MATH 150: MULTIVARIABLE CALCULUS: SPRING 2023
HOMEWORK PROBLEMS AND SOLUTION KEY

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ABSTRACT. A key part of any math course is doing the homework. This ranges from reading the material in the book so that you can do the problems to thinking about the problem statement, how you might go about solving it, and why some approaches work and others don't. Another important part, which is often forgotten, is how the problem fits into math. Is this a cookbook problem with made up numbers and functions to test whether or not you've mastered the basic material, or does it have important applications throughout math and industry? Below I'll try and provide some comments to place the problems and their solutions in context.

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2. Chapter 11: Parametric Equations, Polar Coordinates, and Conic Sections

2.1. 11.1: Parametric Equations – Problems.

10. Express the following parametric equation in the form $y = f(x)$:
$$x = \frac{1}{1+t}, \quad y = te^t.$$
22. Find an interval of t-values such that $c(t) = (2t+1, 4t-5)$ parametrizes the segment from $(0, -7)$ to $(7, 7)$.
28. Find a parametric equation for the curve
$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{12}\right)^2 = 1.$$
53. Given $(x(t), y(t)) = (t^{-1} - 3t, t^3)$, find $\frac{dy}{dx}$ at $t = -1$.
68. Given $c(t) = (t^2 - 9, t^2 - 8t)$, find the equation of the tangent line at $t = 4$.

2.2. 11.2: Arc Length and Speed – Problems.

5. Find the length of the path over the given interval:
$$(3t^2, 4t^3), \quad 1 \leq t \leq 4.$$
19. Find the speed $\frac{ds}{dt}$ of (t^2, e^t) at $t = 0$.
33. Find the surface area of the surface generated by revolving the curve $c(t) = (t^2, t)$ around the x-axis for $0 \leq t \leq 1$.

2.3. 11.3: Polar Coordinates – Problems.

The key identities to remember are:
$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2.$$
9. Find an equation in polar coordinates of the line through the origin with slope $\frac{1}{\sqrt{3}}$.
19. Convert the equation $x^2 + y^2 = 5$ into polar coordinates.
21. Convert the equation $y = x^2$ into polar coordinates.
41. Show that $r = a\cos\theta + b\sin\theta$ is the equation of a circle passing through the origin, and write the equation for the circle in rectangular (x-y) coordinates.

2.4. 11.4: Area and Arc Length in Polar Coordinates – Problems.

3. Calculate the area of the circle $r = 4\sin\theta$ using polar integration.
11. Find the area of the intersection of the circles $r = 2\sin\theta$ and $r = 2\cos\theta$.
27. Find the arc length of the curve $r = \theta^2$ for $0 \leq \theta \leq \pi$.

2.5. 11.5: Conic Sections – Problems.

A lot of this material is kind of niche so I'm not sure if its necessary to learn (ask Miller).

11. Find the equation of the ellipse with vertices at $(\pm 3, 0)$ and $(0, \pm 5)$.
15. Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and foci $(\pm 5, 0)$.
21. Find the equation of the parabola with vertex $(0, 0)$ and focus $(\frac{1}{12}, 0)$
29. Find the vertices, foci and center of the ellipse $x^2 + 4y^2 = 16$.
55. Identify the type (ellipse, hyperbola, parabola) and eccentricity of the conic defined by
$$r = \frac{8}{1 + 4\cos\theta}$$

3. Chapter 12: Vector Geometry

3.1. **12.1: Vectors in the Plane – Problems.** #1: Exercise 12.1.44: Determine the unit vector e_w, where $w = \langle 24, 7\rangle$. #2: Exercise 12.1.49: Determine the unit vector that makes an angle of $4\pi/7$ with the x-axis. #3: Exercise 12.1.52: Determine the unit vector that points in the direction from $(-3, 4)$ to the origin.

3.2. **12.2: Three-Dimensional Space: Surfaces, Vectors, and Curves – Problems.** #1: Exercise 12.2.34: Describe the surface given by the equation $x^2 + y^2 + z^2 = 9$, with $x, y, z \geq 0$. #2: Exercise 12.2.38: Give an equation for the sphere centered at the origin passing through $(1, 2, -3)$. #3: Exercise 12.2.50: Find a vector parametrization for the line passing through $(1, 1, 1)$ which is parallel to the line passing through $(2, 0, -1)$ and $(4, 1, 3)$.

3.3. **12.3: Dot Product and the Angle Between Two Vectors – Problems.** #0: Exercise 12.3.13: Determine whether $\langle 1, 1, 1\rangle$ and $\langle 1, -2, -2\rangle$ are orthogonal, and, if not, whether the angle between them is acute or obtuse. #1: Exercise 12.3.25: Find the angle between $\langle 1, 1, 1\rangle$ and $\langle 1, 0, 1\rangle$. #2: Exercise 12.3.57: Find the projection of $u = \langle -1, 2, 0\rangle$ along $v = \langle 2, 0, 1\rangle$. #3: Exercise 12.3.64: Compute the component of $u = \langle 3, 0, 9\rangle$ along $v = \langle 1, 2, 2\rangle$.

3.4. **12.4: The Cross Product – Problems.** #0: Preliminary Question 12.4.6: When is the cross product $v \times w$ equal to zero? #1: Exercise 12.4.16: Calculate $(j - k) \times (j + k)$. #2: Exercise 12.4.30: What are the possible angles θ between two unit vectors e and f if $||e \times f|| = 1/2$?

3.5. **12.5: Planes in 3-Space – Problems.** #1: Exercise 12.5.13: Find a vector normal to the plane specified by $9x - 4y - 11z = 2$. #2: Exercise 12.5.18: Find the equation of the plane that passes through $(4, 1, 9)$ and is parallel to $x + y + z = 3$. #3: Exercise 12.5.48: Find the trace of the plane specified by $3x + 4z = -2$ in the xz coordinate plane. #4: Exercise 12.5.63: Find an equation of a plane making an angle of $\pi/2$ with the plane $3x + y - 4z = 2$.

3.6. **12.6: A Survey of Quadric Surfaces – Problems.** #1: Exercise 12.6.30: Sketch $(x/4)^2 + (y/8)^2 + (z/12)^2 = 1$. #2: Exercise 12.6.33: Sketch $z^2 = (x/4)^2 + (y/8)^2$. #3: Exercise 12.6.39: Sketch $x = 1 + y^2 + z^2$. #4: Exercise 12.6.42: Sketch $y^2 - 4x^2 - z^2 = 4$.

3.7. **12.7: Cylindrical and Spherical Coordinates – Problems.** #1: Exercise 12.7.12: Describe $x^2 + y^2 + z^2 \leq 10$ in cylindrical coordinates. #2: Exercise 12.7.15: Describe $x^2 + y^2 \leq 9$, with $x \geq y$, in cylindrical coordinates. #3: Exercise 12.7.50: Describe $x^2 + y^2 + z^2 = 1$, with $z \geq 0$, in spherical coordinates. #4: Exercise 12.7.54: Describe $x^2 + y^2 = 3z^2$ in spherical coordinates.

4. Chapter 13: Calculus of Vector-Valued Functions

4.1. **13.1: Vector-Valued Functions – Problems.**

5. Find a vector parametization of the line through $P = (3, -5, 7)$ in the direction $v = \langle 3, 0, 1\rangle$.
19. The function $\mathbf{r}(t) = (9\cos t)\mathbf{i} + (9\sin t)\mathbf{j}$ traces a circle. Determine the radius and center of the circle.
29. Parametrize the intersection of the surfaces
$$y^2 - z^2 = x - 2, \qquad y^2 + z^2 = 9$$
where $z \geq 0$. Use $t = y$ as the parameter.
37. Determine whether the two curves
$$\mathbf{r}_1(t) = \langle t^2 + 3, t + 1, 6t^{-1}\rangle, \qquad \mathbf{r}_2(t) = \langle 4t, 2t - 2, t^2 - 7\rangle.$$

4.2. **13.2: Calculus of Vector-Valued Functions – Problems.**

7. Find the derivative of $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$.
25. Find the derivative of $\mathbf{r}(g(t))$, where $\mathbf{r}(t) = \langle t^2, 1 - t\rangle$ and $g(t) = e^t$ (use the chain rule).
31. Find a parameterization of the tangent line to $\mathbf{r}(t) = \langle t^2, t^4\rangle$ at $t = -2$.
43. Evaluate the integral
$$\int_{-2}^{2} (u^3\mathbf{i} + u^5\mathbf{j})du.$$

4.3. **13.3: Arc Length and Speed – Problems.**

3. Compute the arc length of the curve
$$\mathbf{r}(t) = \langle 2t, \ln t, t^2\rangle \qquad , 1 \leq t \leq 4.$$
5. Compute the arc length of the curve
$$\mathbf{r}(t) = \langle t, 4t^{3/2}, 2t^{3/2}\rangle \qquad , 0 \leq t \leq 3.$$
17. Find the speed of
$$\mathbf{r}(t) = \langle \sin 3t, \cos 4t, \cos 5t\rangle \qquad \text{at } t = \frac{\pi}{2}.$$

4.4. **13.4: Curvature – Problems.**

3. Calculate $\mathbf{r}'(t)$ and $\mathbf{T}(t)$ for the curve $\mathbf{r}(t) = \langle 3 + 4t, 3 - 5t, 9t\rangle$.
17. Find the curvature of the curve $y = t^4$ at $t = 2$.
37. Find the normal vector $\mathbf{N}(t)$ to $\mathbf{r}(t) = \langle 4, \sin 2t, \cos 2t\rangle$.

