MATH 150: MULTIVARIABLE CALCULUS: SPRING 2023
HOMEWORK PROBLEMS AND SOLUTION KEY

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ABSTRACT. A key part of any math course is doing the homework. This ranges from reading the material in the book so that you can do the problems to thinking about the problem statement, how you might go about solving it, and why some approaches work and others don't. Another important part, which is often forgotten, is how the problem fits into math. Is this a cookbook problem with made up numbers and functions to test whether or not you've mastered the basic material, or does it have important applications throughout math and industry? Below I'll try and provide some comments to place the problems and their solutions in context.

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1. Chapter 10: Infinite Series

1.1. **10.1: Sequences – Problems.** #1: Exercise 10.1.24: Determine the limit of $a_n = \frac{n}{\sqrt{n^3+1}}$. #2: Exercise 10.1.62. Find the limit of $b_n = n!/\pi^n$. #3: Exercise 10.1.68. Use L'Hopital's Rule to find the limit of $b_n = \sqrt{n}\ln\left(1+\frac{1}{n}\right)$.

1.2. **10.2: Summing an Infinite Series – Problems.** #1: Exercise 10.2.15: Find the sum of $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots$.#2: Exercise 10.2.18: Use the nth Term Divergence Test to prove that $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^2+1}}$ diverges. #3: Exercise 10.2.27: Evaluate $\sum_{n=3}^{\infty}\left(\frac{3}{11}\right)^{-n}$. #4: Exercise 10.2.37: Evaluate $\frac{7}{8}-\frac{49}{64}+\frac{343}{512}-\frac{2401}{4096}+\cdots$.

1.3. **10.3: Convergence of Series with Positive Terms – Problems.** #1: Exercise 10.3.10: Use the Integral Test to determine whether $\sum_{n=1}^{\infty} ne^{-n^2}$ is a convergent infinite series. #2: Exercise 10.3.25. Use the Direct Comparison Test to determine whether $\sum_{n=1}^{\infty}\frac{2}{3^n+3^{-n}}$ is a convergent infinite series. #3: Exercise 10.3.57: Determine convergence or divergence for $\sum_{k=1}^{\infty} 4^{1/k}$. #4: Exercise 10.3.68: Determine convergence or divergence for $\sum_{n=1}^{\infty}\frac{\sin(1/n)}{\sqrt{n}}$.

1.4. **10.4: Absolute and Conditional Convergence – Problems.** #1: Exercise 10.4.7: Determine whether the series converges absolutely, conditionally, or not at all: $\sum_{n=1}^{\infty}(-1)^n e^{-n}/n^2$. #2: Exercise 10.4.26: Determine whether the series converges absolutely, conditionally, or not at all: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}$. #3: Exercise 10.4.36: Determine whether the following series converges conditionally: $1-\frac{1}{3}+\frac{1}{2}-\frac{1}{5}+\frac{1}{3}-\frac{1}{7}+\frac{1}{4}-\frac{1}{9}+\frac{1}{5}-\frac{1}{11}+\cdots$.

1.5. **10.5: The Ratio and Root Tests and Strategies for Choosing Tests – Problems.** #1: Exercise 10.5.18: Use the Ratio Test to evaluate $\sum_{n=1}^{\infty}\frac{(n!)^3}{(3n)!}$. #2: Exercise 10.5.25: Show that $\sum_{n=1}^{\infty}\frac{r^n}{n}$ converges if $|r|<1$. #3: Exercise 10.5.40: Use the Root Test to evaluate $\sum_{n=1}^{\infty}\left(2+\frac{1}{n}\right)^{-n}$. #4: Exercise 10.5.60: Evaluate $\sum_{n=1}^{\infty}(-1)^n\cos(\frac{1}{n})$.

