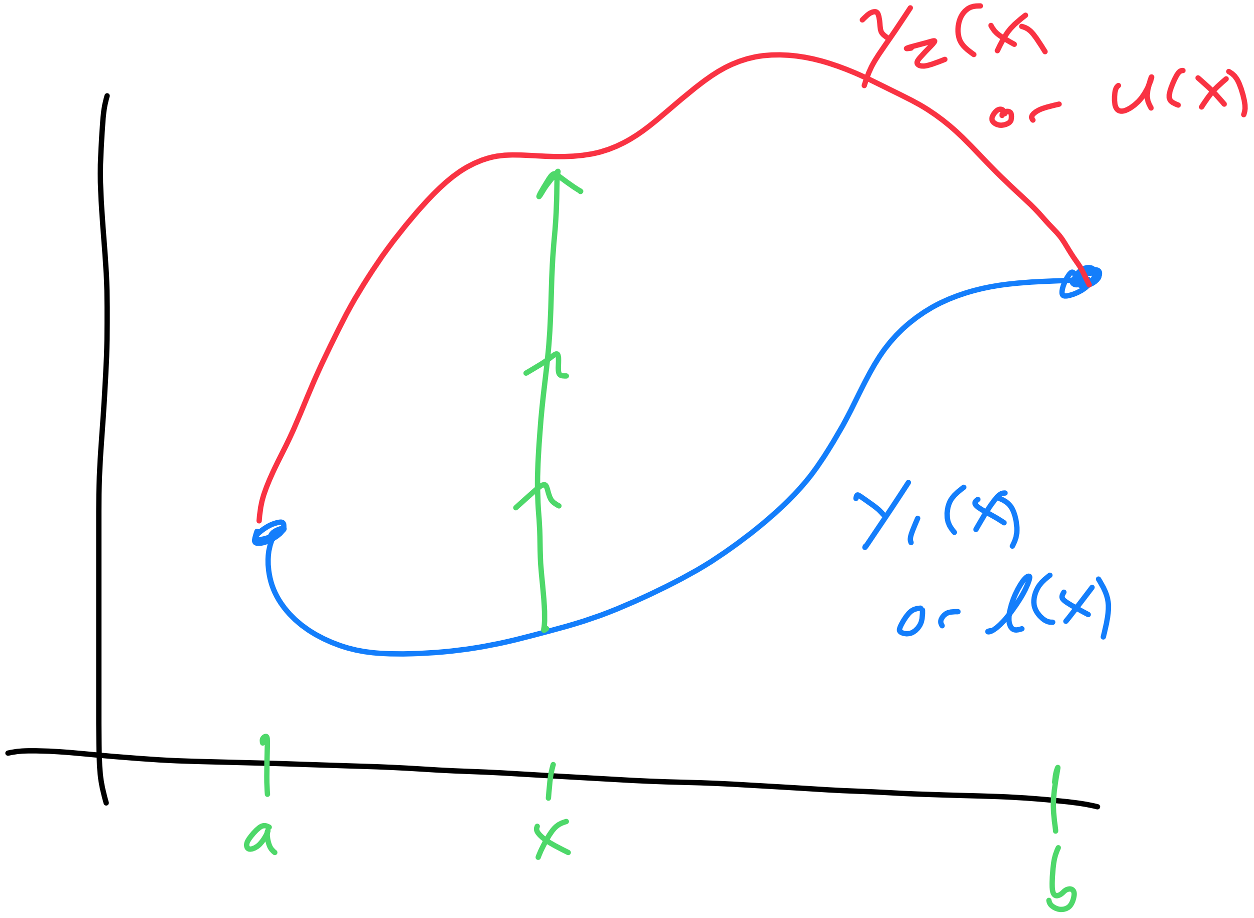
Good f to take:

- $f(x,y)=1$ area of circle
- $f(x,y)=\sqrt{1-x^2-y^2}$ hemisphere

$$x^2+y^2=R^2$$

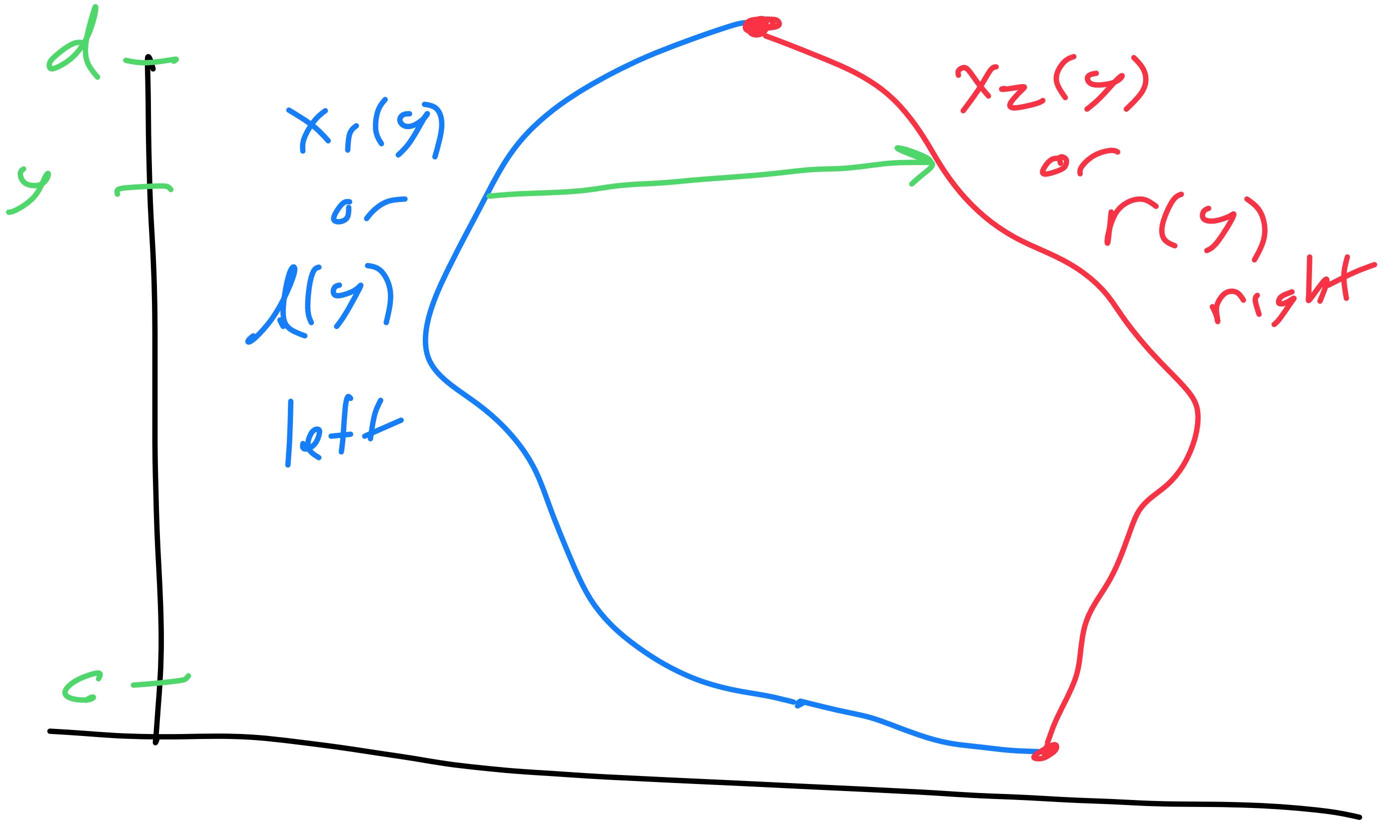
$$y=\pm\sqrt{R^2-x^2}$$



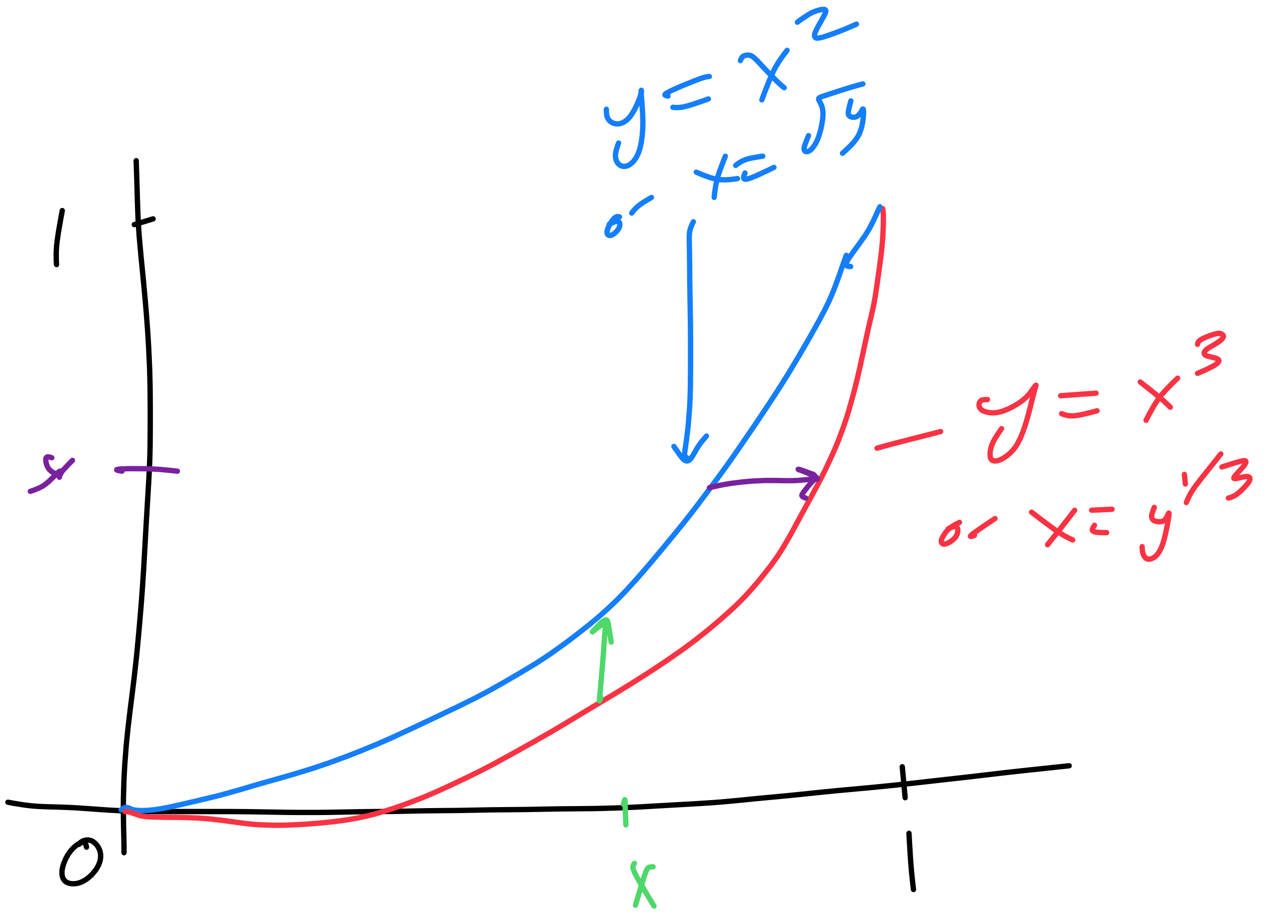


$$\int_{x=a}^{b}\left[\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\right]dx$$