4.5. **13.5: Motion in 3-Space – Problems.**

3. Find the velocity, acceleration, and speed of $\mathbf{r}(t) = \langle t^3, 1 - t, 4t^2\rangle$ at $t = 1$.
11. Take some curve $\mathbf{r}(t)$ such that the acceleration vector is $\mathbf{a}(t) = \langle t, 4\rangle$ and the initial velocity is $\mathbf{v}(0) = \langle \frac{1}{3}, -2\rangle$. What is the velocity vector $\mathbf{v}(t)$?
37. Given that $\mathbf{r}(t) = \langle \frac{1}{3}t^3, 1 - 3t\rangle$, find the decomposition of the acceleration $\mathbf{a}(t)$ into tangential and non-tangential components at $t = -1$.

5. Chapter 14: Differentiation in Several Variables

5.1. **14.1: Functions of Two or More Variables – Problems.** #1: Exercise 14.1.18: Describe the domain and range of $g(r,s) = \cos^{-1}(rs)$. #2: Exercise 14.1.21: Matching functions with their graphs, see book. #3: Exercise 14.1.22: Matching functions with their contour maps, see book.

5.2. **14.2: Limits and Continuity in Several Variables – Problems.** #1: Exercise 14.2.5: Using continuity, evaluate $\lim_{(x,y)\to(\pi/4,0)} \tan x \cos y$. #2: Exercise 14.2.32: Evaluate $\lim_{(x,y)\to(0,0)} xy/(\sqrt{x^2+y^2})$. #3: Exercise 14.2.40: Evaluate $\lim_{(x,y)\to(0,0)}(x+y+2)e^{-1/(x^2+y^2)}$.

5.3. **14.3: Partial Derivatives – Problems.** #1: Exercise 14.3.20: Compute the first-order partial derivatives of $z = x/(x-y)$. #2: Exercise 14.3.23: Compute the first-order partial derivatives of $z = (\sin x)(\cos y)$. #3: Exercise 14.3.35: Compute the first-order partial derivatives of $U = e^{-rt}/r$. #4: Exercise 14.3.58: Compute the derivative $g_{xy}(-3,2)$ of $g(x,y) = xe^{-xy}$. #5: Exercise 14.3.69: Find a function such that $\partial f/\partial x = 2xy$ and $\partial f/\partial y = x^2$.

5.4. **14.4: Differentiability, Tangent Planes, and Linear Approximation – Problems.** #1: Exercise 14.4.5: Find an equation of the tangent plane at $(4,1)$ of $f(x,y) = x^2 + y^{-2}$. #2: Exercise 14.4.14: Find the points on the graph of $f(x,y) = (x+1)y^2$ at which the tangent plane is horizontal. #3: Exercise 14.4.23: Estimate $f(2.1, 3.8)$ assuming that $f(2,4) = 5$, $f_x(2,4) = 0.3$, and $f_y(2,4) = -0.2$.

5.5. **14.5: The Gradient and Directional Derivatives – Problems.** #1: Exercise 14.5.24: Calculate the directional derivative of $\sin(x-y)$ at $P = (\pi/2, \pi/6)$ in the direction of $v = \langle 1,1\rangle$. #2: Exercise 14.5.35: Determine the direction in which $f(x,y,z) = xy/z$ has maximum rate of increase from $P = (1,-1,3)$, and give the rate of change in that direction. #3: Exercise 14.5.41: Find a vector normal to the surface $x^2+y^2-z^2 = 6$ at $P = (3,1,2)$. #4: Exercise 14.5.55: Find a function $f(x,y,z)$ such that $\nabla f = \langle z, 2y, x\rangle$.

5.6. **14.6: Multivariable Calculus Chain Rules – Problems.** #1: Exercise 14.6.8: Use the Chain Rule to calculate $\partial f/\partial u$ for $f(x,y) = x^2+y^2$, $x = e^{u+v}$, $y = u+v$. #2: Exercise 14.6.12: Use the Chain Rule to evaluate $\partial f/\partial s$ at $(r,s) = (1,0)$, where $f(x,y) = \ln(xy)$, $x = 3r+2s$, and $y = 5r+3s$. #3: Exercise 14.6.31: Use implicit differentiation to calculate $\partial z/\partial y$ for $e^{xy} + \sin(xz) + y = 0$.

5.7. **14.7: Optimization in Several Variables – Problems.** #1: Exercise 14.7.12: Find the critical points of $f(x,y) = x^3+y^4-6x-2y^2$, then apply the Second Derivative Test. #2: Exercise 14.7.17: Find the critical points of $f(x,y) = \sin(x+y) - \cos x$, then apply the Second Derivative Test. #3: Exercise 14.7.24: Show that $f(x,y) = x^2$ has infinitely many critical points (as a function of two variables) and that the Second Derivative Test fails for all of them. What is the minimum value of f? Does $f(x,y)$ have an local maxima?

5.8. **14.8: Lagrange Multipliers: Optimizing with a Constraint – Problems.** #1: Exercise 14.8.10: Find the minimum and maximum vales of $f(x,y) = x^2y^4$, subject to the constraint $x^2+2y^2 = 6$. #2: Exercise 14.8.15: Find the minimum and maximum vales of $f(x,y) = xy + xz$, subject to the constraint $x^2+y^2+z^2 = 4$. #3: Exercise 14.8.18: Find the rectangular box of maximum volume if the sum of the lengths of the edges is 300 cm.

6. Chapter 15: Multiple Integration

6.1. **15.1: Integration in Two Variables – Problems.**

15. Evaluate the integral
$$\int\int_{\mathcal{R}} x^3 dA,$$
where $\mathcal{R} = [-4,4]\times[0,5]$.
27. Evaluate
$$\int_0^1\int_0^2 x + 4y^3 dxdy.$$
31. Evaluate
$$\int_1^2\int_2^4 e^{3x-y} dydx.$$
41. Evaluate
$$\int\int_{\mathcal{R}} e^x \sin y dA,$$
where $\mathcal{R} = [0,2]\times[0,\frac{\pi}{4}]$.

6.2. **15.2: Double Integrals over More General Regions – Problems.**

13. Calculate the double integral of $f(x, y) = x + y$ over the domain $\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 4, \quad y \geq 0\}$ (this is a semicircle of radius 2).
17. Calculate the double integral of $f(x, y) = x^3 y$ over the domain $\mathcal{D} = \{(x, y) : 0 \leq x \leq 5, \quad x \leq y \leq 2x+3\}$.
45. Find the volume of the region bounded by $z = 40 - 10y$, $z = 0$, $y = 0$, $y = 4 - x^2$.

6.3. **15.3: Triple Integrals – Problems.**

1. Integrate $f(x, y, z) = xz + yz^2$ over the region
$$0 \leq x \leq 2, \quad 2 \leq y \leq 4, \quad 0 \leq z \leq 4.$$
11. Integrate $f(x, y, z) = xyz$ over the region
$$0 \leq z \leq 1, \quad 0 \leq y \leq \sqrt{1 - x^2}, \quad 0 \leq x \leq 1.$$
33. Let $\mathcal{W}$ be the region bounded by $z = 1 - y^2$, $y = x^2$ and the plane $z = 0$. Calculate the volume of $\mathcal{W}$ as a triple integral.

6.4. **15.4: Integration in Polar, Cylindrical, and Spherical Coordinates – Problems.**

7. Calculate the following integral by changing to polar coordinates:
$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} x^2 + y^2 dydx.$$
27. Integrate $f(x, y, z) = x^2 + y^2$ over the region $x^2 + y^2 \leq 9, 0 \leq z \leq 5$ by changing to cylindrical coordinates.
45. Integrate $f(x, y, z) = y$ over the region $x^2 + y^2 + z^2 \leq 1, x, y, z \leq 0$ by changing to spherical coordinates.

6.5. **15.6: Change of Variables – Problems.**

7. Let $G(u, v) = (2u + v, 5u + 3v)$ be a map from the uv-plane to the xy-plane. Describe the image of the line $v = 4u$ under G.
13. Calculate the Jacobian of $G(u, v) = (3u + 4v, u - 2v)$.
17. Calculate the Jacobian of $G(r, \theta) = (r \cos\theta, r \sin\theta)$.
35. Calculate
$$\int \int_{\mathcal{D}} e^{9x^2+4y^2} dxdy,$$
where $\mathcal{D}$ is the interior of the ellipse
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \leq 1.$$

Homework Solution Key

7. Chapter 10: Infinite Series

7.1. **10.1: Sequences – Problems.** #1: Exercise 10.1.24: Determine the limit of $a_n = \frac{n}{\sqrt{n^3+1}}$. #2: Exercise 10.1.62. Find the limit of $b_n = n!/\pi^n$. #3: Exercise 10.1.68. Use L'Hopital's Rule to find the limit of $b_n = \sqrt{n}\ln\left(1+\frac{1}{n}\right)$.