1.6. **10.6: Power Series – Problems.** #1: Exercise 10.6.14: Find the interval of convergence: $\sum_{n=8}^{\infty} n^7x^n$. #2: Exercise 10.6.29: Find the interval of convergence: $\sum_{n=1}^{\infty}\frac{2^n}{3n}(x+3)^n$. #3: Exercise 10.6.59: Find all values of x such that $\sum_{n=1}^{\infty}\frac{x^{n^2}}{n!}$ converges.

1.7. **10.7: Taylor Polynomials – Problems.** #1: Exercise 10.7.9: Calculate the Taylor polynomials T_2 and T_3 for $f(x)=\tan(x)$ centered at $x=0$. #2: Exercise 10.7.29: Find T_n for all n for $f(x)=\cos x$ centered at $x=\frac{\pi}{4}$. #3: Exercise 10.7.33: Find T_2 and use a calculator to compute the error $|f(x)-T_2(x)|$ for $a=1$, $x=1.2$, and $f(x)=x^{-2/3}$.

1.8. **10.8: Taylor Series – Problems.** #1: Exercise 10.8.13: Find the Maclaurin series and find the interval on which the expansion is valid for $f(x)=e^{x-2}$. #2: Exercise 10.8.15: Find the Maclaurin series and find the interval on which the expansion is valid for $f(x)=\ln(1-5x)$. #3: Exercise 10.8.37: Find the Taylor series centered at $c=4$ and the interval on which the expansion is valid for $f(x)=1/x^2$. #4: Exercise 10.8.70: Find the function with $f(x)=x^4-\frac{x^{12}}{3}+\frac{x^{20}}{5}-\frac{x^{28}}{7}+\cdots$ as its Maclaurin series. #5: Exercise 10.8.90: Use Euler's Formula to demonstrate $\cos z=\left(e^{iz}+e^{-iz}\right)/2$.

2. Chapter 11: Parametric Equations, Polar Coordinates, and Conic Sections

2.1. 11.1: Parametric Equations – Problems.

10. Express the following parametric equation in the form $y = f(x)$:
$$x = \frac{1}{1+t}, \quad y = te^t.$$
22. Find an interval of t-values such that $c(t) = (2t+1, 4t-5)$ parametrizes the segment from $(0,-7)$ to $(7,7)$.
28. Find a parametric equation for the curve
$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{12}\right)^2 = 1.$$
53. Given $(x(t), y(t)) = (t^{-1} - 3t, t^3)$, find $\frac{dy}{dx}$ at $t = -1$.
68. Given $c(t) = (t^2 - 9, t^2 - 8t)$, find the equation of the tangent line at $t = 4$.

2.2. 11.2: Arc Length and Speed – Problems.

5. Find the length of the path over the given interval:
$$(3t^2, 4t^3), \quad 1 \leq t \leq 4.$$
19. Find the speed $\frac{ds}{dt}$ of (t^2, e^t) at $t = 0$.
33. Find the surface area of the surface generated by revolving the curve $c(t) = (t^2, t)$ around the x-axis for $0 \leq t \leq 1$.

2.3. 11.3: Polar Coordinates – Problems.

The key identities to remember are:
$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2.$$

9. Find an equation in polar coordinates of the line through the origin with slope $\frac{1}{\sqrt{3}}$.
19. Convert the equation $x^2 + y^2 = 5$ into polar coordinates.
21. Convert the equation $y = x^2$ into polar coordinates.
41. Show that $r = a\cos\theta + b\sin\theta$ is the equation of a circle passing through the origin, and write the equation for the circle in rectangular (x-y) coordinates.

2.4. 11.4: Area and Arc Length in Polar Coordinates – Problems.

3. Calculate the area of the circle $r = 4\sin\theta$ using polar integration.
11. Find the area of the intersection of the circles $r = 2\sin\theta$ and $r = 2\cos\theta$.
27. Find the arc length of the curve $r = \theta^2$ for $0 \leq \theta \leq \pi$.

2.5. 11.5: Conic Sections – Problems.