iterated integrals

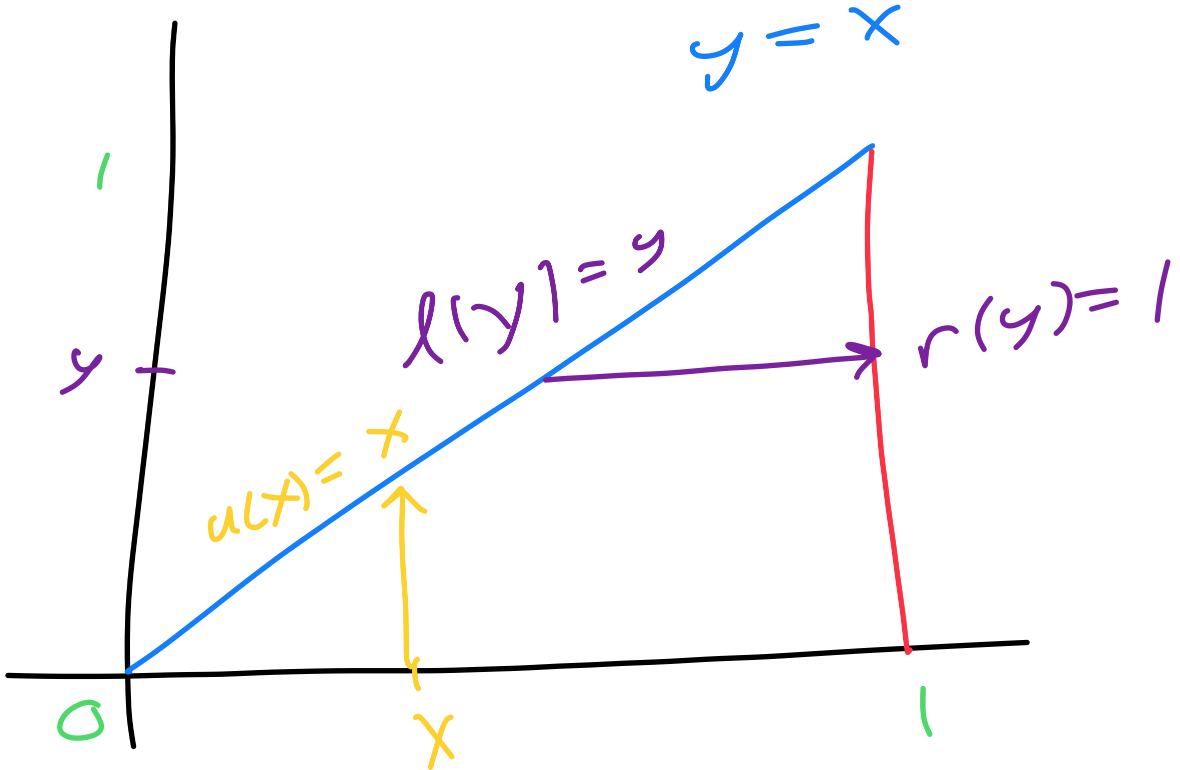


$$\int_{y=c}^{d} \left[\int_{x=l(y)}^{r(y)} f(x,y)\, dx \right] dy$$



$$\int_{x=0}^{1}\int_{y=x^3}^{x^2} f(x,y)\,dy\,dx$$

$$\int_{y=0}^{1}\int_{x=y^{1/2}}^{y^{1/3}} f(x,y)\,dx\,dy$$



$$f(x, y) = e^{-y^2}$$

$$\int_{y=0}^{1}\left[\int_{x=y}^{1} e^{-y^2}\,dx\right]dy$$

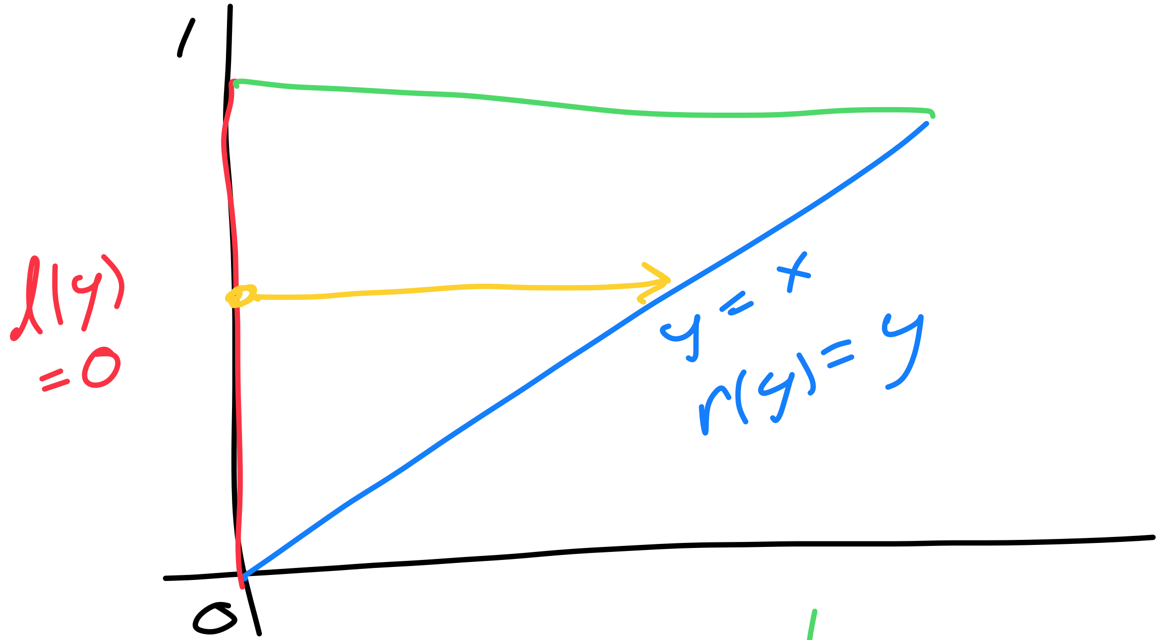
$$= \int_{y=0}^{1} e^{-y^2}\, x\Big|_{y}^{1}\, dy$$

$$= \int_{y=0}^{1} e^{-y^2}(1-y)\,dy \quad \text{Tmble}$$

$\ell(x) = 0$

$$\int_{x=0}^{1}\left[\int_{y=0}^{x} e^{-y^2}\,dy\right]dx$$

Tmble!



$$f(x,y) = e^{-y^2} > 0$$

$$\int_{y=0}^{1} \left[\int_{x=0}^{y} e^{-y^2}\, dx \right] dy$$

$$= \int_{y=0}^{1} e^{-y^2} x \Big|_0^y \, dy$$

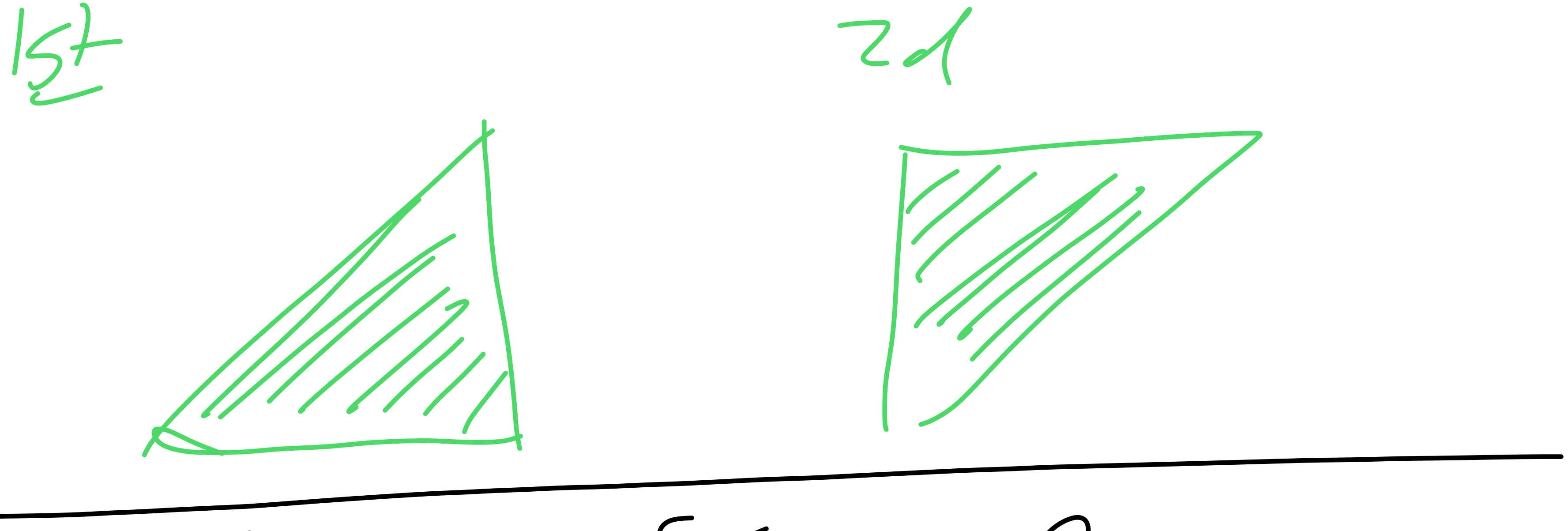
$u = y^2$ $\quad y: 0 \to 1$

$u: 0 \to 1$

$du = 2y\,dy$ or $y\,dy = \frac{1}{2} du$

$$= \int_{y=0}^{1} e^{-y^2} y\, dy = \int_{u=0}^{1} e^{-u} \cdot \frac{1}{2} du = -\frac{1}{2} e^{-u} \Big|_0^1$$

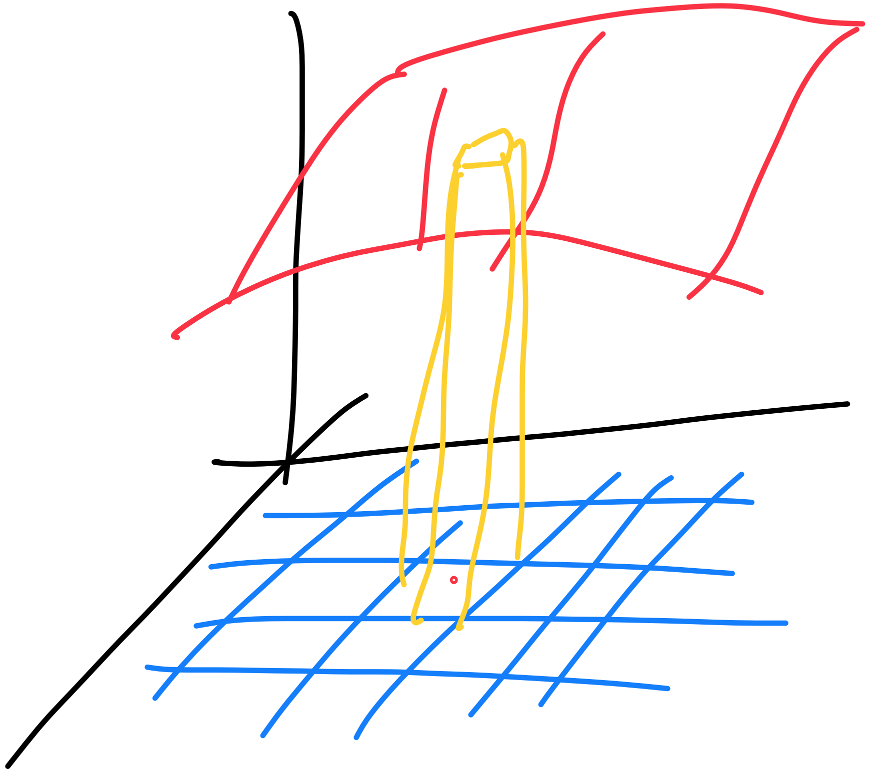
$$= \frac{1}{2} - \frac{1}{2e} > 0$$

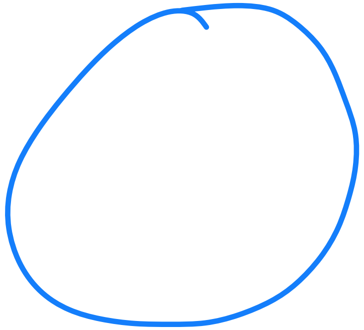


often let $z = f(x,y)$ then

$$\iint_{R \text{ in } xy\text{-plane}} f(x,y)\, dA$$

is the volume under the surface $z = f(x,y)$

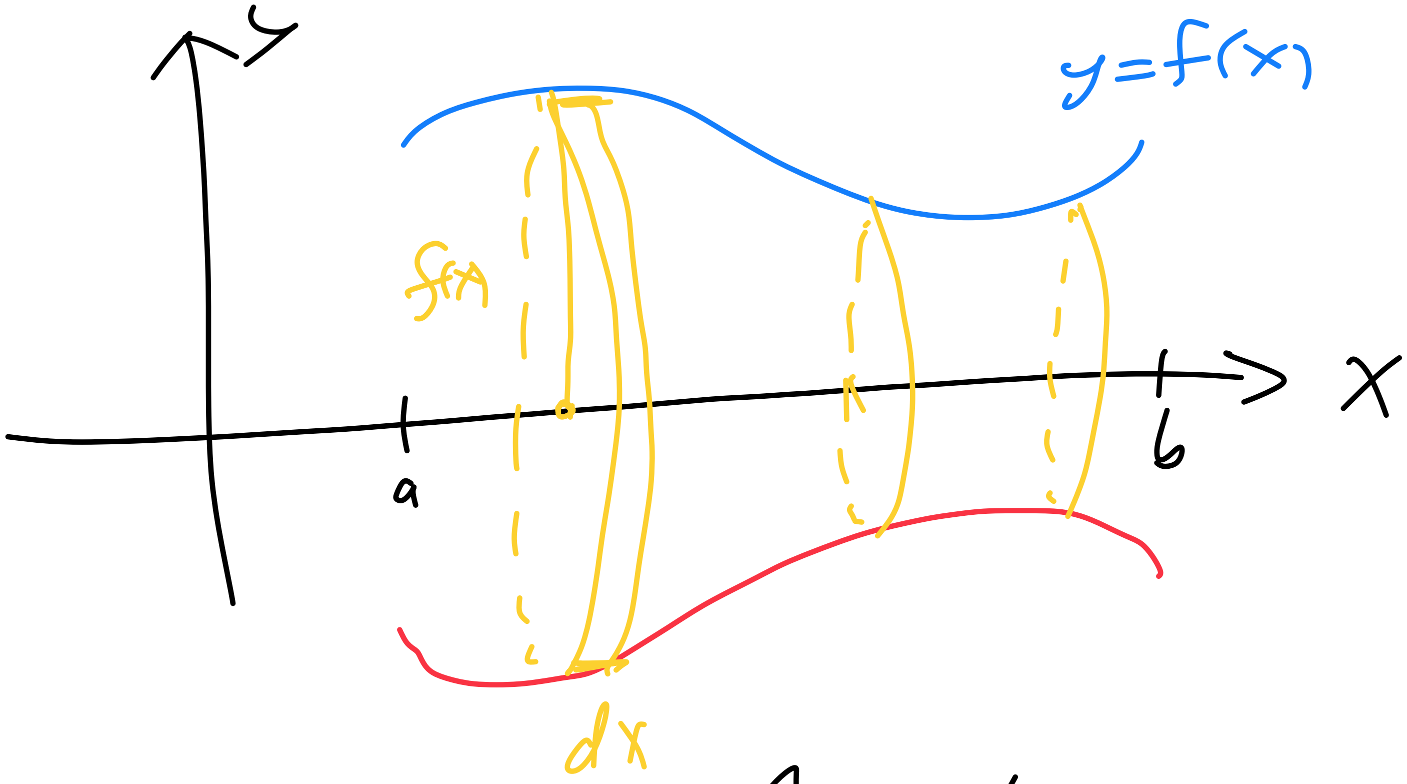




$x^2 + y^2 = R^2$

$z = \sqrt{1 - x^2 - y^2}$

hemisphere



Volume of rotate about x-axis

$$\int_{x=a}^{b} \pi f(x)^2 \, dx$$

using vol of a cylinder is $\pi r^2 h$

Math 150: Multivariable Calculus: Spring 2023:
Lecture 31: Triple Integrals: https://youtu.be/hD3qal6H1gg

Plan for the day:

- Triple Integrals
- Start of Polar Integration
- Gaussian Integral (if time permits)

15.3: Triple Integrals – Problems.