7.2. **Solutions.** #1: The limit is zero, as we may consider the denominator as $n^{3/2}$, such that the denominator will dominate as $n \to \infty$.

#2: This limit diverges to infinity, as the factorial in the numerator will dominate the exponential in the denominator.

#3: The sequences converges to zero. Rewrite the term as $\ln(1+1/n)/(n^{-1/2})$ so as to apply L'Hopital's Rule. By taking the derivatives of the numerator and denominator we obtain:

$$\frac{\frac{-1}{n^2+n}}{\frac{-n^{-3/2}}{2}} = \frac{2n^{3/2}}{n^2+n}.$$

Thus, the squared term forces the denominator to dominate the expression as $n \to \infty$.

7.3. **10.2: Summing an Infinite Series – Problems.** #1: Exercise 10.2.15: Find the sum of $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$.#2: Exercise 10.2.18: Use the nth Term Divergence Test to prove that $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$ diverges. #3: Exercise 10.2.27: Evaluate $\sum_{n=3}^{\infty}\left(\frac{3}{11}\right)^{-n}$. #4: Exercise 10.2.37: Evaluate $\frac{7}{8} - \frac{49}{64} + \frac{343}{512} - \frac{2401}{4096} + \cdots$.

7.4. **Solutions.** #1: **1/2.** We can utilize the fact that the two numbers in the denominators differ by 2 by writing:

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \cdots$$

By pulling out the $1/2$, the series of partial sums telescopes to:

$$\frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \to \frac{1}{2}, \text{ as } n \to \infty.$$

#2: The limit of the term is 1 as $n \to \infty$, so the sum diverges.

#3: The sum diverges, as we may rewrite it as a geometric series with $11/3$, which is greater than 1.

#4: **7/15.** This is a geometric series with $c = 7/8$ and $r = -7/8$. Thus,

$$\frac{\frac{7}{8}}{1 - \left(-\frac{7}{8}\right)} = \frac{7}{15}.$$

7.5. **10.3: Convergence of Series with Positive Terms – Problems.** #1: Exercise 10.3.10: Use the Integral Test to determine whether $\sum_{n=1}^{\infty} ne^{-n^2}$ is a convergent infinite series. #2: Exercise 10.3.25. Use the Direct Comparison Test to determine whether $\sum_{n=1}^{\infty} \frac{2}{3^n+3^{-n}}$ is a convergent infinite series. #3: Exercise 10.3.57: Determine convergence or divergence for $\sum_{k=1}^{\infty} 4^{1/k}$. #4: Exercise 10.3.68: Determine convergence or divergence for $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$.

7.6. **Solutions.** #1: Let $f(x) = xe^{-x^2}$. This function is positive, decreasing and continuous for $x \geq 1$, so the integral test applies. Furthermore we have

$$\int_1^{\infty} xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}\Big|_1^{\infty} = 0 - \left(\frac{1}{2e}\right) = \frac{-1}{2e}.$$

The integral converges, so the series must also converge.

#2: Note that $3^{-n} > 0, \forall n$. This implies that

$$\frac{2}{3^n+3^{-n}} \leq \frac{2}{3^n} = 2\cdot\frac{1}{3}^n.$$

Then the series $\sum_1^{\infty} 2\cdot\frac{1}{3}^n$ is a geometric series with $r = \frac{1}{3}$, so it converges. Thus, by the direct comparison test, we may also conclude that the given series converges.

#3: Note that as $k \to \infty$, we have $4^{1/k} \to 1$. Thus the series diverges since the terms do not converge to zero.

#4: We can apply the integral test, and the integral converges, so the series converges. To estimate the integral, consider

$$\int_1^{\infty} \frac{\sin(1/n)}{\sqrt{n}} = 2\sqrt{n}\sin(1/n) + \text{constant}$$

and as $n \to \infty$, we have $\sqrt{n}\sin(1/n) \to 0$.

7.7. **10.4: Absolute and Conditional Convergence – Problems.** #1: Exercise 10.4.7: Determine whether the series converges absolutely, conditionally, or not at all: $\sum_{n=1}^{\infty} (-1)^n e^{-n}/n^2$. #2: Exercise 10.4.26: Determine whether the series converges absolutely, conditionally, or not at all: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$. #3: Exercise 10.4.36: Determine whether the following series converges conditionally: $1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{7} + \frac{1}{4} - \frac{1}{9} + \frac{1}{5} - \frac{1}{11} + \cdots$.

7.8. **Solutions.** #1: Note that
$$\left|\frac{(-1)^n e^{-n}}{n^2}\right| = \frac{e^{-n}}{n^2} < \frac{1}{n^2}.$$
Thus, the given series is bounded above by the convergent series $\sum_1^{\infty} \frac{1}{n^2}$ and thus converges absolutely.

#2: This series converges absolutely because, for example,
$$\left|\frac{(-1)^n}{(2n+1)!}\right| = \frac{(-1)^n}{(2n+1)!} < \frac{1}{n^2}$$
.

#3: This series diverges. We can write it as
$$\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{2n+1}\right] = \sum_{n=1}^{\infty}\left[\frac{n+1}{n(2n+1)}\right].$$
We can use the integral test to see that this diverges.

7.9. **10.5: The Ratio and Root Tests and Strategies for Choosing Tests – Problems.** #1: Exercise 10.5.18: Use the Ratio Test to evaluate $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$. #2: Exercise 10.5.25: Show that $\sum_{n=1}^{\infty} \frac{r^n}{n}$ converges if $|r| < 1$. #3: Exercise 10.5.40: Use the Root Test to evaluate $\sum_{n=1}^{\infty} \left(2 + \frac{1}{n}\right)^{-n}$. #4: Exercise 10.5.60: Evaluate $\sum_{n=1}^{\infty} (-1)^n \cos(\frac{1}{n})$.

7.10. **Solutions.** #1: The series converges absolutely. We have
$$\frac{((n+1)!)^3}{(3n+3)!}\frac{(3n)!}{(n!)^3} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} < 1.$$

#2: The series does converge since
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{n+1}\cdot\frac{n}{|r|^n} = |r|\cdot\frac{n}{n+1} < 1 \text{ as } n \to \infty \text{ , since } |r| < 1$$

#3: The series converges since using the root test we have
$$\sqrt{(2+1/n)^{-n}} = \frac{n}{2n+1} < 1.$$

#4: The series diverges as $\cos(1/n) \to \cos 1 > 0$ as $n \to \infty$.

7.11. **10.6: Power Series – Problems.** #1: Exercise 10.6.14: Find the interval of convergence: $\sum_{n=8}^{\infty} n^7 x^n$. #2: Exercise 10.6.29: Find the interval of convergence: $\sum_{n=1}^{\infty} \frac{2^n}{3n} (x+3)^n$. #3: Exercise 10.6.59: Find all values of x such that $\sum_{n=1}^{\infty} \frac{x^{n^2}}{n!}$ converges.

7.12. **Solutions.** #1: The radius of convergence is $|x| < 1$ as we use the ratio test to see that
$$\left|\frac{(n+1)^7 x^{n+1}}{n^7 x^n}\right| = \left|\left(\frac{n+1}{n}\right)^7 \cdot x\right| \to |x| \text{ as } n \to \infty.$$

#2: We have
$$\lim_{n\to\infty}\left|\frac{2^{n+1}(x+3)^{n+1}}{3(n+1)}\cdot\frac{3n}{2^n(x+3)^n}\right| = |x+3|\lim_{n\to\infty}\left|\frac{6n}{3n+3}\right| = 2|x+3|.$$
Thus the series converges absolutely on the interval $-7/2 < x < -5/2$. For the endpoint $-5/2$, the series becomes $\sum_1^{\infty} 1/3n$, which is divergent as it is a multiple of the harmonic series. However, for the endpoint $-7/2$, the series becomes $\sum_1^{\infty}(-1)^n/3n$, which converges by the Alternating Series Test. Thus the radius of convergence is
$$\frac{-7}{2} \le x < \frac{-5}{2}.$$

#3: We have
$$\frac{|x|^{(n+1)^2}}{(n+1)!}\cdot\frac{n!}{|x|^{n^2}} = \frac{|x|^{2n+1}}{n+1}.$$

Clearly $\lim_{n\to\infty} \frac{|x|^{2n+1}}{n+1} = 0$ is satisfied when $|x| \leq 1$.