A lot of this material is kind of niche so I'm not sure if its necessary to learn (ask Miller).

11. Find the equation of the ellipse with vertices at $(\pm 3, 0)$ and $(0, \pm 5)$.
15. Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and foci $(\pm 5, 0)$.
21. Find the equation of the parabola with vertex $(0,0)$ and focus $(\frac{1}{12}, 0)$
29. Find the vertices, foci and center of the ellipse $x^2 + 4y^2 = 16$.
55. Identify the type (ellipse, hyperbola, parabola) and eccentricity of the conic defined by
$$r = \frac{8}{1 + 4\cos\theta}$$

3. Chapter 12: Vector Geometry

3.1. **12.1: Vectors in the Plane – Problems.** #1: Exercise 12.1.44: Determine the unit vector e_w, where $w = \langle 24, 7 \rangle$. #2: Exercise 12.1.49: Determine the unit vector that makes an angle of $4\pi/7$ with the x-axis. #3: Exercise 12.1.52: Determine the unit vector that points in the direction from $(-3, 4)$ to the origin.

3.2. **12.2: Three-Dimensional Space: Surfaces, Vectors, and Curves – Problems.** #1: Exercise 12.2.34: Describe the surface given by the equation $x^2 + y^2 + z^2 = 9$, with $x, y, z \geq 0$. #2: Exercise 12.2.38: Give an equation for the sphere centered at the origin passing through $(1, 2, -3)$. #3: Exercise 12.2.50: Find a vector parametrization for the line passing through $(1, 1, 1)$ which is parallel to the line passing through $(2, 0, -1)$ and $(4, 1, 3)$.

3.3. **12.3: Dot Product and the Angle Between Two Vectors – Problems.** #0: Exercise 12.3.13: Determine whether $\langle 1, 1, 1 \rangle$ and $\langle 1, -2, -2 \rangle$ are orthogonal, and, if not, whether the angle between them is acute or obtuse. #1: Exercise 12.3.25: Find the angle between $\langle 1, 1, 1 \rangle$ and $\langle 1, 0, 1 \rangle$. #2: Exercise 12.3.57: Find the projection of $u = \langle -1, 2, 0 \rangle$ along $v = \langle 2, 0, 1 \rangle$. #3: Exercise 12.3.64: Compute the component of $u = \langle 3, 0, 9 \rangle$ along $v = \langle 1, 2, 2 \rangle$.

3.4. **12.4: The Cross Product – Problems.** #0: Preliminary Question 12.4.6: When is the cross product $v \times w$ equal to zero? #1: Exercise 12.4.16: Calculate $(j - k) \times (j + k)$. #2: Exercise 12.4.30: What are the possible angles θ between two unit vectors e and f if $||e \times f|| = 1/2$?

3.5. **12.5: Planes in 3-Space – Problems.** #1: Exercise 12.5.13: Find a vector normal to the plane specified by $9x - 4y - 11z = 2$. #2: Exercise 12.5.18: Find the equation of the plane that passes through $(4, 1, 9)$ and is parallel to $x + y + z = 3$. #3: Exercise 12.5.48: Find the trace of the plane specified by $3x + 4z = -2$ in the xz coordinate plane. #4: Exercise 12.5.63: Find an equation of a plane making an angle of $\pi/2$ with the plane $3x + y - 4z = 2$.

3.6. **12.6: A Survey of Quadric Surfaces – Problems.** #1: Exercise 12.6.30: Sketch $(x/4)^2 + (y/8)^2 + (z/12)^2 = 1$. #2: Exercise 12.6.33: Sketch $z^2 = (x/4)^2 + (y/8)^2$. #3: Exercise 12.6.39: Sketch $x = 1 + y^2 + z^2$. #4: Exercise 12.6.42: Sketch $y^2 - 4x^2 - z^2 = 4$.