1. Integrate $f(x,y,z) = xz + yz^2$ over the region

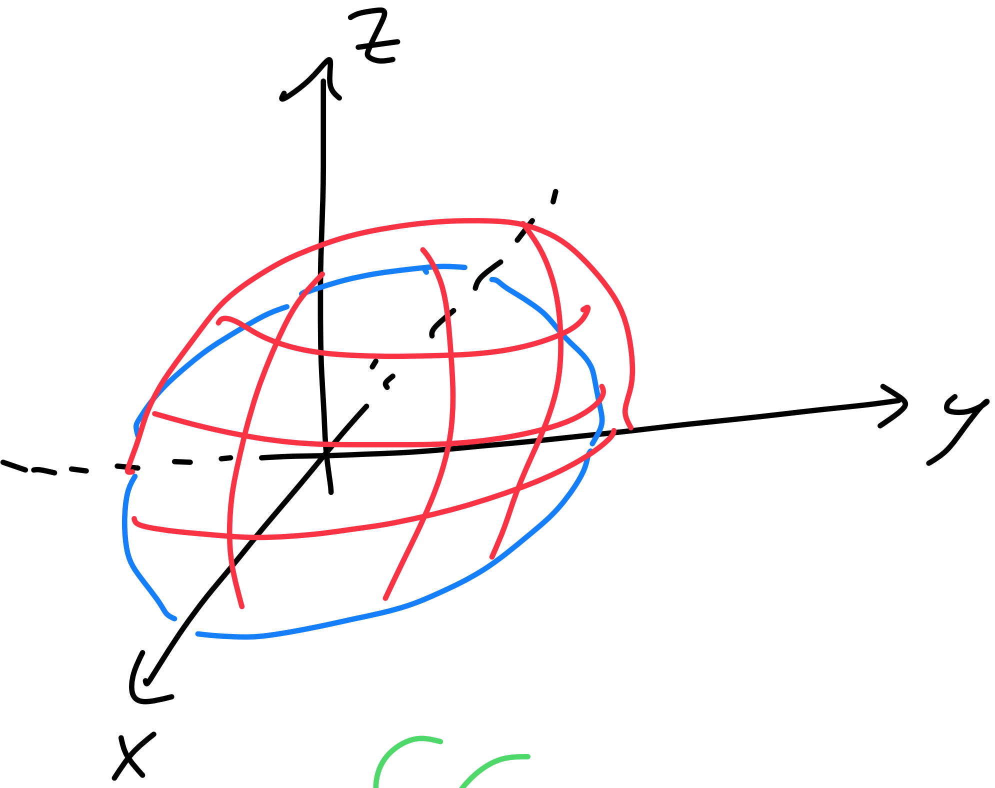
$$0 \le x \le 2, \quad 2 \le y \le 4, \quad 0 \le z \le 4.$$

11. Integrate $f(x,y,z) = xyz$ over the region

$$0 \le z \le 1, \quad 0 \le y \le \sqrt{1-x^2}, \quad 0 \le x \le 1.$$

33. Let $\mathcal{W}$ be the region bounded by $z = 1 - y^2$, $y = x^2$ and the plane $z = 0$. Calculate the volume of $\mathcal{W}$ as a triple integral.

$f(x,y) = \sqrt{1-x^2-y^2}$ $\quad 0 \le x^2+y^2 \le 1$



$z = f(x,y)$

$z = \sqrt{1-x^2-y^2}$

$z^2 = 1-x^2-y^2$

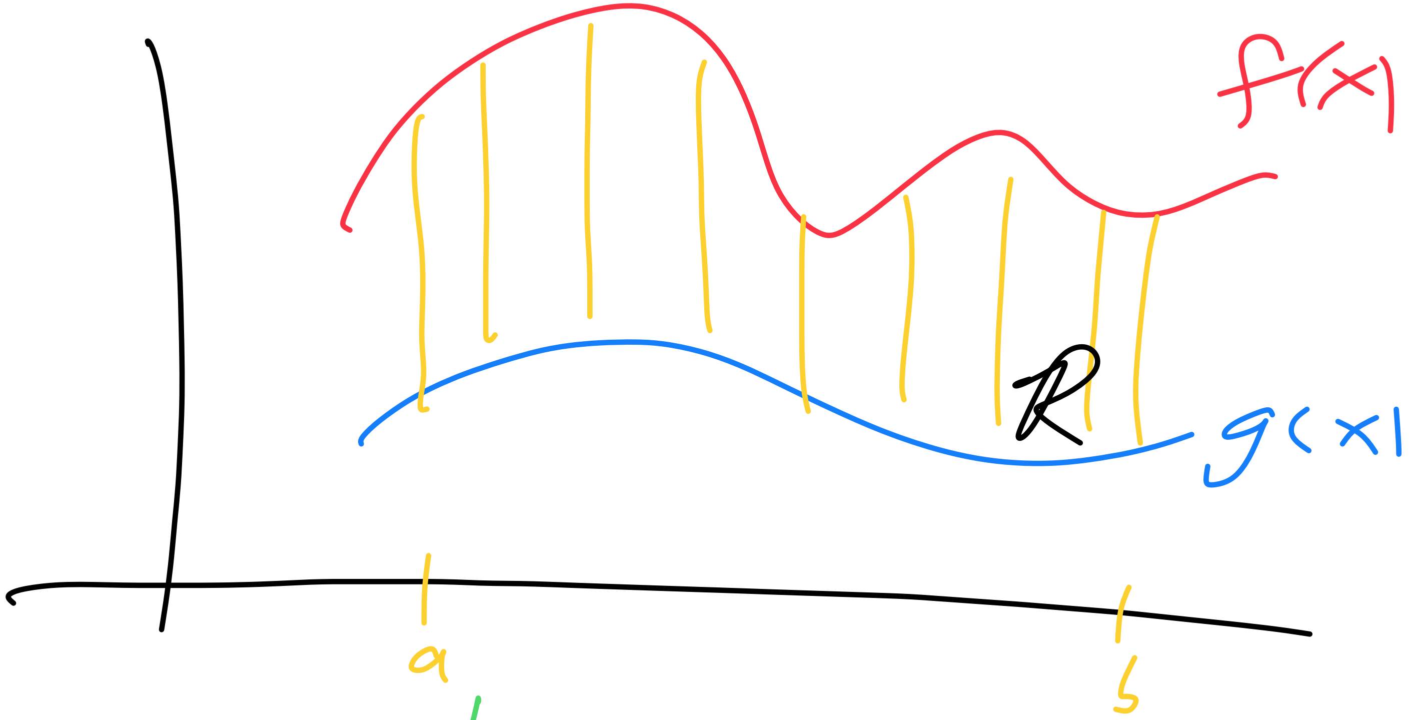
$x^2+y^2+z^2=1$ HEMI-SPHERE

$$\iint_{0 \le x^2+y^2 \le 1} f(x,y)\,dA = \iiint_{\substack{0 \le x^2+y^2+z^2 \le 1 \\ z \geq 0}} 1\,dV$$

$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx$$

$$=\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\;dy\,dx$$

$$=\iint_{0\le x^2+y^2\le 1} f(x,y)\,dA$$



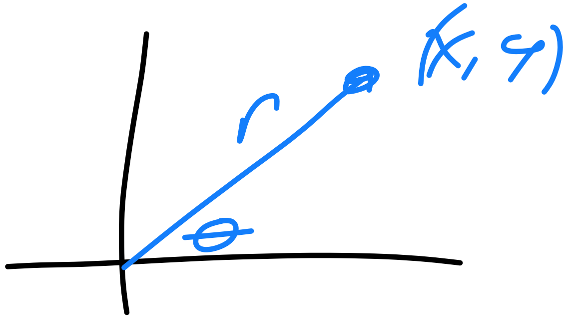
Area: $\displaystyle\int_{x=a}^{b} [f(x) - g(x)]\,dx$

$$= \iint_{(x,y)\in R} 1\, dA = \int_{x=a}^{b} \int_{y=g(x)}^{f(x)} 1\, dy\, dx = \int_{x=a}^{b} [f(x) - g(x)]\,dx$$

Change of Variables

$x = r\cos\theta$ $\qquad r = \sqrt{x^2+y^2}$

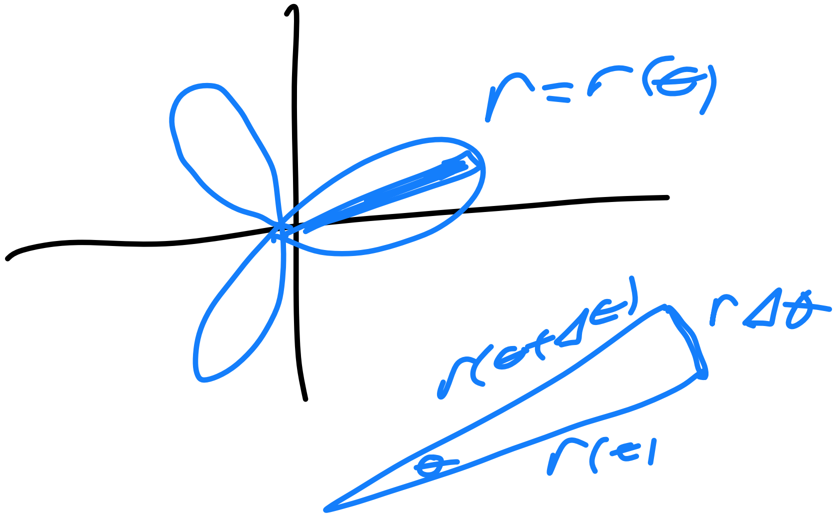
$y = r\sin\theta$ $\qquad \theta = \arctan(y/x)$



$dx\,dy \longleftrightarrow \boxed{?}\, dr\, d\theta$

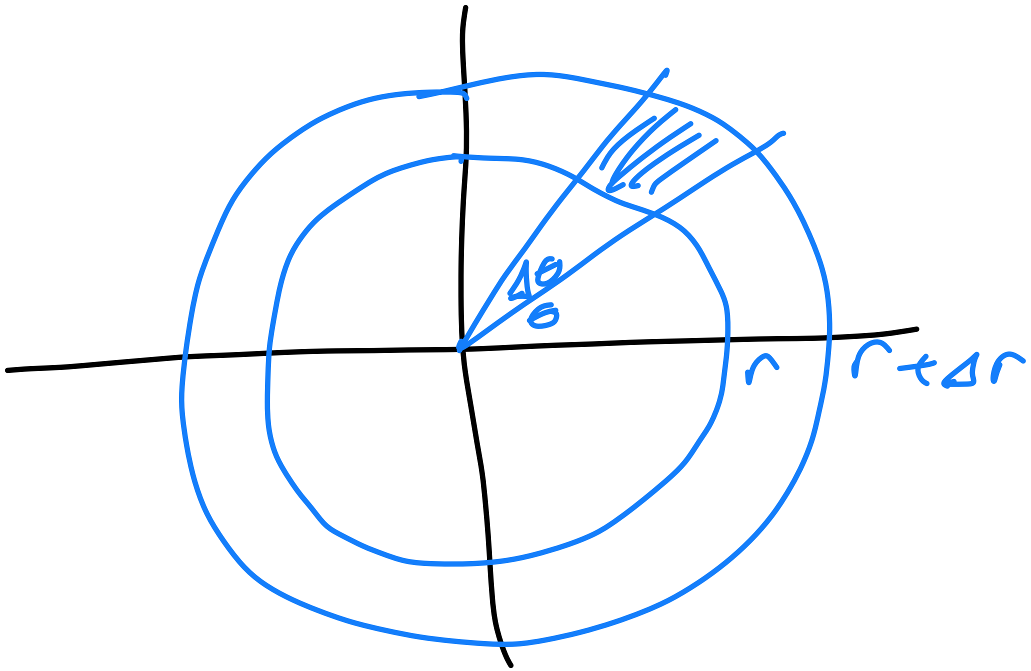
"Recall" if given $r(\theta)$ then area is

$$\int_{\theta=\theta_1}^{\theta_2} \frac{1}{2} r(\theta)^2 \, d\theta$$



height $\approx r(\theta)$

base $\approx r(\theta)\Delta\theta$



$$\left[\pi(r+\Delta r)^2 - \pi r^2\right]\frac{\Delta\theta}{2\pi}$$

$$= \left[\pi r^2 + 2\pi r\Delta r + \pi(\Delta r)^2 - \pi r^2\right]\frac{\Delta\theta}{2\pi}$$

$$= r\Delta r\Delta\theta + \frac{1}{2}(\Delta r)^2\Delta\theta$$

$$= \underbrace{r\Delta r\Delta\theta}_{\text{2 factors}} + \underbrace{\frac{1}{2}\Delta r\Delta r\Delta\theta}_{\text{3 infinitesimal factors}}$$