7.13. **10.7: Taylor Polynomials – Problems.** #1: Exercise 10.7.9: Calculate the Taylor polynomials T_2 and T_3 for $f(x) = \tan(x)$ centered at $x = 0$. #2: Exercise 10.7.29: Find T_n for all n for $f(x) = \cos x$ centered at $x = \frac{\pi}{4}$. #3: Exercise 10.7.33: Find T_2 and use a calculator to compute the error $|f(x) - T_2(x)|$ for $a = 1$, $x = 1.2$, and $f(x) = x^{-2/3}$.

7.14. **Solutions.** #1: By calculating the first three derivatives of f we can piece together the following Taylor polynomials: $T_2(x) = x$ and $T_3(x) = x + 1/3x^3$.

#2: By taking repeated derivatives of $\cos x$, we find the pattern that

$$f^{(n)}(\pi/4) = \begin{cases} (-1)^{(n+1)/2}\frac{1}{\sqrt{2}}, & \text{if } n \text{ odd} \\ (-1)^{n/2}\frac{1}{\sqrt{2}}, & n \text{ even} \end{cases}$$

from this, it follows that

$$T_n(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3$$

where in general the coefficient of $(x - \pi/4)^n$ is $\pm\frac{1}{(\sqrt{2})n!}$, with the pattern of the sign being $+, -, -, +, +, -, -, \dots$.

#3: We calculate the various derivatives to obtain $T_2(x) = 1 - 2/3(x-1) + 5/9(x-1)^2$. Thus, we have $T_2(1.2) = 8/9 \approx 0.88889$. A calculator tells us that $f(1.2) = (1.2)^{-2.3} \approx 0.88555$. Hence we have $|T_2(1.2) - f(1.2)| \approx 0.00334$.

7.15. **10.8: Taylor Series – Problems.** #1: Exercise 10.8.13: Find the Maclaurin series and find the interval on which the expansion is valid for $f(x) = e^{x-2}$. #2: Exercise 10.8.15: Find the Maclaurin series and find the interval on which the expansion is valid for $f(x) = \ln(1 - 5x)$. #3: Exercise 10.8.37: Find the Taylor series centered at $c = 4$ and the interval on which the expansion is valid for $f(x) = 1/x^2$. #4: Exercise 10.8.70: Find the function with $f(x) = x^4 - \frac{x^{12}}{3} + \frac{x^{20}}{5} - \frac{x^{28}}{7} + \cdots$ as its Maclaurin series. #5: Exercise 10.8.90: Use Euler's Formula to demonstrate $\cos z = \left(e^{iz} + e^{-iz}\right)/2$.

7.16. **Solutions.** #1; The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{x^n}{e^2 n!}.$$

It is valid for all x.

#2: The Maclaurin series is

$$-\sum_{n=1}^{\infty} \frac{5^n x^n}{n}$$

. It is valid for $|x| < 1/5$, and for $x = -1/5$.

#3: We can obtain the Taylor series for $1/x^2$ by using the Taylor series for $1/x$ and then differentiating this term-by-term. From this we obtain the answer to be

$$\sum_{n=0}^{\infty} (-1)^n (n+1) \frac{(x-4)^n}{4^{n+2}}.$$

This series is valid for $|x - 4| < 4$.

#4: The function that satisfies is $\tan^{-1}(x^4)$.

#5:

$$\begin{aligned} \left(e^{iz} + e^{-iz}\right)/2 &= \left[\cos z + i\sin z + \cos(-z) + i\sin(-z)\right]/2 \\ &= \left[\cos z + i\sin z + \cos(-z) - i\sin(-z)\right]/2 \\ &= 2\cos z/2 \\ &= \cos z \end{aligned}$$

8. Chapter 12: Vector Geometry

8.1. **12.1: Vectors in the Plane – Problems.** #1: Exercise 12.1.44: Determine the unit vector e_w, where $w = \langle 24, 7\rangle$. #2: Exercise 12.1.49: Determine the unit vector that makes an angle of $4\pi/7$ with the x-axis. #3: Exercise 12.1.52: Determine the unit vector that points in the direction from $(-3, 4)$ to the origin.

8.2. **Solutions.** #1: The length of w is $\sqrt{24^2 + 7^2} = 25$. Thus we have

$$e_w = \left\langle \frac{24}{25}, \frac{7}{25} \right\rangle.$$

#2: We have

$$e = \left\langle \cos\frac{4\pi}{7}, \sin\frac{4\pi}{7} \right\rangle = \langle -0.22, 0.97\rangle$$

.

#3: First we compute $\langle 0, 0\rangle - \langle -3, 4\rangle = \langle 3, -4\rangle$. Thus we obtain:

$$\frac{1}{\sqrt{3^2 + (-4)^2}} \langle 3, -4\rangle = \left\langle \frac{3}{\sqrt{5}}, \frac{-4}{\sqrt{5}} \right\rangle.$$

8.3. **12.2: Three-Dimensional Space: Surfaces, Vectors, and Curves – Problems.** #1: Exercise 12.2.34: Describe the surface given by the equation $x^2 + y^2 + z^2 = 9$, with $x, y, z \geq 0$. #2: Exercise 12.2.38: Give an equation for the sphere centered at the origin passing through $(1, 2, -3)$. #3: Exercise 12.2.50: Find a vector parametrization for the line passing through $(1, 1, 1)$ which is parallel to the line passing through $(2, 0, -1)$ and $(4, 1, 3)$.

8.4. **Solutions.** #1: This is a sphere of radius 3 centered at the origin, except since we have the restriction that x, y, z must all be nonnegative, this cuts it down to just one-eighth of the sphere.

#2: The solution is simply $x^2 + y^2 + z^2 = 14$.

#3: The direction can be found by

$$\langle 4, 1, 3\rangle - \langle 2, 0, -1\rangle = \langle 2, 1, 4\rangle$$

Hence we obtain the parametrization

$$r(t) = \langle 1, 1, 1\rangle + t\,\langle 2, 1, 4\rangle = \langle 1 + 2t, 1 + t, 1 + 4t\rangle\,.$$

8.5. **12.3: Dot Product and the Angle Between Two Vectors – Problems.** #0: Exercise 12.3.13: Determine whether $\langle 1, 1, 1\rangle$ and $\langle 1, -2, -2\rangle$ are orthogonal, and, if not, whether the angle between them is acute or obtuse. #1: Exercise 12.3.25: Find the angle between $\langle 1, 1, 1\rangle$ and $\langle 1, 0, 1\rangle$. #2: Exercise 12.3.57: Find the projection of $u = \langle -1, 2, 0\rangle$ along $v = \langle 2, 0, 1\rangle$. #3: Exercise 12.3.64: Compute the component of $u = \langle 3, 0, 9\rangle$ along $v = \langle 1, 2, 2\rangle$.

8.6. **Solutions.** #0: We compute the dot product $\langle 1,1,1\rangle \cdot \langle 1,-2,-2\rangle = -3$ which being negative tells us that the two vectors are not orthogonal, but rather that the angle between them is obtuse.

#1: Using the formula we obtain the solution as $\cos^{-1}(\sqrt{6}/3) \approx 0.615$.

#2: We compute

$$\frac{u \cdot v}{v \cdot v} = -2/5$$

and thus the solution is $\langle -4/5, 0, -2/5\rangle$.

#3: First compute the following dot products

$$u \cdot v = \langle 3,0,9\rangle \cdot \langle 1,2,2\rangle = 21$$

$$v \cdot v = ||v||^2 = 9$$

Then we compute

$$\left|\left|\left(\frac{u \cdot v}{v \cdot v}\right) v\right|\right| = \frac{21}{9}||v|| = 7.$$

8.7. **12.4: The Cross Product – Problems.** #0: Preliminary Question 12.4.6: When is the cross product $v \times w$ equal to zero? #1: Exercise 12.4.16: Calculate $(j-k) \times (j+k)$. #2: Exercise 12.4.30: What are the possible angles θ between two unit vectors e and f if $||e \times f|| = 1/2$?

8.8. **Solutions.** #0: The cross product $v \times w$ is equal to zero if one of the vectors v or w (or both) is the zero vector, or if v and w are parallel vectors. #1: **2i**

$$(j-k) \times j + (j-k) \times k = j \times j - k \times j + j \times k - k \times k = 2(j \times k) = 2i$$

#2: On pg. 707 a consequence of the final proof is that $||v \times w|| = ||v||\ ||w|| \sin\theta$. Thus, since e and f are unit vectors we know that $1/2 = \sin\theta$. Thus $\theta = \pi/6$ or $5\pi/6$.

8.9. **12.5: Planes in 3-Space – Problems.** #1: Exercise 12.5.13: Find a vector normal to the plane specified by $9x - 4y - 11z = 2$. #2: Exercise 12.5.18: Find the equation of the plane that passes through $(4,1,9)$ and is parallel to $x + y + z = 3$. #3: Exercise 12.5.48: Find the trace of the plane specified by $3x + 4z = -2$ in the xz coordinate plane. #4: Exercise 12.5.63: Find an equation of a plane making an angle of $\pi/2$ with the plane $3x + y - 4z = 2$.