3.7. **12.7: Cylindrical and Spherical Coordinates – Problems.** #1: Exercise 12.7.12: Describe $x^2 + y^2 + z^2 \leq 10$ in cylindrical coordinates. #2: Exercise 12.7.15: Describe $x^2 + y^2 \leq 9$, with $x \geq y$, in cylindrical coordinates. #3: Exercise 12.7.50: Describe $x^2 + y^2 + z^2 = 1$, with $z \geq 0$, in spherical coordinates. #4: Exercise 12.7.54: Describe $x^2 + y^2 = 3z^2$ in spherical coordinates.

4. Chapter 13: Calculus of Vector-Valued Functions

4.1. **13.1: Vector-Valued Functions – Problems.**

5. Find a vector parametization of the line through $P = (3, -5, 7)$ in the direction $v = \langle 3, 0, 1\rangle$.
19. The function $\mathbf{r}(t) = (9\cos t)\mathbf{i} + (9\sin t)\mathbf{j}$ traces a circle. Determine the radius and center of the circle.
29. Parametrize the intersection of the surfaces
$$y^2 - z^2 = x - 2, \qquad y^2 + z^2 = 9$$
where $z \geq 0$. Use $t = y$ as the parameter.
37. Determine whether the two curves
$$\mathbf{r}_1(t) = \langle t^2 + 3, t + 1, 6t^{-1}\rangle, \qquad \mathbf{r}_2(t) = \langle 4t, 2t - 2, t^2 - 7\rangle.$$

4.2. **13.2: Calculus of Vector-Valued Functions – Problems.**

7. Find the derivative of $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$.
25. Find the derivative of $\mathbf{r}(g(t))$, where $\mathbf{r}(t) = \langle t^2, 1 - t\rangle$ and $g(t) = e^t$ (use the chain rule).
31. Find a parameterization of the tangent line to $\mathbf{r}(t) = \langle t^2, t^4\rangle$ at $t = -2$.
43. Evaluate the integral
$$\int_{-2}^{2} (u^3\mathbf{i} + u^5\mathbf{j})du.$$

4.3. **13.3: Arc Length and Speed – Problems.**

3. Compute the arc length of the curve
$$\mathbf{r}(t) = \langle 2t, \ln t, t^2\rangle \qquad , 1 \leq t \leq 4.$$
5. Compute the arc length of the curve
$$\mathbf{r}(t) = \langle t, 4t^{3/2}, 2t^{3/2}\rangle \qquad , 0 \leq t \leq 3.$$
17. Find the speed of
$$\mathbf{r}(t) = \langle \sin 3t, \cos 4t, \cos 5t\rangle \qquad \text{at } t = \frac{\pi}{2}.$$

4.4. **13.4: Curvature – Problems.**

3. Calculate $\mathbf{r}'(t)$ and $\mathbf{T}(t)$ for the curve $\mathbf{r}(t) = \langle 3 + 4t, 3 - 5t, 9t\rangle$.
17. Find the curvature of the curve $y = t^4$ at $t = 2$.
37. Find the normal vector $\mathbf{N}(t)$ to $\mathbf{r}(t) = \langle 4, \sin 2t, \cos 2t\rangle$.

4.5. **13.5: Motion in 3-Space – Problems.**

3. Find the velocity, acceleration, and speed of $\mathbf{r}(t) = \langle t^3, 1 - t, 4t^2\rangle$ at $t = 1$.
11. Take some curve $\mathbf{r}(t)$ such that the acceleration vector is $\mathbf{a}(t) = \langle t, 4\rangle$ and the initial velocity is $\mathbf{v}(0) = \langle \frac{1}{3}, -2\rangle$. What is the velocity vector $\mathbf{v}(t)$?
37. Given that $\mathbf{r}(t) = \langle \frac{1}{3}t^3, 1 - 3t\rangle$, find the decomposition of the acceleration $\mathbf{a}(t)$ into tangential and non-tangential components at $t = -1$.