2 factors
This is what matters in the limit

3 infinitesimal factors
doesn't contribute in limit
like

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(\,)\frac{R}{n}\frac{R}{n}\frac{2\pi}{m}$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} 1\,\frac{1}{n}\frac{1}{m}$$

n × m summands

Polar Conversion Factor

$$dx\,dy \longleftrightarrow r\,dr\,d\theta = dr \cdot r\,d\theta$$

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dx\,dy = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 1 \cdot r\,dr\,d\theta$$

$$\parallel$$

$$\int_{x=-1}^{1} 2\sqrt{1-x^2}\,dx \qquad = \int_{\theta=0}^{2\pi} d\theta \cdot \int_{r=0}^{1} r\,dr$$

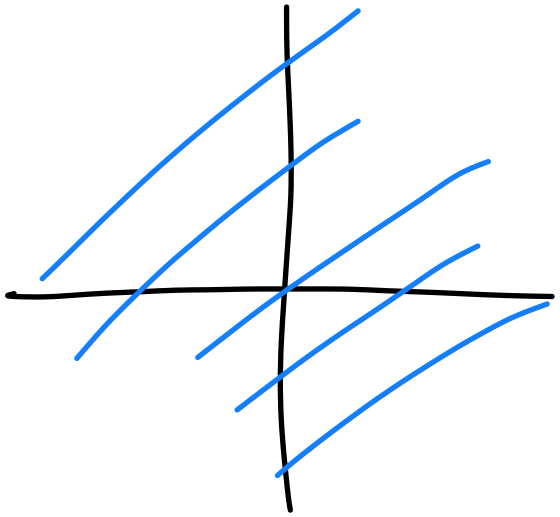
$$= 4\int_0^1 \sqrt{1-x^2}\,dx \qquad = 2\pi \cdot \frac{r^2}{2}\Big|_0^1 = \pi$$

Gaussian Integral : Normal $\mu=0$, $\sigma=1$

$$I := \int_{-\infty}^{\infty} e^{-x^2/2}\,dx > 0$$

$$I^2 = \int_{x=-\infty}^{\infty} e^{-x^2/2}\,dx \int_{y=-\infty}^{\infty} e^{-y^2/2}\,dy$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dy\,dx$$



$$f(x,y) = e^{-(x^2+y^2)/2}$$

$$f(r\cos\theta, r\sin\theta) = e^{-r^2/2}$$

$u = r^2$

$du = -2r\,dr$

so $r\,dr = -\frac{1}{2}du$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2/2}\, r\,dr\,d\theta = \int_{\theta=0}^{2\pi} d\theta \cdot \int_{r=0}^{\infty} e^{-r^2/2}\, r\,dr$$

$$= 2\pi \left(-e^{-r^2/2}\right)\Big|_0^{\infty} = 2\pi \implies I = \sqrt{2\pi}$$

as $I > 0$, take positive root

For fun:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

Show equals 1.

Volume of a sphere

$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\,dy\,dx$$ Volume of a unit sphere

$$f(x,y)=2\sqrt{1-x^2-y^2} = 2\sqrt{1-(x^2+y^2)}$$

$$f(r\cos\theta, r\sin\theta) = 2\sqrt{1-r^2}$$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1} 2\sqrt{1-r^2}\, r\,dr\,d\theta = 2\pi\int_{r=0}^{1}\sqrt{1-r^2}\,2r\,dr$$

$u=1-r^2$ $\quad r: 0\to 1$ Then $u: 1\to 0$

$du=-2r\,dr$ so $2r\,dr=-du$

$$= 2\pi\int_{u=1}^{0} u^{1/2}(-1)\,du = 2\pi\int_{u=0}^{1} u^{1/2}\,du = 2\pi\left.\frac{u^{3/2}}{3/2}\right|_0^1 = \frac{4}{3}\pi$$

Area Circle

$$4\int_{x=0}^{1} \sqrt{1-x^2}\, dx$$

$x = \sin\theta$ $\quad$ $x: 0 \to 1$

$\theta: 0 \to \pi/2$

$dx = \cos\theta\, d\theta$

$$= 4\int_{\theta=0}^{\pi/2} \sqrt{1-\sin^2\theta} \cdot \cos\theta\, d\theta$$

$$= 4\int_{\theta=0}^{\pi/2} \cos^2\theta\, d\theta$$

$\cos(\theta+\theta) = \cos^2\theta - \sin^2\theta$

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}(\cos(2\theta)+1)$

$$= 4\cdot\frac{1}{2}\int_{\theta=0}^{\pi/2} [\cos^2\theta + \sin^2\theta]\, d\theta = 2\cdot\int_{\theta=0}^{\pi/2} 1\, d\theta = 2\cdot\frac{\pi}{2}$$

$= \pi$

Math 150: Multivariable Calculus: Spring 2023:
Lecture 32: Polar, Cylindrical, Spherical Integrals, and the Gamma Function:
https://youtu.be/Pjp19j-R4dw

Plan for the day:

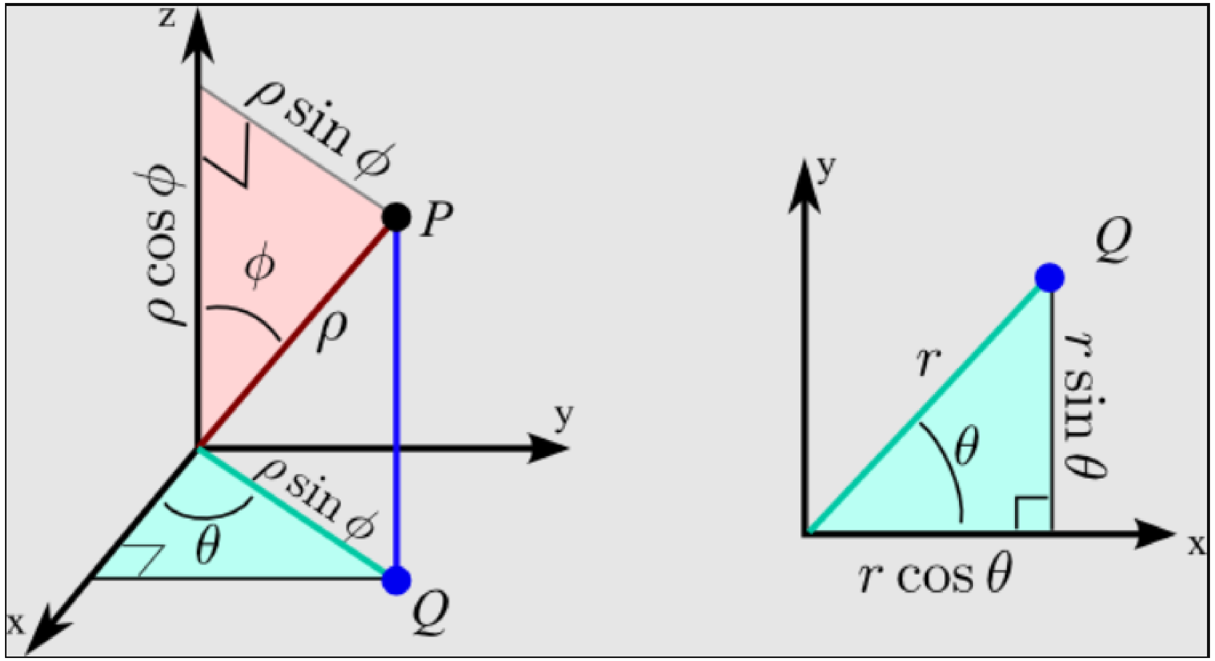
- Review Polar
- Cylindrical Integrals
- Spherical Integrals

15.4: Integration in Polar, Cylindrical, and Spherical Coordinates – Problems.

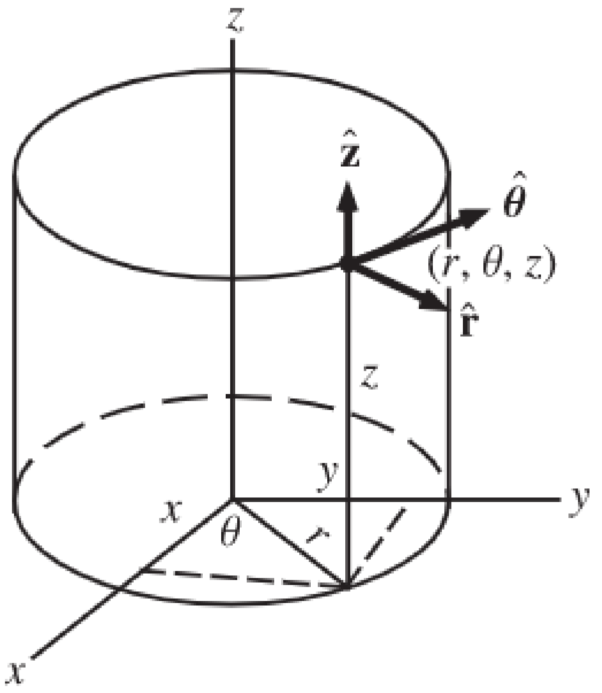
7. Calculate the following integral by changing to polar coordinates:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx.$$

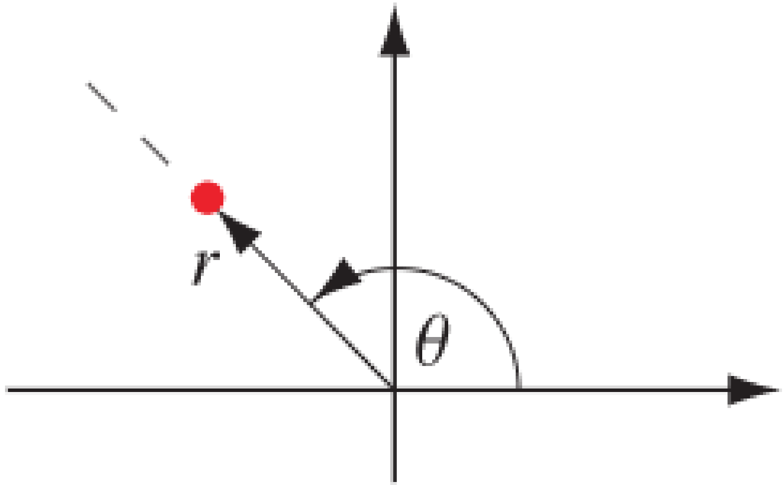
27. Integrate $f(x, y, z) = x^2 + y^2$ over the region $x^2 + y^2 \leq 9, 0 \leq z \leq 5$ by changing to cylindrical coordinates.
45. Integrate $f(x, y, z) = y$ over the region $x^2 + y^2 + z^2 \leq 1, x, y, z \leq 0$ by changing to spherical coordinates.



https://mathinsight.org/media/image/image/spherical_coordinates_cartesian.png



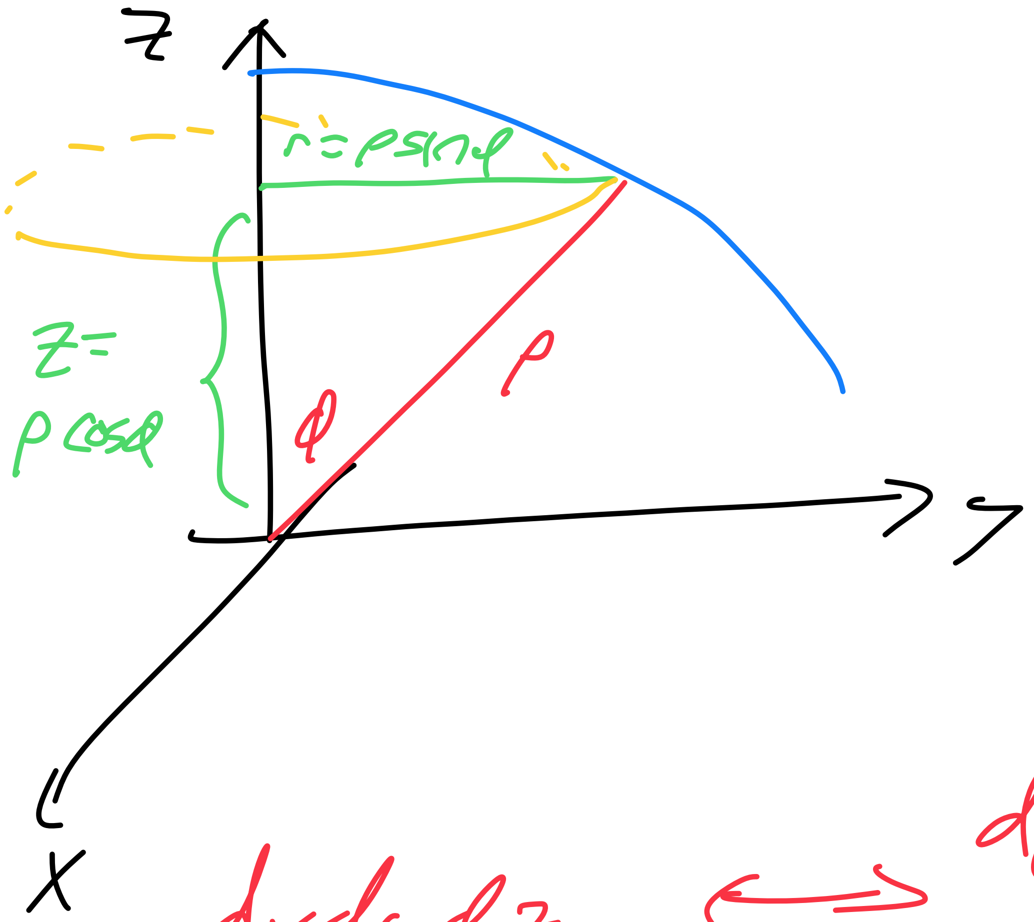
https://mathworld.wolfram.com/CylindricalCoordinates.html