8.10. **Solutions.** #1: Using the scalar form of the equation of the plane, a vector normal to the plane is the coefficients vector $n = \langle 9, -4, -11\rangle$.

#2: The vector form of the plane $x + y + z = 3$ is

$$\langle 1,1,1\rangle \cdot \langle x,y,z\rangle = 3.$$

Hence, we can use the normal vector $n = \langle 1,1,1\rangle$ to obtain

$$\langle 1,1,1\rangle \cdot \langle x,y,z\rangle = \langle 1,1,1\rangle \cdot \langle 4,1,9\rangle = 14 \Longrightarrow x + y + z = 14.$$

#3: The trace of the plane $3x + 4z = -2$ in the xz coordinate plane is the set of all points that satisfy the equation of the plane and the equation $y = 0$ of the xz plane. That is, the set of all points $(x, 0, z)$ such that

$$3x + 4z = -2 \to z = -\frac{3}{4}x - \frac{1}{2}$$

which is a line in the xz plane.

#4: The angle between two planes is defined as the angle between their normal vectors. We have the vector $n_1 = \langle 3, 1, -4\rangle$ as normal to the given plane.

Now, let $n \cdot \langle x,y,z\rangle = d$ denote the equation of a plane making an angle of $\pi/2$ with the given plane, with $n = \langle a,b,c\rangle$. Since the two planes are perpendicular, the dot product of their normal vectors is zero. Thus,

$$n \cdot n_1 = \langle a,b,c\rangle \cdot \langle 3,1,-4\rangle = 3a + b - 4c = 0 \Longrightarrow b = -3a + 4c.$$

Now we now that $n = \langle a, -3a + 4c, c\rangle$, which implies an equation of the form $ax + (4c - 3a)y + cz = d$. We can set a, c, d to whatever we like. So we could set $a = c = d = 1$ to obtain the answer

$$x + y + z = 1.$$

8.11. **12.6: A Survey of Quadric Surfaces – Problems.** #1: Exercise 12.6.30: Sketch $(x/4)^2 + (y/8)^2 + (z/12)^2 = 1$. #2: Exercise 12.6.33: Sketch $z^2 = (x/4)^2 + (y/8)^2$. #3: Exercise 12.6.39: Sketch $x = 1 + y^2 + z^2$. #4: Exercise 12.6.42: Sketch $y^2 - 4x^2 - z^2 = 4$.

8.12. **Solutions.** #1: This function is an ellipsoid.

#2: This is an elliptic cone centered at the origin, opening along the z-axis, and having lengths of 8 in the y direction, 4 in the x direction, and 1 in the z direction.

#3: This is the equation of an elliptic paraboloid oriented along the x-axis, and encircling the y-axis.

#4: This is a two-sheeted hyperboloid encircling the y-axis (i.e. the two sheets are opening in the y direction).

8.13. **12.7: Cylindrical and Spherical Coordinates – Problems.** #1: Exercise 12.7.12: Describe $x^2 + y^2 + z^2 \leq 10$ in cylindrical coordinates. #2: Exercise 12.7.15: Describe $x^2 + y^2 \leq 9$, with $x \geq y$, in cylindrical coordinates. #3: Exercise 12.7.50: Describe $x^2 + y^2 + z^2 = 1$, with $z \geq 0$, in spherical coordinates. #4: Exercise 12.7.54: Describe $x^2 + y^2 = 3z^2$ in spherical coordinates.

8.14. **Solutions.** #1: This is a sphere and we can rewrite in cylindrical coordinates as $r^2 + z^2 \leq 10$.

#2: The equation $x^2 + y^2 \leq 9$ in cylindrical coordinates becomes $r^2 \leq 9$, which is just $r \leq 3$. However, since $x \geq y$, the projection of the set onto the xy plane must be below and to the right of the line $y = x$. Thus, θ must be restricted to $-3\pi/4 \leq \theta \leq \pi/4$.

#3: This is the unit ball, so we simply substitute in ρ to get $\rho = 1$.

#4: By adding z^2 to both sides, we can substitute in ρ^2 to obtain $\rho^2 = 4z^2 \Longrightarrow \rho = 2z$. We then can substitute in $z = \rho \cos \phi$ and simplify to obtain $\cos \phi = 1/2$.

$$\Longrightarrow \phi = \pi/3.$$

Intuitively, we can see that this is correct because the rectangular coordinate equation is a cone.

9. Chapter 13: Calculus of Vector-Valued Functions

9.1. **13.1: Vector-Valued Functions – Problems.**

5. Find a vector parametization of the line through $P = (3, -5, 7)$ in the direction $v = \langle 3, 0, 1\rangle$.
19. The function $\mathbf{r}(t) = (9\cos t)\mathbf{i} + (9\sin t)\mathbf{j}$ traces a circle. Determine the radius and center of the circle.
29. Parametrize the intersection of the surfaces
$$y^2 - z^2 = x - 2, \qquad y^2 + z^2 = 9$$
where $z \geq 0$. Use $t = y$ as the parameter.
37. Determine whether the two curves
$$\mathbf{r}_1(t) = \langle t^2 + 3, t + 1, 6t^{-1}\rangle, \qquad \mathbf{r}_2(t) = \langle 4t, 2t - 2, t^2 - 7\rangle.$$

9.2. **Solutions.**

5. The line is parameterized as $\mathbf{r}(t) = \langle 3, -5, 7\rangle + t\langle 3, 0, 1\rangle = \langle 3 + 3t, -5, 7 + t\rangle$. We can also write
$$\mathbf{r}(t)(3 + 3t)\mathbf{i} - 5\mathbf{j} + (7 + t)\mathbf{k}$$
19. We use that $x(t) = 9\cos t$ and $y(t) = 9\sin t$, so $x^2 + y^2 = 81$. This is a circle centered at the origin with radius 9
29. We need to solve for z and x in terms of y. Adding the two equations gives $2y^2 = x+7$, so $x = 2y^2-7$. Solving for z using the second equation gives $z = \sqrt{9 - y^2}$, so the parameterization becomes $\langle 2t^2 - 7, t, \sqrt{9 - t^2}$. Since $z \geq 0$, we have that $-3 \leq t \leq 3$.
37. If the curves intersect, then there exists some t for which the three equations are the same. This means that $t + 1 = 2t - 2$, so $t = 3$. Plugging in $t = 3$ to the other coordinates, we see that the curves intersect at $(12, 4, 2)$.

9.3. **13.2: Calculus of Vector-Valued Functions – Problems.**

7. Find the derivative of $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$.
25. Find the derivative of $\mathbf{r}(g(t))$, where $\mathbf{r}(t) = \langle t^2, 1 - t\rangle$ and $g(t) = e^t$ (use the chain rule).
31. Find a parameterization of the tangent line to $\mathbf{r}(t) = \langle t^2, t^4\rangle$ at $t = -2$.
43. Evaluate the integral
$$\int_{-2}^{2} (u^3\mathbf{i} + u^5\mathbf{j})du.$$

9.4. **Solutions.**

7. Differentiating each component gives
$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle.$$
25. We first differentiate each function:
$$\mathbf{r}'(t) = \langle 2t, -1\rangle.$$
$$g'(t) = e^t.$$
Now, using the chain rule gives
$$(\mathbf{r}(g(t)))' = g'(t)\mathbf{r}'(g(t)) = e^t\langle 2e^t, -1\rangle = \langle 2e^{2t}, -e^t\rangle.$$
31. First we calculate
$$\mathbf{r}'(t) = \langle 2t, 4t^3\rangle.$$
The tangent line has the parameterization
$$\ell(t) = \mathbf{r}(-2) + t\mathbf{r}'(-2),$$
and plugging in our equations gives
$$\ell(t) = \langle 4, 16\rangle + t\langle -4, -32\rangle = \langle 4 - 4t, 16 - 32t\rangle.$$

43. We integrate each component individually:

$$\int_{-2}^{2}(u^3\mathbf{i}+u^5\mathbf{j})du=\int_{-2}^{2}u^3du\mathbf{i}+\int_{-2}^{2}u^5\mathbf{j}du.$$

We have

$$\int_{-2}^{2}u^3du=0,\quad \int_{-2}^{2}u^5du=0,$$

so the answer is $0\mathbf{i}+0\mathbf{j}$.

9.5. **13.3: Arc Length and Speed – Problems.**

3. Compute the arc length of the curve

$$\mathbf{r}(t)=\langle 2t,\ln t,t^2\rangle \qquad ,1\le t\le 4.$$

5. Compute the arc length of the curve

$$\mathbf{r}(t)=\langle t,4t^{3/2},2t^{3/2}\rangle \qquad ,0\le t\le 3.$$

17. Find the speed of

$$\mathbf{r}(t)=\langle \sin 3t,\cos 4t,\cos 5t\rangle \qquad \text{at } t=\frac{\pi}{2}.$$

9.6. **Solutions.**

3. We have that

$$\mathbf{r}'(t)=\langle 2,\frac{1}{t},2t\rangle.$$

Using the arc length formula gives

$$\begin{aligned}L&=\int_1^4||\mathbf{r}'(t)||dt\\&=\int_1^4\sqrt{2^2+\left(\frac{1}{t}\right)^2(2t)^2}dt\\&=\int_1^4\sqrt{4t^2+4+\frac{1}{t^2}}dt\\&=\int_1^4\sqrt{\left(2t+\frac{1}{t}\right)^2}dt\\&=\int_1^4 2t+\frac{1}{t}dt\end{aligned}$$

This is an ordinary integral which we can evaluate directly to find that $L=15+\ln 4$.