5. Chapter 14: Differentiation in Several Variables

5.1. **14.1: Functions of Two or More Variables – Problems.** #1: Exercise 14.1.18: Describe the domain and range of $g(r,s) = \cos^{-1}(rs)$. #2: Exercise 14.1.21: Matching functions with their graphs, see book. #3: Exercise 14.1.22: Matching functions with their contour maps, see book.

5.2. **14.2: Limits and Continuity in Several Variables – Problems.** #1: Exercise 14.2.5: Using continuity, evaluate $\lim_{(x,y)\to(\pi/4,0)} \tan x \cos y$. #2: Exercise 14.2.32: Evaluate $\lim_{(x,y)\to(0,0)} xy/(\sqrt{x^2+y^2})$. #3: Exercise 14.2.40: Evaluate $\lim_{(x,y)\to(0,0)}(x+y+2)e^{-1/(x^2+y^2)}$.

5.3. **14.3: Partial Derivatives – Problems.** #1: Exercise 14.3.20: Compute the first-order partial derivatives of $z = x/(x-y)$. #2: Exercise 14.3.23: Compute the first-order partial derivatives of $z = (\sin x)(\cos y)$. #3: Exercise 14.3.35: Compute the first-order partial derivatives of $U = e^{-rt}/r$. #4: Exercise 14.3.58: Compute the derivative $g_{xy}(-3,2)$ of $g(x,y) = xe^{-xy}$. #5: Exercise 14.3.69: Find a function such that $\partial f/\partial x = 2xy$ and $\partial f/\partial y = x^2$.

5.4. **14.4: Differentiability, Tangent Planes, and Linear Approximation – Problems.** #1: Exercise 14.4.5: Find an equation of the tangent plane at $(4,1)$ of $f(x,y) = x^2 + y^{-2}$. #2: Exercise 14.4.14: Find the points on the graph of $f(x,y) = (x+1)y^2$ at which the tangent plane is horizontal. #3: Exercise 14.4.23: Estimate $f(2.1, 3.8)$ assuming that $f(2,4) = 5$, $f_x(2,4) = 0.3$, and $f_y(2,4) = -0.2$.

5.5. **14.5: The Gradient and Directional Derivatives – Problems.** #1: Exercise 14.5.24: Calculate the directional derivative of $\sin(x-y)$ at $P = (\pi/2, \pi/6)$ in the direction of $v = \langle 1, 1\rangle$. #2: Exercise 14.5.35: Determine the direction in which $f(x,y,z) = xy/z$ has maximum rate of increase from $P = (1,-1,3)$, and give the rate of change in that direction. #3: Exercise 14.5.41: Find a vector normal to the surface $x^2 + y^2 - z^2 = 6$ at $P = (3,1,2)$. #4: Exercise 14.5.55: Find a function $f(x,y,z)$ such that $\nabla f = \langle z, 2y, x\rangle$.

5.6. **14.6: Multivariable Calculus Chain Rules – Problems.** #1: Exercise 14.6.8: Use the Chain Rule to calculate $\partial f/\partial u$ for $f(x,y) = x^2 + y^2$, $x = e^{u+v}$, $y = u + v$. #2: Exercise 14.6.12: Use the Chain Rule to evaluate $\partial f/\partial s$ at $(r,s) = (1,0)$, where $f(x,y) = \ln(xy)$, $x = 3r + 2s$, and $y = 5r + 3s$. #3: Exercise 14.6.31: Use implicit differentiation to calculate $\partial z/\partial y$ for $e^{xy} + \sin(xz) + y = 0$.

5.7. **14.7: Optimization in Several Variables – Problems.** #1: Exercise 14.7.12: Find the critical points of $f(x,y) = x^3 + y^4 - 6x - 2y^2$, then apply the Second Derivative Test. #2: Exercise 14.7.17: Find the critical points of $f(x,y) = \sin(x+y) - \cos x$, then apply the Second Derivative Test. #3: Exercise 14.7.24: Show that $f(x,y) = x^2$ has infinitely many critical points (as a function of two variables) and that the Second Derivative Test fails for all of them. What is the minimum value of f? Does $f(x,y)$ have an local maxima?