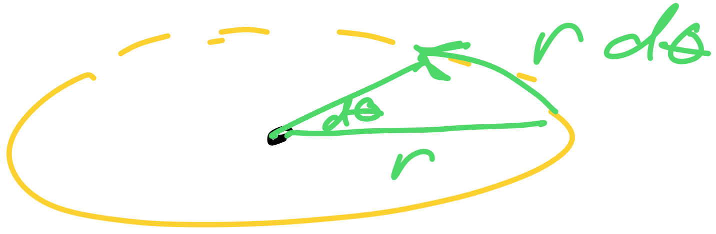


https://mathworld.wolfram.com/PolarCoordinates.html

 $\}dz$

$$dx\,dy\,dz \leftrightarrow r\,dr\,d\theta\,dz$$
$$= dr \cdot r\,d\theta \cdot dz$$





$r = \rho\sin\varphi$

infinitesimal arc is $\rho\sin\varphi\,d\theta$

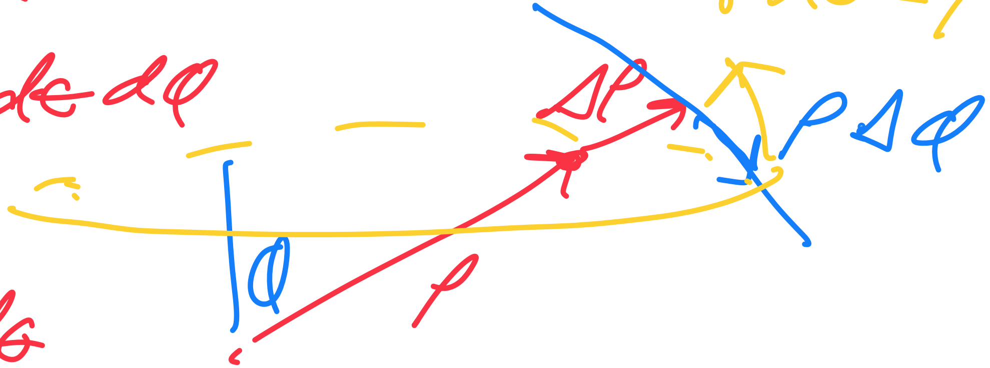
$$dx\,dy\,dz \longleftrightarrow d\rho \cdot \rho\,d\varphi \cdot \rho\sin\varphi\,d\theta$$

$$r\,d\theta = \rho\sin\varphi\,d\theta$$

$$= \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

or

$$\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$



$$\iiint_{0 \le x^2+y^2+z^2 \le R} f(x, y, z)\, dx\, dy\, dz$$

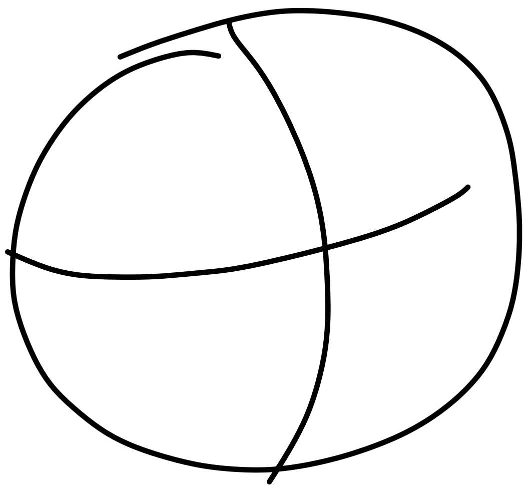
$$= \int_{\varphi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{R} f(\rho\cos\theta\sin\varphi,\ \rho\sin\theta\sin\varphi,\ \rho\cos\varphi)\, \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

$z = \rho\cos\varphi$, $r = \rho\sin\varphi$

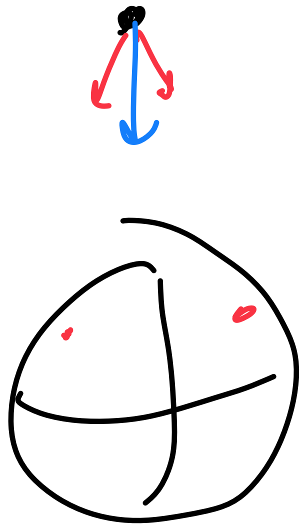
$x^2+y^2+z^2 = \rho^2$

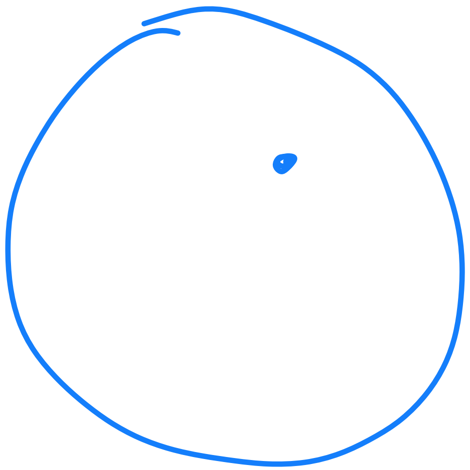
$x = r\cos\theta = \rho\cos\theta\sin\varphi$

$y = r\sin\theta = \rho\sin\theta\sin\varphi$



wlog





no force!

$$\iiint_{\substack{\text{sphere of} \\ \text{radius } R}} e^{-(x^2+y^2+z^2)/2}\, dx\, dy\, dz$$

$$e^{-(x^2+y^2+z^2/2)}$$

$$= e^{-x^2/2} e^{-y^2/2} e^{-z^2/2}$$

$x^2+y^2+z^2 \le R^2$

$$\int_{x=-R}^{R} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{z=-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} e^{-(x^2+y^2+z^2)/2}\, dz\, dy\, dx$$

$$\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{R} \underbrace{e^{-\rho^2/2}\cdot\rho^2\sin\phi}_{A(\rho)\cdot B(\theta)\cdot C(\phi)}\,d\rho\,d\theta\,d\phi$$

$$A(\rho) = e^{-\rho^2/2}\cdot\rho^2$$

$$B(\theta) = 1$$

$$C(\phi) = \sin\phi$$

$$= \underbrace{\int_{\phi=0}^{\pi}\sin\phi\,d\phi}_{2}\cdot\int_{\theta=0}^{2\pi}1\,d\theta\cdot\int_{\rho=0}^{R}e^{-\rho^2/2}\rho^2\,d\rho$$

$$2\cdot 2\pi\cdot\int_{\rho=0}^{R}e^{-\rho^2/2}\rho^2\,d\rho$$

Gamma Function

$$\Gamma(s) := \int_{x=0}^{\infty} e^{-x} x^{s-1}\,dx = \int_{x=0}^{\infty} e^{-x} x^{s} \frac{dx}{x}$$

Claim integral converges if $Re(s) > 0$

Danger near 0, Danger near ∞

x large, integrand looks like $e^{-x} \cdot x^{s-1} = \dfrac{x^{s-1}}{e^{x}}$

$$\lim_{x\to\infty} \frac{x^{47-1}}{e^{x}} \underset{\text{L'Hopital}}{=} \lim_{x\to\infty} \frac{46 \cdot x^{45}}{e^{x}} = \cdots = \lim_{x\to\infty} \frac{46!}{e^{x}} = 0$$

Take $s = 44$, $e^{-x} \cdot x^{44-1} = \dfrac{e^{-x} \cdot x^{47-1}}{x^{3}} \underset{x \text{ big}}{\leq} \dfrac{1}{x^{3}}$

Near 0: $e^{-x} \cdot x^{s-1}$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \approx 1$$

x close to zero

So near zero, $e^{-x} \cdot x^{s-1} \approx x^{s-1}$. Take ε small:

$$\int_0^{\varepsilon} e^{-x} x^{s-1}\,dx \approx \int_0^{\varepsilon} x^{s-1}\,dx$$

$s=0$ get $\int_0^{\varepsilon} \frac{1}{x}\,dx = \ln(x)\Big|_0^{\varepsilon} = \ln(\varepsilon) - \ln(0)$

infinity!

$s>0$ get $\int_0^{\varepsilon} x^{s-1}\,dx = \frac{x^s}{s}\Big|_0^{\varepsilon} = \frac{\varepsilon^s}{s} - 0$ finite

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$$

Take $s = 1$

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = e^{-x}\Big|_\infty^0 = 1$$

$$\Gamma(2) = \int_0^\infty e^{-x} \cdot x \, dx$$

$u = x$ $\quad$ $dv = e^{-x} dx$

$du = 1 \cdot dx$ $\quad$ $v = -e^{-x}$

$\lim_{x\to\infty} x e^{-x}$

$= \lim_{x\to\infty} \frac{x}{e^{x}}$

$$= uv\Big|_0^\infty - \int_0^\infty v \, du = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx$$

$$= 1$$

$$\Gamma(3) = \int_0^\infty e^{-x} \cdot x^2 \, dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} \cdot x^{n+1-1} \, dx$$

$$= \int_0^\infty e^{-x} \cdot x^{n} \, dx$$

$$u = x^n \qquad dv = e^{-x} dx$$

$$du = n x^{n-1} dx \qquad v = -e^{-x}$$

$$= uv\Big|_0^\infty - \int_0^\infty v \, du$$

$$= -x^n e^{-x}\Big|_0^\infty + \int_0^\infty n e^{-x} \cdot x^{n-1} \, dx$$

$$= n \int_0^\infty e^{-x} x^{n-1} \, dx = n\,\Gamma(n)$$

$\Gamma(n+1) = n\Gamma(n)$ $\quad \Gamma(1) = 1, \quad \Gamma(2) = 1$

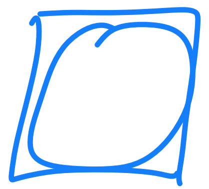
n	$\Gamma(n)$
1	$\rightarrow 1 = 0!$
2	$\rightarrow 1 = 1!$
$2+1 = 3$	$\rightarrow 2\cdot\Gamma(2) = 2\cdot 1 = 2!$
$3+1 = 4$	$\rightarrow 3\cdot\Gamma(3) = 3\cdot 2\cdot 1 = 3!$
$4+1 = 5$	$\rightarrow 4\cdot\Gamma(4) = 4\cdot 3\cdot 2\cdot 1 = 4!$

$\Gamma(3) = \Gamma(2+1) = 2\cdot\Gamma(2)$

$\Gamma(n+1) = n!$ if $n \geq 0$ integer

n	box	sphere	$\frac{\text{sphere}}{\text{box}}$
1	2	2	1
2	2^2	π	$\frac{\pi}{4} \approx 75\%$
3	2^3	$\frac{4}{3}\pi$	$\frac{\pi}{6}$ = less!

radius = 1
box has sides of length 2



Math 150: Multivariable Calculus: Spring 2023:

Lecture 33: Hypersphere Integrals, Ellipse Area: https://youtu.be/z7wfwZ9Lr0s

Plan for the day:

- Gamma Function
- Generalized Spherical Coordinates?
- Volume of the n-sphere

Gamma Function

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx \quad \text{converges if } Re(s) > 0$$

Showed: $\Gamma(n+1) = n\Gamma(n)$ if $n > 0$ pos (integer)

Ex: $\Gamma(1) = 1$, $\Gamma(2) = 1$

$$\Gamma(3) = \Gamma(2+1) \underset{n=2}{=} 2\cdot\Gamma(2) = 2\cdot 1 = 1!$$

$$\Gamma(4) = \Gamma(3+1) \underset{n=3}{=} 3\cdot\Gamma(3) = 3\cdot 2! = 3!$$

So $\Gamma(n+1) = n!$ if $n \geq 0$ integer

Gaussian Integral

$$2\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1 \qquad \text{(prob density)}$$

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1}\,du$$

Change variables: $u = x^2/2$ so $du = x\,dx$

so $dx = x^{-1}du = (2u)^{-1/2}du$

x: 0 to ∞

u: 0 to ∞

Get: $2\int_{u=0}^\infty \frac{1}{\sqrt{2\pi}} e^{-u}(2u)^{-1/2}\,du$

$$= \pi^{-1/2}\int_{u=0}^\infty e^{-u} u^{\frac{1}{2}-1}\,du = \Gamma\left(\tfrac{1}{2}\right)\pi^{-\frac{1}{2}} = 1$$

So $\Gamma\left(\frac{1}{2}\right) = \pi^{1/2} = \sqrt{\pi}$

Consider: $f(x_1, \ldots, x_n) = \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_k^2/2}$

$$= (2\pi)^{-n/2} e^{-(x_1^2 + \cdots + x_n^2)/2}$$

$$\int_{x_1=-\infty}^{\infty} \cdots \int_{x_n=-\infty}^{\infty} (2\pi)^{-n/2} e^{-(x_1^2+\cdots+x_n^2)/2} \, dx_n \cdots dx_1$$

$$= \underbrace{\int_{x_1=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \, dx_1}_{=1} \cdots \underbrace{\int_{x_n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x_n^2/2} \, dx_n}_{=1} = 1$$

$$\int_{x_1=-\infty}^{\infty} \cdots \int_{x_n=-\infty}^{\infty} (2\pi)^{-n/2} e^{-(x_1^2+\cdots+x_n^2)/2} \, dx_n \cdots dx_1$$

only depends on distance from origin...

n-dim spherical coordinates

$$(x_1, \ldots, x_n) \longleftrightarrow \rho, \theta_1, \theta_2, \ldots, \theta_{n-1}$$

$$dx_1 \cdots dx_n \longleftrightarrow g(\rho, \theta_1, \ldots, \theta_{n-1}) \, d\rho \, d\theta_1 \cdots d\theta_{n-1}$$

$$= \rho^{n-1} h(\theta_1, \ldots, \theta_{n-1}) \, d\rho \, d\theta_1 \cdots d\theta_{n-1}$$