5. We have that

$$\mathbf{r}'(t)=\langle 1,6t^{1/2},3t^{1/2}\rangle.$$

Using the arc length formula gives

$$\begin{aligned}L&=\int_1^4||\mathbf{r}'(t)||dt\\&=\int_0^3\sqrt{1^2\left(6t^{1/2}\right)^2,\left(3t^{1/2}\right)^2}dt\\&=\int_0^3\sqrt{1+45t}dt\\&=\frac{2}{135}(1+45t)^{3/2}|_0^3\\&+\frac{2}{135}\left(136^{3/2}-1\right)\end{aligned}$$

17. The derivative (velocity) is
$$\mathbf{r}'(t) = \langle 3\cos 3t, -4\sin 4t, -5\sin 5t\rangle.$$
At $t = \frac{\pi}{2}$ we have
$$\mathbf{r}'(\frac{\pi}{2}) = \langle 0, 0, -5\rangle.$$
The norm of this vector is $\sqrt{0^2+0^2+5^2} = 5$, so the speed at $t = \pi/2$ is 5 units/second.

9.7. 13.4: Curvature – Problems.

3. Calculate $\mathbf{r}'(t)$ and $\mathbf{T}(t)$ for the curve $\mathbf{r}(t) = \langle 3+4t, 3-5t, 9t\rangle$.
17. Find the curvature of the curve $y = t^4$ at $t = 2$.
37. Find the normal vector $\mathbf{N}(t)$ to $\mathbf{r}(t) = \langle 4, \sin 2t, \cos 2t\rangle$.

9.8. Solutions.

3. We have that $\mathbf{r}'(t) = \langle 4, -5, 9\rangle$ so its norm is
$$||\mathbf{r}'(t)|| = \sqrt{4^2+(-5)^2+9^2} = \sqrt{122}.$$
We then calculate the unit tangent vector using the formula:
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{1}{\sqrt{122}}\langle 4, -5, 9\rangle.$$
17. We use the formula for curvature of a graph in the plane:
$$\kappa(t) = \frac{|f''(t)|}{(1+f'(t)^2)^{3/2}}.$$
We have $f(t) = t^4$, so $f'(t) = 4t^3$ and $f''(t) = 12t^2$. So
$$\kappa(t) = \frac{12t^2}{(1+(4t^3)^2)^{3/2}} = \frac{12t^2}{(1+16t^6)^{3/2}}.$$
When $t = 2$ we have
$$\kappa(2) = \frac{12\cdot 2^2}{(1+16\cdot 2^6)^{3/2}} = \frac{48}{1025^{3/2}} \approx 0.0015.$$
37. First we need to find the unit tangent vector. We have $\mathbf{r}'(t) = \langle 0, 2\cos 2t, -2\sin 2t\rangle$ so
$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}\\ &= \frac{\langle 0, 2\cos 2t, -2\sin 2t\rangle}{\sqrt{0^2+(2\cos 2t)^2+(-2\sin 2t)^2}}\\ &= \frac{\langle 0, 2\cos 2t, -2\sin 2t\rangle}{\sqrt{4(\cos^2 2t+\sin^2 2t)}}\\ &= \frac{\langle 0, 2\cos 2t, -2\sin 2t\rangle}{\sqrt{4}}\\ &= \langle 0, \cos 2t, -\sin 2t\rangle\end{aligned}$$
The normal vector is then
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}.$$
We have that $\mathbf{T}'(t) = \langle 0, -2\sin 2t, -2\cos 2t$. Note that this looks very similar to $\mathbf{r}'(t)$. Calculating the normal vector similarly to above, we have
$$\mathbf{N}(t) = \langle 0, -\sin 2t, -\cos 2t\rangle.$$

9.9. **13.5: Motion in 3-Space – Problems.**

3. Find the velocity, acceleration, and speed of $\mathbf{r}(t) = \langle t^3, 1-t, 4t^2\rangle$ at $t = 1$.
11. Take some curve $\mathbf{r}(t)$ such that the acceleration vector is $\mathbf{a}(t) = \langle t, 4\rangle$ and the initial velocity is $\mathbf{v}(0) = \langle \frac{1}{3}, -2\rangle$. What is the velocity vector $\mathbf{v}(t)$?
37. Given that $\mathbf{r}(t) = \langle \frac{1}{3}t^3, 1-3t\rangle$, find the decomposition of the acceleration $\mathbf{a}(t)$ into tangential and non-tangential components at $t = -1$.

9.10. **Solutions.**

3. We have that
$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 3t^2, -1, 8t\rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 6t, 0, 8\rangle.$$
11. We find $\mathbf{v}(t)$ by integrating $\mathbf{a}(t)$:
$$\begin{aligned}\mathbf{v}(t) &= \mathbf{v}(0) + \int_0^t \mathbf{a}(u)du \\ &= \langle \frac{1}{3}, -2\rangle + \int_0^t \langle u, 4\rangle du \\ &= \langle \frac{1}{3}, -2\rangle + \langle \frac{1}{2}u^2, 4u\rangle\big|_0^t \\ &= \langle \frac{1}{3}, -2\rangle + \langle \frac{1}{2}t^2, 4t\rangle \\ &= \langle \frac{1}{3} + \frac{1}{2}t^2, 4t-2\rangle.\end{aligned}$$
37. We have $\mathbf{v}(t) = \mathbf{r}'(t) = \langle t^2, -3\rangle$ and $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 2t, 0\rangle$, so the velocity and acceleration at $t = -1$ are
$$\mathbf{v} = \langle 1, -3\rangle, \qquad \mathbf{a} = \langle -2, 0\rangle.$$
Therefor we can calculate the normal tangent vector:
$$\mathbf{T} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{1}{\sqrt{10}}\langle 1, -3\rangle.$$
We have that the acceleration along the path is given by
$$a_T = \mathbf{a}\cdot\mathbf{T} = -\frac{2}{\sqrt{10}}.$$
The acceleration in the normal direction is given by
$$a_N\mathbf{N} = \mathbf{a} - a_t\mathbf{T} = \langle -\frac{9}{5}, -\frac{3}{5}\rangle.$$
Since $\mathbf{N}$ has norm 1, the length of this vector will be a_N:
$$a_N = ||a_N\mathbf{N}|| = \frac{3\sqrt{10}}{5}$$
Thus we have the decomposition
$$\mathbf{a} = \langle -2, 0\rangle = -\frac{2}{\sqrt{10}}\mathbf{T} + \frac{3\sqrt{10}}{5}\mathbf{N}$$

10. Chapter 14: Differentiation in Several Variables

10.1. **14.1: Functions of Two or More Variables – Problems.** #1: Exercise 14.1.18: Describe the domain and range of $g(r,s) = \cos^{-1}(rs)$. #2: Exercise 14.1.21: Matching functions with their graphs, see book. #3: Exercise 14.1.22: Matching functions with their contour maps, see book.

10.2. **Solutions.** #1: The domain is where $|rs| \leq 1$. The range would be between 0 and π (inclusive).
#2: a.) $\to$ D
b.) $\to$ C
c.) $\to$ E
d.) $\to$ B
e.) $\to$ A
f.) $\to$ F.
#3: a.) $\to$ B
b.) $\to$ A
c.) $\to$ C
d.) $\to$ D

10.3. **14.2: Limits and Continuity in Several Variables – Problems.** #1: Exercise 14.2.5: Using continuity, evaluate $\lim_{(x,y)\to(\pi/4,0)} \tan x \cos y$. #2: Exercise 14.2.32: Evaluate $\lim_{(x,y)\to(0,0)} xy/(\sqrt{x^2+y^2})$. #3: Exercise 14.2.40: Evaluate $\lim_{(x,y)\to(0,0)} (x+y+2)e^{-1/(x^2+y^2)}$.

10.4. **Solutions.** #1: We use the continuity of $\tan x \cos y$ at the point $(\pi/4, 0)$ to evaluate the limit by substitution:

$$\lim_{(x,y)\to(\pi/4,0)} \tan x \cos y = \tan\frac{\pi}{4}\cos 0 = 1\cdot 1 = 1.$$

#2: Since we cannot simply use substitution, try converting to polar coordinates:

$$\left|xy/(\sqrt{x^2+y^2})\right| = \left|\frac{r\cos\theta \cdot r\sin\theta}{r}\right| = r\cos\theta\sin\theta \leq r.$$

Thus, since as $(x,y)\to(0,0)$, $r\to 0$, the squeeze theorem ensures that the given limit is zero.