5.8. **14.8: Lagrange Multipliers: Optimizing with a Constraint – Problems.** #1: Exercise 14.8.10: Find the minimum and maximum vales of $f(x,y) = x^2y^4$, subject to the constraint $x^2 + 2y^2 = 6$. #2: Exercise 14.8.15: Find the minimum and maximum vales of $f(x,y) = xy + xz$, subject to the constraint $x^2 + y^2 + z^2 = 4$. #3: Exercise 14.8.18: Find the rectangular box of maximum volume if the sum of the lengths of the edges is 300 cm.

5.9. **15.1: Integration in Two Variables – Problems.**

15. Evaluate the integral
$$\int\int_{\mathcal{R}} x^3 dA,$$
where $\mathcal{R} = [-4,4] \times [0,5]$.
27. Evaluate
$$\int_0^1 \int_0^2 x + 4y^3 dxdy.$$
31. Evaluate
$$\int_1^2 \int_2^4 e^{3x-y} dydx.$$
41. Evaluate
$$\int\int_{\mathcal{R}} e^x \sin y dA,$$
where $\mathcal{R} = [0,2] \times [0, \frac{\pi}{4}]$.

5.10. **15.2: Double Integrals over More General Regions – Problems.**

13. Calculate the double integral of $f(x,y) = x + y$ over the domain $\mathcal{D} = \{(x,y) : x^2 + y^2 \le 4, \quad y \ge 0\}$ (this is a semicircle of radius 2).
17. Calculate the double integral of $f(x,y) = x^3 y$ over the domain $\mathcal{D} = \{(x,y) : 0 \le x \le 5, \quad x \le y \le 2x+3\}$.
45. Find the volume of the region bounded by $z = 40 - 10y$, $z = 0$, $y = 0$, $y = 4 - x^2$.

5.11. **15.3: Triple Integrals – Problems.**

1. Integrate $f(x,y,z) = xz + yz^2$ over the region
$$0 \le x \le 2, \quad 2 \le y \le 4, \quad 0 \le z \le 4.$$
11. Integrate $f(x,y,z) = xyz$ over the region
$$0 \le z \le 1, \quad 0 \le y \le \sqrt{1-x^2}, \quad 0 \le x \le 1.$$
33. Let $\mathcal{W}$ be the region bounded by $z = 1 - y^2$, $y = x^2$ and the plane $z = 0$. Calculate the volume of $\mathcal{W}$ as a triple integral.

5.12. **15.4: Integration in Polar, Cylindrical, and Spherical Coordinates – Problems.**

7. Calculate the following integral by changing to polar coordinates:
$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} x^2 + y^2 dy dx.$$
27. Integrate $f(x,y,z) = x^2 + y^2$ over the region $x^2 + y^2 \le 9, 0 \le z \le 5$ by changing to cylindrical coordinates.
45. Integrate $f(x,y,z) = y$ over the region $x^2 + y^2 + z^2 \le 1, x, y, z \le 0$ by changing to spherical coordinates.

5.13. **15.6: Change of Variables – Problems.**

7. Let $G(u,v) = (2u + v, 5u + 3v)$ be a map from the uv-plane to the xy-plane. Describe the image of the line $v = 4u$ under G.
13. Calculate the Jacobian of $G(u,v) = (3u + 4v, u - 2v)$.
17. Calculate the Jacobian of $G(r,\theta) = (r\cos\theta, r\sin\theta)$.
35. Calculate
$$\int \int_{\mathcal{D}} e^{9x^2 + 4y^2} dx dy,$$
where $\mathcal{D}$ is the interior of the ellipse
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \le 1.$$