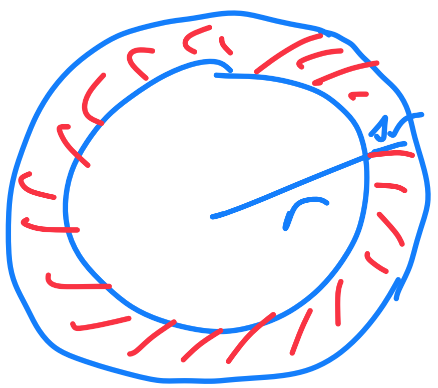
$$1 = \int_{x_1=-\infty}^{\infty} \cdots \int_{x_n=-\infty}^{\infty} (2\pi)^{-n/2} e^{-(x_1^2+\cdots+x_n^2)/2} \, dx_n \cdots dx_1$$

$$= \underbrace{\int_{\rho=0}^{\infty} (2\pi)^{-n/2} e^{-\rho^2/2} \rho^{n-1} d\rho}_{\substack{\text{Show this is } \Gamma \text{ of} \\ \text{something depending} \\ \text{on } n}} \underbrace{\int \cdots \int_{\theta_1, \ldots, \theta_{n-1} \text{ ranges}} h(\theta_1, \ldots, \theta_{n-1}) \, d\theta_1 \cdots d\theta_{n-1}}_{\substack{?? \\ \text{Surface area of the} \\ n\text{-dimensional} \\ \text{unit sphere}}}$$

Try $u = \rho^2/2$

$n=2$: circle: perim is $2\pi r$ area πr^2

$n=3$: sphere: area is $4\pi r^2$ vol $\frac{4}{3}\pi r^3$

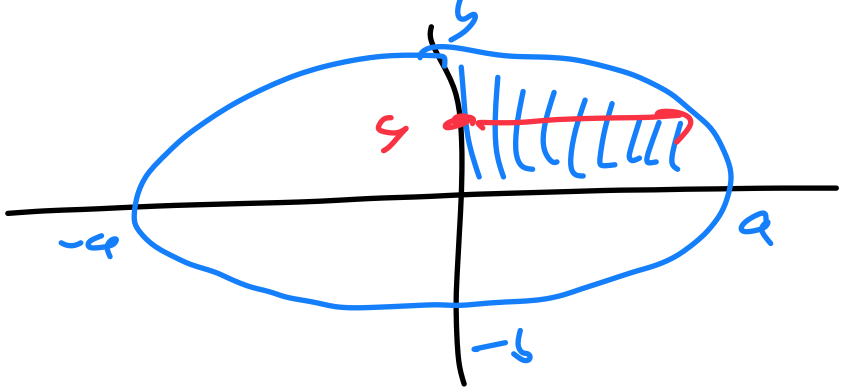


$$\frac{A(r+\Delta r) - A(r)}{\Delta r} \approx \frac{\text{perim}(r)\cdot\Delta r}{\Delta r}$$

take lim as $\Delta r \to 0$

$A'(r) = \text{perim}(r)$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



Area

$$4\int_{y=0}^{b}\int_{x=0}^{\sqrt{1-(y/b)^2}} 1\,dx\,dy$$

Change variables:

$u = x/a$ $\quad$ $v = y/b$

So $au = x$ $\quad$ $bv = y$

$dx = a\,du$ $\quad$ $dy = b\,dv$

$dx\,dy = ab\,du\,dv$

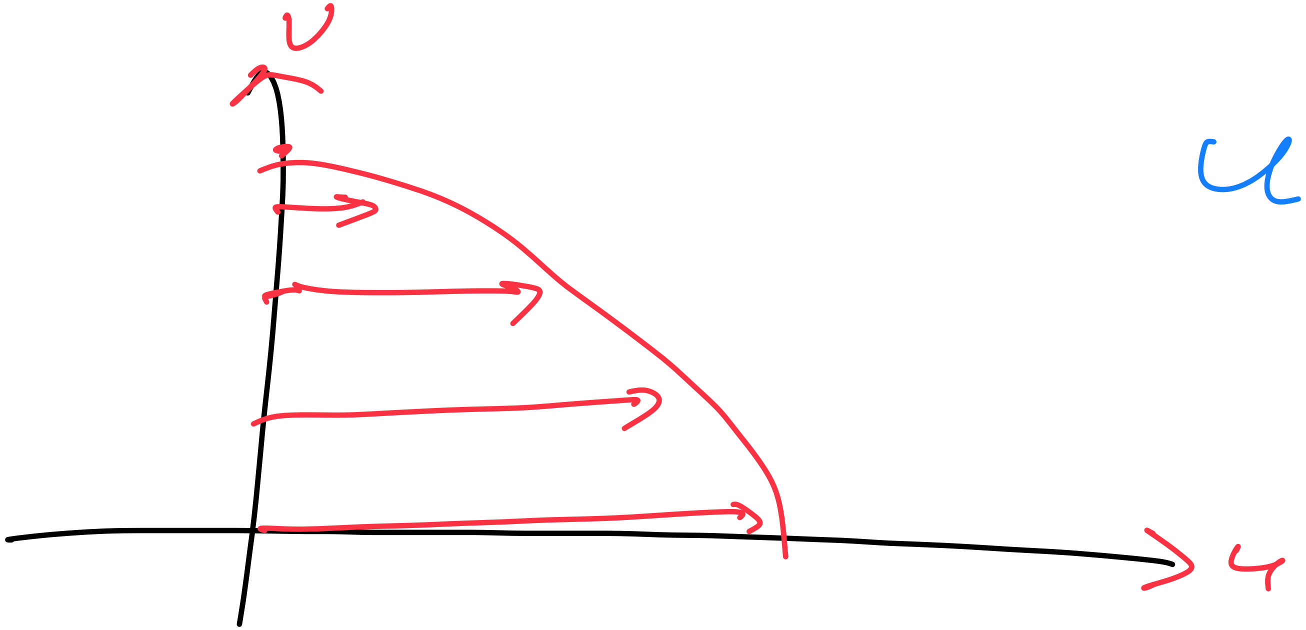
Circle: $u^2 + v^2 = 1$

$$= 4\int_{v=0}^{1}\int_{u=0}^{\sqrt{1-v^2}} 1\,ab\,du\,dv$$

$$= 4ab^{*}\int_{v=0}^{1}\int_{u=0}^{\sqrt{1-v^2}} 1\,du\,dv$$

$$= 4ab\cdot\frac{1}{4}\pi 1^2 = \pi ab$$

$$u^2 + v^2 = 1$$



$v: 0 \longrightarrow 1$

$u: 0 \longrightarrow \sqrt{1-v^2}$

Math 150: Multivariable Calculus: Spring 2023:
Lecture 34: Change of Variables, Newton's Law of Gravity:
https://youtu.be/ZEQJc6BtHrU

Plan for the day:

- Change of Variables
- Newton's Law of Gravity
- Dropped a term in class today: **For the correct calculation see:** **https://www.youtube.com/watch?v=3Pt4E1BeUTw&t=104s**

15.6: Change of Variables – Problems.

7. Let $G(u,v) = (2u+v, 5u+3v)$ be a map from the uv-plane to the xy-plane. Describe the image of the lin $v = 4u$ under G.
13. Calculate the Jacobian of $G(u,v) = (3u+4v, u-2v)$.
17. Calculate the Jacobian of $G(r,\theta) = (r\cos\theta, r\sin\theta)$.
35. Calculate

$$\int\int_{\mathcal{D}} e^{9x^2+4y^2}\,dx dy,$$

where $\mathcal{D}$ is the interior of the ellipse

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \le 1.$$

Change of variable:

$x = x(u,v) \quad y = y(u,v)$

$$\iint_R f(x,y)\,dx\,dy \longleftrightarrow \iint_{R^*} f(x(u,v), y(u,v))\,du\,dv$$

$g(u,v)$

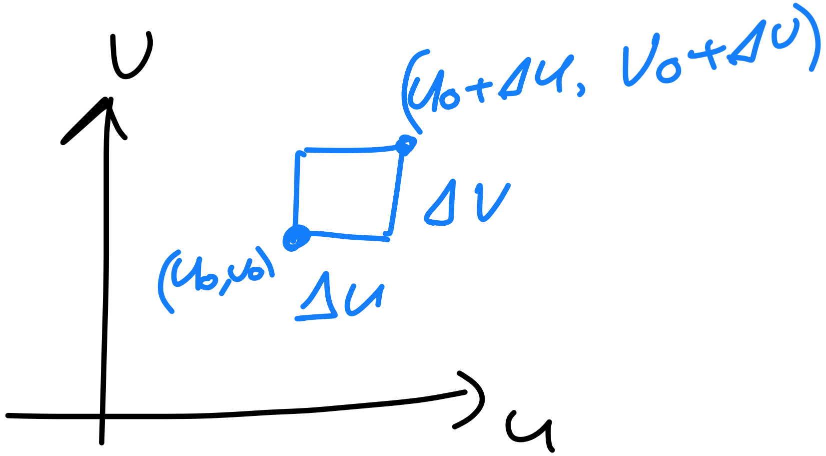
absolute value of the det of the Jacobian

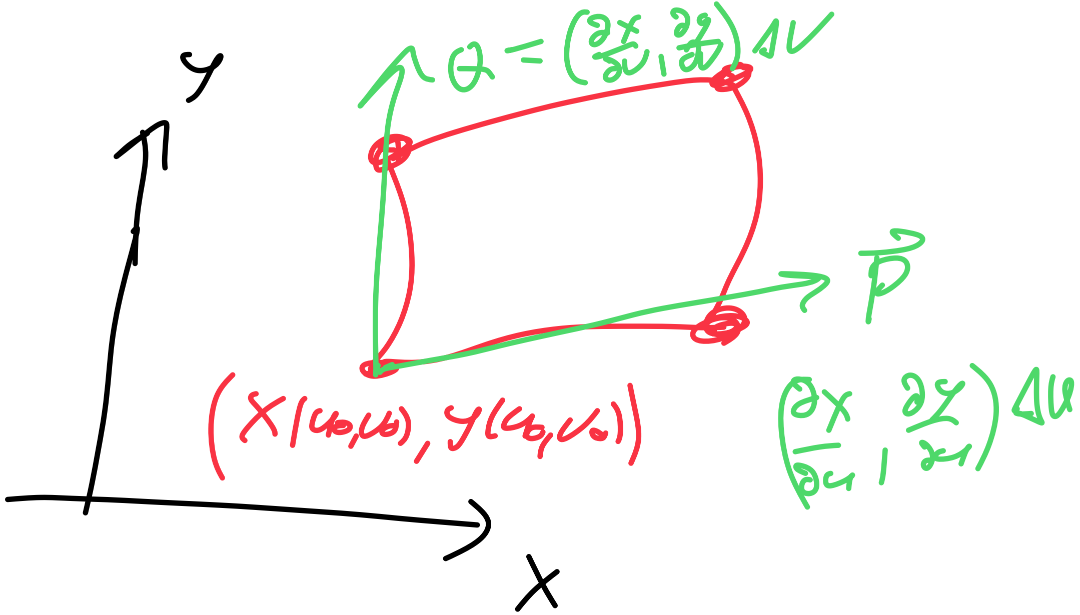
$$dx\,dy \longleftrightarrow g(u,v)\,du\,dv$$

Polar: $dx\,dy \longleftrightarrow r\,dr\,d\theta$

Spherical: $dx\,dy\,dz \longleftrightarrow \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

Ellipses: $dx\,dy \longleftrightarrow ab\,du\,dv$ =

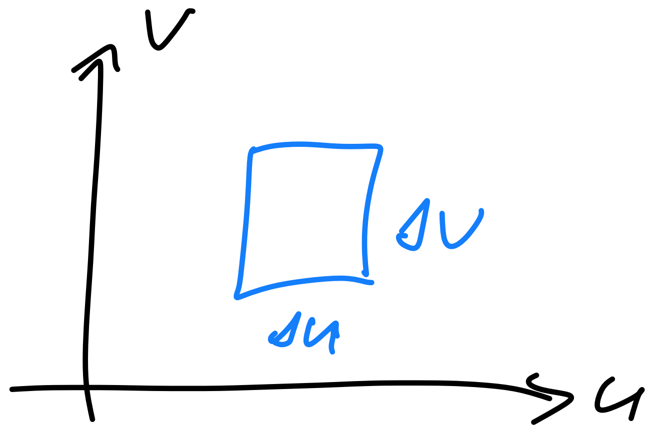


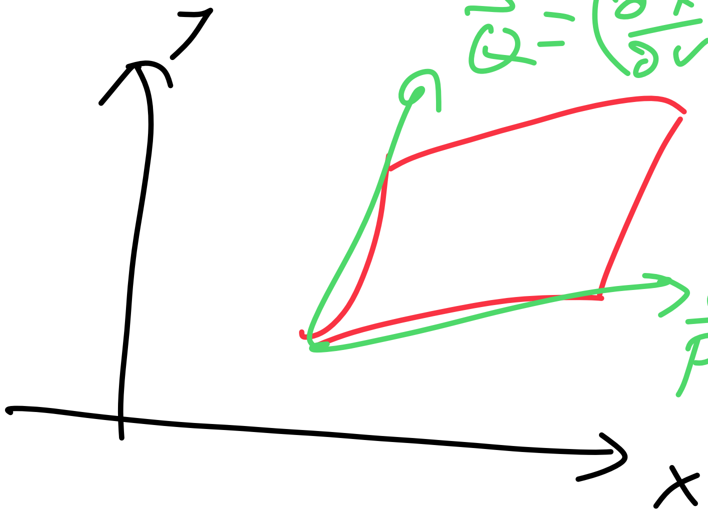


$$x(u_0+\Delta u, v_0) = \underbrace{x(u_0,v_0)}_{\text{Start}} + \underbrace{\frac{\partial x}{\partial u}(u_0,v_0)\,\Delta u}_{\text{Speed * time}}$$