#3: Consider that as x and y get very small, the fraction $-1/(x^2+y^2)\to-\infty$. This will force $e^{-1/(x^2+y^2)}\to 0$. Thus:

$$\lim_{(x,y)\to(0,0)} (x+y+2)e^{-1/(x^2+y^2)} = 0$$

10.5. **14.3: Partial Derivatives – Problems.** #1: Exercise 14.3.20: Compute the first-order partial derivatives of $z = x/(x-y)$. #2: Exercise 14.3.23: Compute the first-order partial derivatives of $z = (\sin x)(\cos y)$. #3: Exercise 14.3.35: Compute the first-order partial derivatives of $U = e^{-rt}/r$. #4: Exercise 14.3.58: Compute the derivative $g_{xy}(-3,2)$ of $g(x,y) = xe^{-xy}$. #5: Exercise 14.3.69: Find a function such that $\partial f/\partial x = 2xy$ and $\partial f/\partial y = x^2$.

10.6. **Solutions.** #1:

$$\frac{\partial}{\partial x}\left(\frac{x}{x-y}\right) = \frac{-y}{(x-y)^2},\ \frac{\partial}{\partial y}\left(\frac{x}{x-y}\right) = \frac{x}{(x-y)^2}$$

#2:

$$\frac{\partial}{\partial x}(\sin x\cos y) = (\cos x)(\cos y),\ \frac{\partial}{\partial y}(\sin x\cos y) = -(\sin x)(\sin y)$$

#3:

$$\frac{\partial U}{\partial r} = \frac{-(1+rt)e^{-rt}}{r^2},\ \frac{\partial U}{\partial t} = -e^{-rt}$$

#4:

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(xe^{-xy}\right)\right] = \frac{\partial}{\partial y}\left[(1-xy)e^{-xy}\right] = xe^{-xy}(xy-2)$$

#5: The function $f(x,y) = x^2y$ works.

10.7. **14.4: Differentiability, Tangent Planes, and Linear Approximation – Problems.** #1: Exercise 14.4.5: Find an equation of the tangent plane at $(4,1)$ of $f(x,y) = x^2+y^{-2}$. #2: Exercise 14.4.14: Find the points on the graph of $f(x,y) = (x+1)y^2$ at which the tangent plane is horizontal. #3: Exercise 14.4.23: Estimate $f(2.1, 3.8)$ assuming that $f(2,4) = 5$, $f_x(2,4) = 0.3$, and $f_y(2,4) = -0.2$.

10.8. **Solutions.** #1: Using the formula for tangent plane and calculating the first-order partial derivatives at $(4, 1)$, we obtain the equation

$$z = 8x - 2y - 13.$$

#2: We calculate

$$\frac{\partial}{\partial x}\left((x+1)y^2\right) = y^2,\ \frac{\partial}{\partial y}\left((x+1)y^2\right) = 2(x+1)y.$$

The system $y^2 = 0$ and $2(x+1)y = 0$ is satisfied when $y = 0$.

#3: Use the linear approximation of f at the point $(2, 4)$, which is

$$f(2+h, 4+k) \approx f(2,4) + f_x(2,4)h + f_y(2,4)k.$$

Substituting the given values and $h = .1$, $k = -.2$ we obtain the following approximation:

$$f(2.1, 3.8) \approx 5 + 0.3 \cdot 0.1 + 0.2 \cdot 0.2 = 5.07.$$

10.9. **14.5: The Gradient and Directional Derivatives – Problems.** #1: Exercise 14.5.24: Calculate the directional derivative of $\sin(x-y)$ at $P = (\pi/2, \pi/6)$ in the direction of $v = \langle 1, 1\rangle$. #2: Exercise 14.5.35: Determine the direction in which $f(x, y, z) = xy/z$ has maximum rate of increase from $P = (1, -1, 3)$, and give the rate of change in that direction. #3: Exercise 14.5.41: Find a vector normal to the surface $x^2 + y^2 - z^2 = 6$ at $P = (3, 1, 2)$. #4: Exercise 14.5.55: Find a function $f(x, y, z)$ such that $\nabla f = \langle z, 2y, x\rangle$.

10.10. **Solutions.** #1: We normalize v to obtain a unit vector u in the direction of v:

$$u = \frac{v}{||v||} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle.$$

Next, we compute the gradient of f at the point P:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle = \langle \cos(x-y), -\cos(x-y)\rangle \Longrightarrow \nabla f_{\left(\frac{\pi}{2},\frac{\pi}{6}\right)} = \left\langle \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right), -\cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle.$$

Thus, the directional derivative in the direction of v is

$$D_u f\left(\frac{\pi}{2}, \frac{\pi}{6}\right) = \nabla f_{\left(\frac{\pi}{2},\frac{\pi}{6}\right)} \cdot u = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle \cdot \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle = 0.$$

#2: The gradient points in the direction of maximum rate increase, so we compute the gradient at P:

$$\nabla f = \left\langle \frac{y}{z}, \frac{x}{z}, -\frac{xy}{z^2}\right\rangle,\ \nabla f_{(1,-1,3)} = \left\langle -\frac{1}{3}, \frac{1}{3}, \frac{1}{9}\right\rangle.$$

Then the rate of change in this direction is

$$||\nabla f_{(1,-1,3)}|| = \sqrt{\left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{9}\right)^2} = \frac{\sqrt{19}}{9}.$$

#3: The gradient ∇f_P is normal to the level curve $f(x, y, z) = x^2 + y^2 + z^2 = 6$ at P. We compute this vector: $f_x(x, y, z) = 2x$, $f_y(x, y, z) = 2y$, $f_z(x, y, z) = -2z \Longrightarrow \nabla f_P = \langle 6, 2, -4\rangle$.

#4: For example, $f(x, y, z) = xz + y^2$.

10.11. **14.6: Multivariable Calculus Chain Rules – Problems.** #1: Exercise 14.6.8: Use the Chain Rule to calculate $\partial f/\partial u$ for $f(x, y) = x^2 + y^2$, $x = e^{u+v}$, $y = u + v$. #2: Exercise 14.6.12: Use the Chain Rule to evaluate $\partial f/\partial s$ at $(r, s) = (1, 0)$, where $f(x, y) = \ln(xy)$, $x = 3r + 2s$, and $y = 5r + 3s$. #3: Exercise 14.6.31: Use implicit differentiation to calculate $\partial z/\partial y$ for $e^{xy} + \sin(xz) + y = 0$.

10.12. **Solutions.** #1: $2(x^2+y) = 2(e^{2(u+v)}+u+v)$.
#2:

$$\begin{aligned}\frac{\partial f}{\partial s}\left[\ln(xy)\right] &= \frac{\partial f}{\partial s}\left[\ln((3r+2s)(5r+3s))\right]\\ &= \frac{\partial f}{\partial s}\left[\ln(3r+2s)+\ln(5r+3s)\right]\\ &= \frac{\partial f}{\partial s}\left[\ln(3r+2s)\right]+\frac{\partial f}{\partial s}\left[\ln(5r+3s)\right]\\ &= \frac{2}{3r+2s}+\frac{3}{5r+3s}\\ &= \frac{19r+12s}{(3r+2s)(5r+3s)}\\ &\Longrightarrow \frac{19}{15} \text{ at } (r,s)=(1,0).\end{aligned}$$

#3: We use the following relation:

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Thus we calculate

$$F_y = xe^{xy}+1,\ F_z = x\cos(xz).$$

$$\Longrightarrow \frac{\partial z}{\partial y} = -\frac{xe^{xy}+1}{x\cos(xz)}.$$

10.13. **14.7: Optimization in Several Variables – Problems.** #1: Exercise 14.7.12: Find the critical points of $f(x,y) = x^3+y^4-6x-2y^2$, then apply the Second Derivative Test. #2: Exercise 14.7.17: Find the critical points of $f(x,y) = \sin(x+y)-\cos x$, then apply the Second Derivative Test. #3: Exercise 14.7.24: Show that $f(x,y) = x^2$ has infinitely many critical points (as a function of two variables) and that the Second Derivative Test fails for all of them. What is the minimum value of f? Does $f(x,y)$ have an local maxima?