Taylor: $f(x) = f(0) + f'(0)(x-0) + \cdots$

$$y(u_0+\Delta u, v_0) = y(u_0,v_0) + \frac{\partial y}{\partial u}(u_0,v_0)\,\Delta u$$





$\vec{Q} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right)\Delta v$

$\approx$ parallelogram

$\vec{P} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right)\Delta u$

Area in xy-space $\approx \vec{P} \times \vec{Q}$ or det $\begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v \end{vmatrix}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \, \Delta v$$

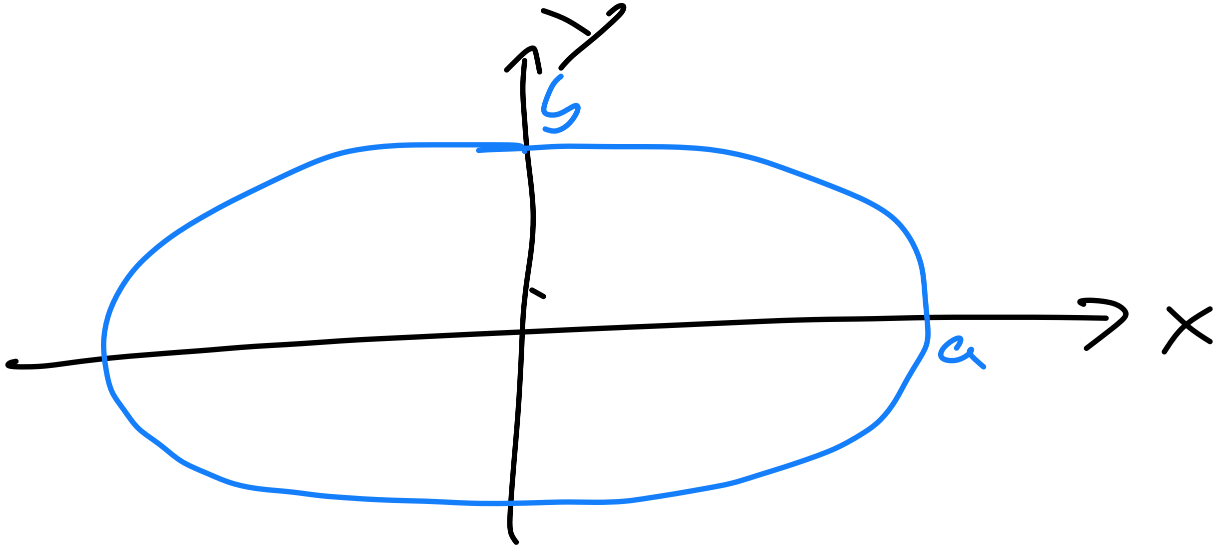
recall $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad$
$= -(ad - bc)$

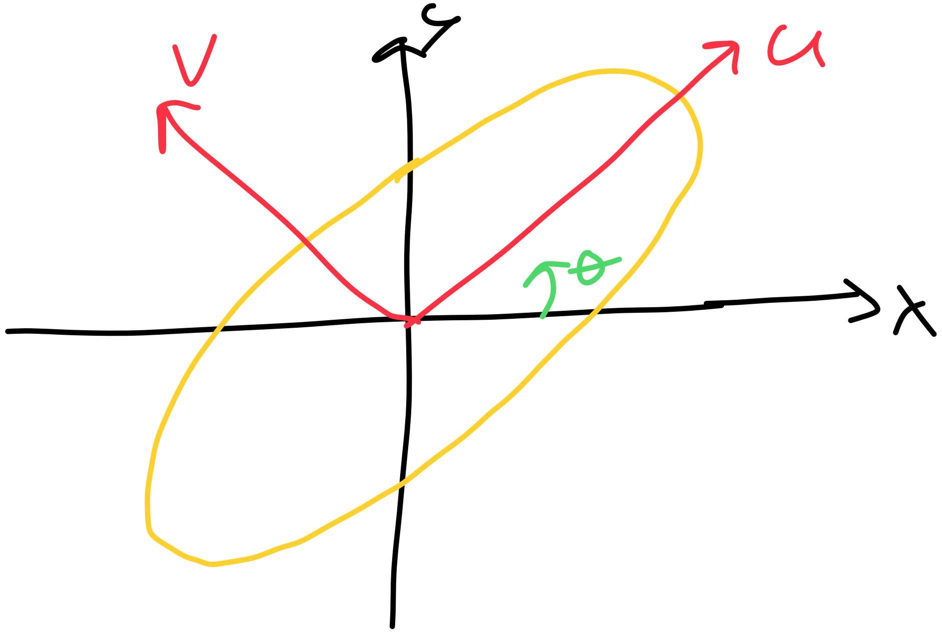
Polar: $x(r,\theta) = r\cos\theta$
$y(r,\theta) = r\sin\theta$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$
$$= r$$

$$dx\,dy \longleftrightarrow \left| \det \begin{pmatrix} x_r & y_r \\ x_\theta & y_\theta \end{pmatrix} \right| dr\,d\theta = r\,dr\,d\theta$$



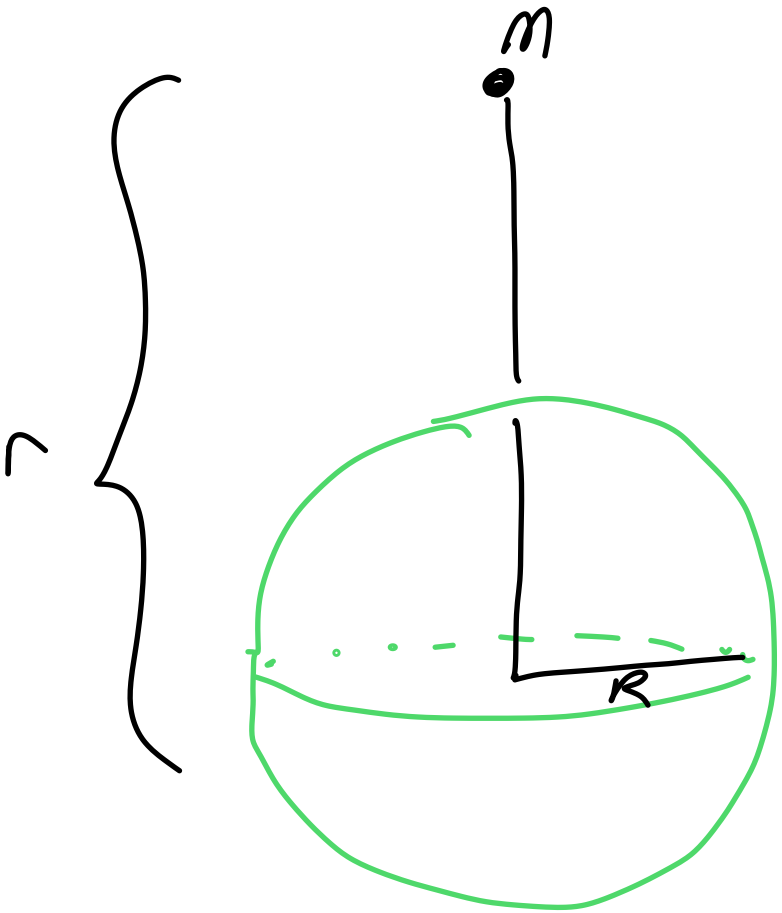
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

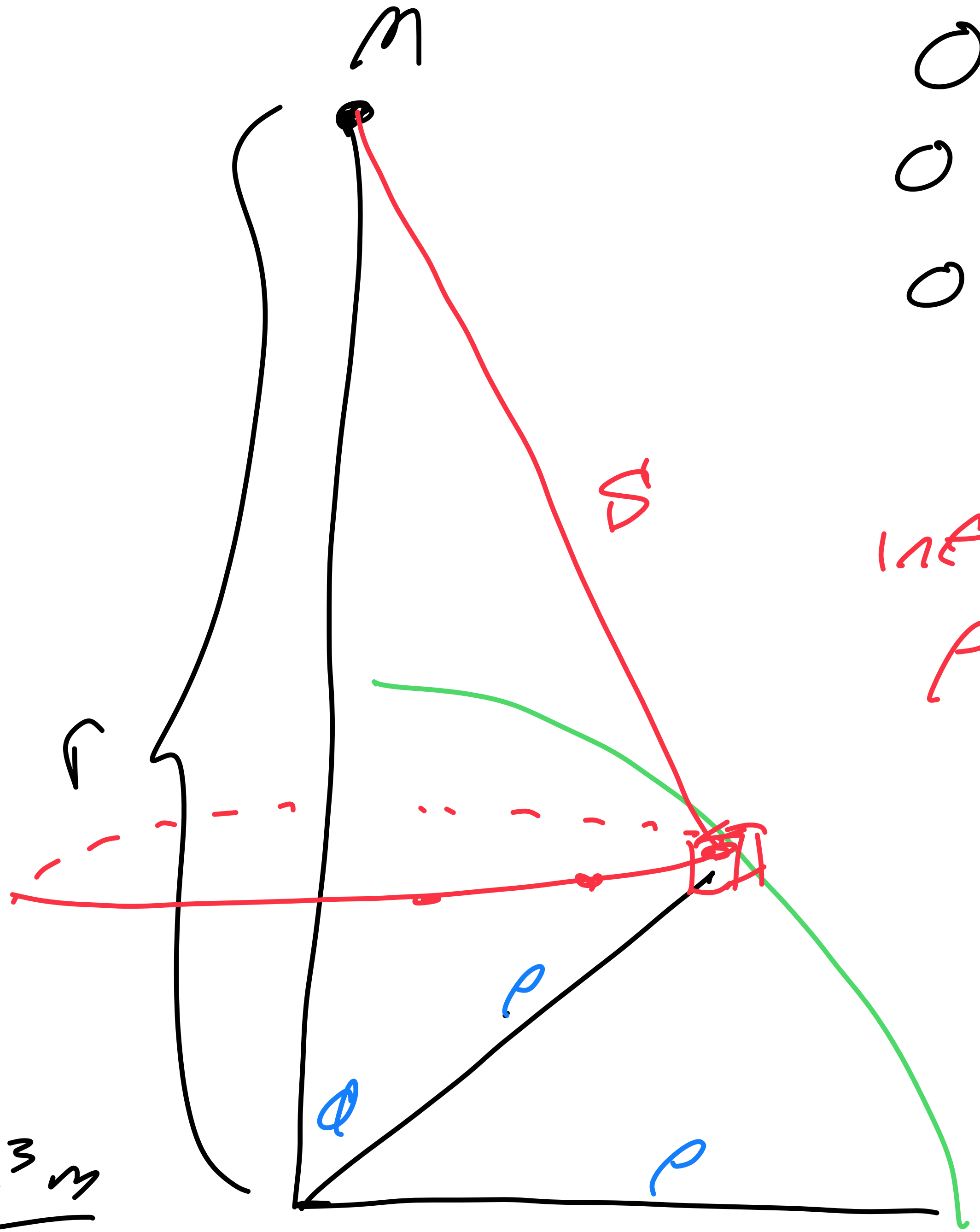


$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

$$u = u(x,y)$$

$$v = v(x,y)$$





$0 \le \rho \le R$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

infinitesimal mass is

$\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

assume density is 1

$\text{Mass} = \frac{4}{3}\pi R^3$

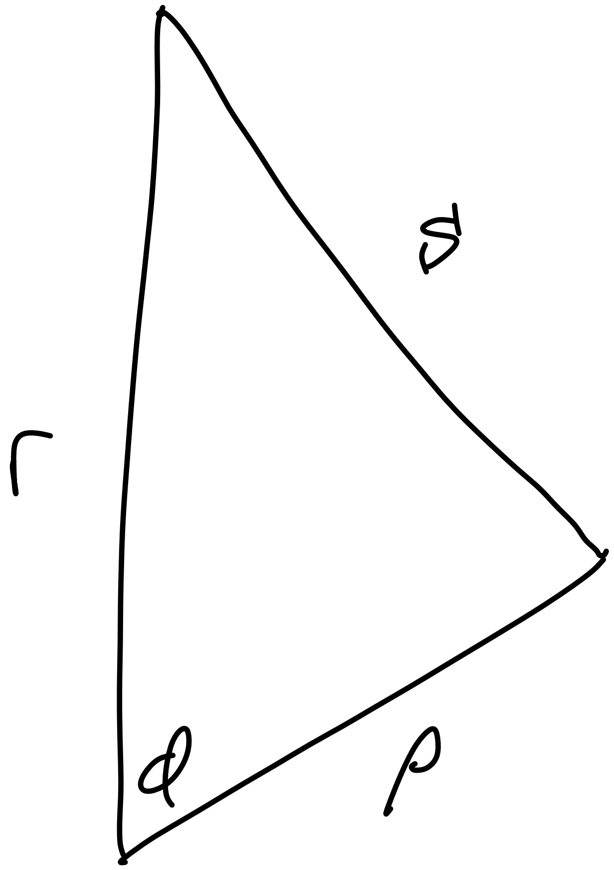
all mass at center: $G \dfrac{\frac{4}{3}\pi R^3 m}{r^2}$

WHOOPS – we forgot a key factor in the analysis!
We calculated the net force – we want just the component down!

We need to multiply by the cosine of the angle between r and s in the triangle!

Thus the calculation on the next few pages is off….

For the correct calculation see: https://www.youtube.com/watch?v=3Pt4E1BeUTw&t=104s



Use Law of Cosines!

$$s^2 = r^2 + \rho^2 - 2r\rho\cos\phi$$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{R} \frac{Gm\rho^2\sin\phi \, d\rho \, d\phi \, d\theta}{r^2+\rho^2-2r\rho\cos\phi}$$

$$2\pi Gm \int_{\rho=0}^{R}\left[\int_{\phi=0}^{\pi}\frac{\rho^2\sin\phi \, d\phi}{r^2+\rho^2-2r\rho\cos\phi}\right]d\rho$$

$$\left[\int_{\phi=0}^{\pi} \frac{\rho^2 \sin\phi \, d\phi}{r^2+\rho^2-2r\rho\cos\phi}\right]$$

$$u = r^2+\rho^2-2r\rho\cos\phi$$

$$du = 2r\rho\sin\phi\, d\phi$$

so $\rho\sin\phi\, d\theta = \frac{1}{2r}\, du$

$\phi: 0 \to \pi$, $u: \underbrace{r^2+\rho^2-2r\rho}_{\underbrace{r^2-2r\rho+\rho^2}_{r-\rho}}$ to .... $r+\rho$

$$= \int_{u=r-\rho}^{r+\rho} \frac{\rho}{2r}\frac{du}{u}$$

$$= \frac{\rho}{2r}\ln\left(\frac{r+\rho}{r-\rho}\right)$$

Need $\displaystyle\int_0^R 2\pi G m \frac{\rho}{2r}\ln\left(\frac{r+\rho}{r-\rho}\right) d\rho$

Math 150: Multivariable Calculus: Spring 2023:

Bonus Lecture: Watch Green's Theorem in a Day:

https://www.youtube.com/watch?v=aQbPrQ82K-Y

Lecture 35: Review Class: https://youtu.be/TL1xHE819-I

Plan for the day:

- Review

15.6: Change of Variables – Problems.

7. Let $G(u, v) = (2u + v, 5u + 3v)$ be a map from the uv-plane to the xy-plane. Describe the image of the line $v = 4u$ under G.
13. Calculate the Jacobian of $G(u, v) = (3u + 4v, u - 2v)$.
17. Calculate the Jacobian of $G(r, \theta) = (r\cos\theta, r\sin\theta)$.
35. Calculate

$$\int\int_{\mathcal{D}} e^{9x^2+4y^2}\,dxdy,$$

where $\mathcal{D}$ is the interior of the ellipse

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \leq 1.$$

$$\iint_R f(x,y)\,dx\,dy$$

(1) Region
(2) function
(3) Jacobean

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Region: $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \leq 1$

Generally: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1$

$(x, y, z) \longleftrightarrow (u, v, w)$

1 $\left.\begin{array}{l} u = x/a \\ v = y/b \\ w = z/c \end{array}\right\}$ or $\begin{array}{l} x = au \\ y = bv \\ z = cw \end{array}$

2 $dx\,dy\,z$

$= abc\,du\,dv\,dw$

3 (unit) sphere

$u^2 + v^2 + w^2 = 1$

$$\iiint_{\underbrace{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\le 1}_{\text{unit sphere}}} e^{\underbrace{-[(x/a)^2+(y/b)^2+(z/c)^2]^{3/2}}_{e^{-(u^2+v^2+w^2)^{3/2}}}} \underbrace{dx\,dy\,dz}_{abc\,du\,dv\,dw}$$

f(x,y,z) replaced with

f(x(u,v,w), y(u,v,w), z(u,v,w))

$$= \iiint_{u^2+v^2+w^2\le 1} e^{-(u^2+v^2+w^2)^{3/2}}\, abc\,du\,dv\,dw$$

$$\iiint e^{-(u^2+v^2+w^2)^{3/2}}\, abc\, du\, dv\, dw$$

$u^2+v^2+w^2 \leq 1$

$$= abc \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{1} e^{-\rho^3} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= abc \int_{\theta=0}^{2\pi} d\theta \int_{\phi=0}^{\pi} \sin\phi\, d\phi \int_{\rho=0}^{1} e^{-\rho^3} \rho^2 \, d\rho$$

$$= abc \cdot 2\pi \cdot 2 \cdot \int_{\rho=0}^{1} e^{-\rho^3} \cdot \rho^2 d\rho$$

$$\int_{\rho=0}^{1} e^{-\rho^3} \rho^2 d\rho$$

$t = \rho^3$

$dt = 3\rho^2 d\rho$

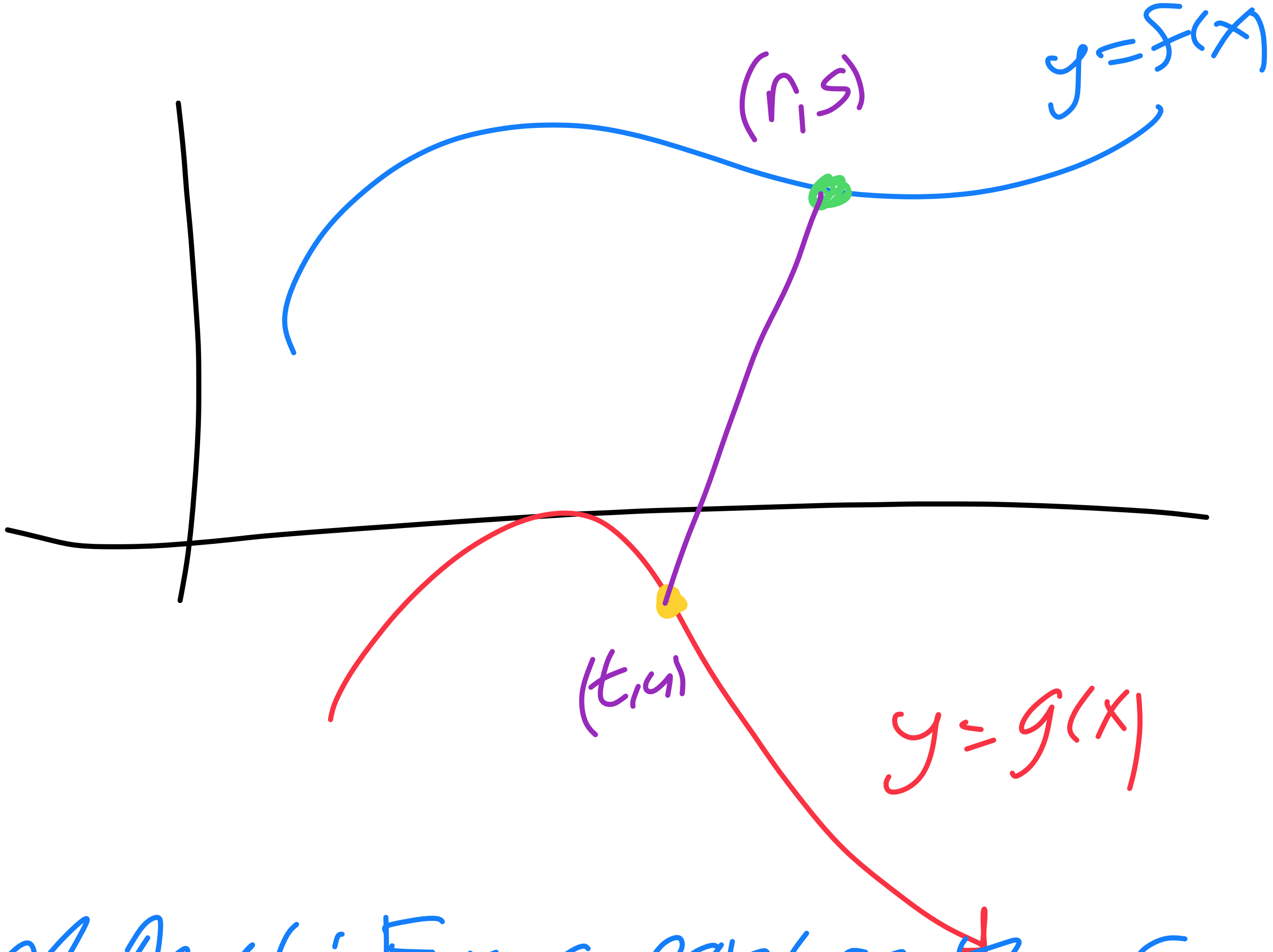
$\rho : 0 \to 1$

$t : 0 \to 1$

$\rho^2 d\rho = \frac{1}{3} dt$

$$= \int_{t=0}^{1} e^{-t} \frac{1}{3} dt = -\frac{1}{3} e^{-t} \Big|_0^1 = \frac{1}{3}\left(1 - \frac{1}{e}\right)$$

Answer: $\frac{4}{3}\pi abc\left(1 - \frac{1}{e}\right)$



distance^2 is

$$\|(r,s)-(t,u)\|^2$$

$$=(r-t)^2+(s-u)^2$$

Method 1: Fix a point on the curve $y=f(x)$, find point on $y=g(x)$ that is closest by using Lagrange Multipliers!

Fix (r,s) on curve $y=f(x)$

Find (t,u) closest with $u=g(t)$

so $\text{distance}^2 = (r-t)^2+(s-u)^2 = L(t,u)$

$c(u,t)$

constraint $u-g(t)=0$

Variables (free): t, u

$$\nabla L = \lambda \nabla C$$

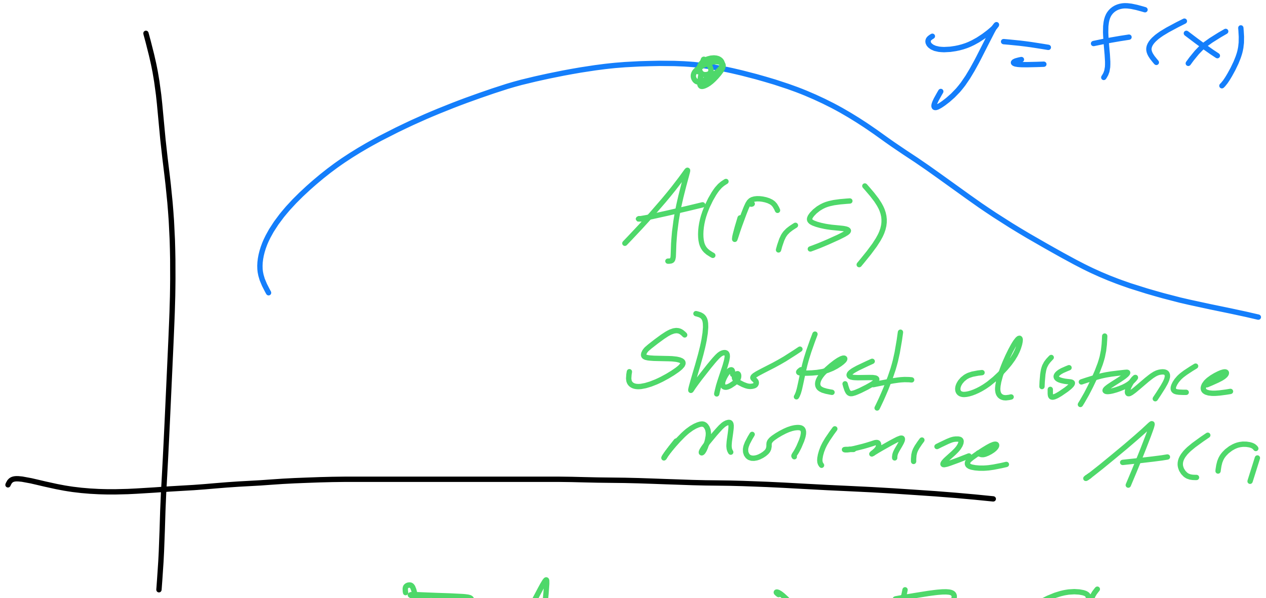
$$u-g(t)=0$$

$$\frac{\partial L}{\partial u} = \lambda \frac{\partial c}{\partial u}$$

$$\frac{\partial L}{\partial t} = \lambda \frac{\partial c}{\partial t}$$

$$u-g(t)=0$$

optimal u,v as a function of the fixed (r,s)



$G(r,s) = s - f(r) = 0$

Shortest distance to the red curve

minimize $A(r,s)$ subject to $s = f(r)$

$\nabla A = \lambda \nabla G$

subject to $s - f(r) = 0$

$$\frac{\partial A}{\partial r} = \lambda \frac{\partial G}{\partial r}$$

$$\frac{\partial A}{\partial s} = \lambda \frac{\partial G}{\partial s}$$

$$s - f(r) = 0$$

Method 2: (r,s) on blue $y=f(x)$

(t,u) on red $y=g(x)$

distance^2 is $(r-t)^2+(s-u)^2$

or $\text{dist}(r,t) = (r-t)^2 + (\underbrace{f(r)}_{=s} - \underbrace{g(t)}_{=u})^2$

Find critical points:

$$\nabla \text{dist} = \left(\frac{\partial \text{dist}}{\partial r}, \frac{\partial \text{dist}}{\partial t} \right) = (0,0)$$

$$\left(2(r-t) + 2(f(r)-g(t)) \cdot f'(r), \ \_\_\_\_ \right)$$

$$\sum_{n=0}^{\infty} \frac{150^n \cdot x^n}{n!} = \sum_{n=0}^{\infty} \frac{(150x)^n}{n!}$$

$$e^u = 1 + u + u^2/2! + u^3/3! + \cdots$$

$u = 150x$ so converges to e^{150x}

Ratio: $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{150^{n+1} x^{n+1} / (n+1)!}{150^n x^n / n!}$

$= \lim_{n\to\infty} \frac{150x}{n+1} = 150x \lim_{n\to\infty} \frac{1}{n+1} = 0$

as $\rho = 0 < 1$, converges for all x