10.14. **Solutions.** #1: By calculating the first-order derivatives and setting them equal to zero, then solve the system of equations to obtain the following 5 critical points:

$$\left(-\sqrt{2},\pm 1\right),\ \left(\pm\sqrt{2},0\right),\ \left(\sqrt{2},-1\right).$$

Next compute the discriminant:

$$D(x,y) = f_{xx}f_{yy}-f_{xy}^2 = (6x)\cdot(12y^2-4)-0 = 6x(12y^2-4).$$

Finally, we apply the second-derivative test:

$$\begin{aligned}D(-\sqrt{2},1) < 0 &\Longrightarrow \text{ Saddle Point}\\ D(-\sqrt{2},-1) < 0 &\Longrightarrow \text{ Saddle Point}\\ D(-\sqrt{2},0) > 0,\ f_{xx}(-\sqrt{2},0) < 0 &\Longrightarrow \text{ Local Maximum}\\ D(\sqrt{2},0) < 0 &\Longrightarrow \text{ Saddle Point}\\ D(\sqrt{2},-1) > 0,\ f_{xx}(\sqrt{2},-1) > 0 &\Longrightarrow \text{ Local Minimum.}\end{aligned}$$

#2: We calculate

$$f_x(x,y) = \cos(x+y)+\sin x,\ f_y(x,y) = \cos(x+y).$$

The equation $\cos(x+y) = 0$ is satisfied when

$$x+y = \frac{(2k+1)\pi}{2} \to y = \frac{(2k+1)\pi}{2}-x,\ k \text{ is an integer.}$$

By setting the two equations equal to one another we get $\sin x = 0$ which is satisfied at the values

$$x = 0, \pm\pi, \pm 2\pi, \dots = \pm k\pi,\ \text{where } k \text{ is an integer}$$

Thus, for integers n and k, the critical points are found at the points where $x = k\pi$ and $y = \frac{(2n+1)\pi}{2}$.
Next we compute the discriminant:

$$f_{xx}(x,y) = -\sin(x+y)+\cos x,\ f_{yy}(x,y) = f_{xy}(x,y) = -\sin(x+y)$$

$$\implies D(x,y) = (-\sin(x+y)+\cos x)(-\sin(x+y)) - \sin^2(x+y) = -\cos x \sin(x+y).$$

Finally, we apply the second-derivative test. We have:

$$D = \begin{cases} +1, & \text{if } y = \frac{4n+3}{2}\pi \\ -1, & \text{if } y = \frac{4n+1}{2}\pi. \end{cases}$$

Thus, the points $\left(k\pi, \frac{4n+1}{2}\pi\right)$ are saddle points, the points $\left(k\pi, \frac{4n+3}{2}\pi\right)$ are local minima if k is even, or they are local maxima if k is odd.

#3: The critical points of f are all points where $x = 0$, so obviously there are infinitely many of them. Then for the discriminant we have:

$$D(x,y) = f_{xx} \cdot f_{yy} - f_{xy}^2 = 2 \cdot 0 - 0 = 0.$$

Thus, since $D = 0$, the second-derivative test fails. Furthermore, the minimum value of f is 0, which of course occurs whenever $x = 0$. Also f does not have any local maxima as the shape of it is a parabolic cylinder opening upwards in the positive z-direction.

10.15. **14.8: Lagrange Multipliers: Optimizing with a Constraint – Problems.** #1: Exercise 14.8.10: Find the minimum and maximum vales of $f(x,y) = x^2y^4$, subject to the constraint $x^2 + 2y^2 = 6$. #2: Exercise 14.8.15: Find the minimum and maximum vales of $f(x,y) = xy + xz$, subject to the constraint $x^2 + y^2 + z^2 = 4$. #3: Exercise 14.8.18: Find the rectangular box of maximum volume if the sum of the lengths of the edges is 300 cm.

10.16. **Solutions.** #1: Rewrite the constraint as $g(x,y) = x^2 + 2y^2 - 6 = 0$. Then we have $\nabla f = \left\langle 2xy^4, 4x^2y^3 \right\rangle$ and $\nabla g = \langle 2x, 4y \rangle$. Hence the Lagrange condition $\nabla f = \lambda \nabla g$ gives

$$\left\langle 2xy^4, 4x^2y^3 \right\rangle = \lambda \langle 2x, 4y \rangle$$
$$\implies \lambda = y^4, \quad \lambda = x^2y^2.$$

Thus, solving for x and y we obtain $y = x$. Substituting back into the constraint we find the 4 critical points $(\pm\sqrt{2}, \pm\sqrt{2})$. These are all maximum values as they give f a value of 8. Furthermore, we must note that we need to include the cases where $x = 0$ or $y = 0$ under the constraint, and get the additional 4 critical points $(0, \pm\sqrt{3})$ and $(\pm\sqrt{6}, 0)$. These are all minimum values as they give f a value of 0.

#2: Rewrite the constraint as $g(x,y) = x^2 + y^2 + z^2 - 4 = 0$. Then we have $\nabla f = \langle y+z, x, x \rangle$ and $\nabla g = \langle 2x, 2y, 2z \rangle$. Hence the Lagrange condition $\nabla f = \lambda \nabla g$ gives

$$\langle y+z, x, x \rangle = \lambda \langle 2x, 2y, 2z \rangle$$
$$\implies \lambda = \frac{y+z}{2x}, \quad \lambda = \frac{x}{2y}, \quad \lambda = \frac{x}{2z}.$$

Thus, solving for x, y, and z we obtain $x = \sqrt{2}y = \sqrt{2}z$. By substituting $x/(\sqrt{2})$ for y and z into the constraint we find the 4 critical points $(\pm\sqrt{2}, 1, 1)$ and $(\pm\sqrt{2}, -1, -1)$. Evaluating f at these points we find that

$$f(\sqrt{2}, 1, 1) = f(-\sqrt{2}, -1, -1) = 2\sqrt{2}$$
$$f(\sqrt{2}, -1, -1) = f(-\sqrt{2}, 1, 1) = -2\sqrt{2}$$

#3: We may set this up as a Lagrange Multiplier problem with $f(x,y,z) = xyz$ under the constraint $4x+4y+4z = 300$. Let's just simplify the constraint to be $x + y + z = 75$.

Rewrite the constraint as $g(x,y) = x + y + z - 75 = 0$. Then we have $\nabla f = \langle yz, xz, xy \rangle$ and $\nabla g = \langle 1, 1, 1 \rangle$. Hence the Lagrange condition $\nabla f = \lambda \nabla g$ gives

$$\langle yz, xz, xy \rangle = \lambda \langle 1, 1, 1 \rangle$$
$$\implies \lambda = yz, \quad \lambda = xz, \quad \lambda = xy.$$

Thus, solving for x, y, and z we obtain $x = y = z$. By substituting into the constraint we find the critical point $(25, 25, 25)$. Evaluating f at this point gives a maximal volume of 15625 for the cube with edge-lengths of 25.

11. Chapter 15: Multiple Integration

11.1. **15.1: Integration in Two Variables – Problems.**

15. Evaluate the integral
$$\int\int_{\mathcal{R}} x^3 dA,$$
where $\mathcal{R} = [-4, 4] \times [0, 5]$.
27. Evaluate
$$\int_0^1 \int_0^2 x + 4y^3 dxdy.$$
31. Evaluate
$$\int_1^2 \int_2^4 e^{3x-y} dydx.$$
41. Evaluate
$$\int\int_{\mathcal{R}} e^x \sin y dA,$$
where $\mathcal{R} = [0, 2] \times [0, \frac{\pi}{4}]$.

11.2. **Solutions.**

15. The integrand is anti-symmetric around the y-axis, because $(-x)^3 = -x^3$. Thus the answer is just 0.
27. We use additivity to write
$$\int_0^1 \int_0^2 x + 4y^3 dxdy = \int_0^1 \int_0^2 xdxdy + \int_0^1 \int_0^2 4y^3 dxdy.$$
Then, we evaluate each integral by factoring
$$\int_0^1 \int_0^2 xdxdy = \int_0^2 xdx \int_0^1 1dy = 2$$
$$\int_0^1 \int_0^2 4y^3 dxdy = \int_0^2 1dx \int_0^1 4y^3 dy = 2$$
so the answer is $2 + 2 = 4$.
31. We evaluate the inner integral first:
$$\int_0^4 \frac{1}{x+y} dy = \ln(x+y)\Big|_{y=0}^4 = \ln(x+4) - \ln x.$$
Plugging this into the main integral gives
$$\int_1^2 \int_0^4 \frac{1}{x+y} dydx = \int_1^2 \ln(x+4) - \ln xdx.$$
We then calculate this integral:
$$\int_1^2 \ln(x+4) - \ln xdx = [(x+4)(\ln(x+4) - 1) - x(\ln x - 1)]_1^2 = 6(\ln 6 - 1) - 2(\ln 2 - 1) - 5(\ln 5 - 1) + (\ln 1 - 1)$$
$$= 6 \ln 6 - 2 \ln 2 - 5 \ln 5 \approx 13.1$$
41. We factor the integral as
$$\int\int_{\mathcal{R}} e^x \sin y dA = \left(\int_0^2 e^x dx\right)\left(\int_0^{\pi/4} \sin y dy\right)$$
$$= \left(e^x\Big|_0^2\right)\left(-\cos y\Big|_0^{\pi/4}\right) = \left(e^2 - 1\right)\left(1 - \frac{\sqrt{2}}{2}\right) \approx 1.87.$$

11.3. **15.2: Double Integrals over More General Regions – Problems.**

13. Calculate the double integral of $f(x, y) = x + y$ over the domain $\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 4, \quad y \geq 0\}$ (this is a semicircle of radius 2).
17. Calculate the double integral of $f(x, y) = x^3 y$ over the domain $\mathcal{D} = \{(x, y) : 0 \leq x \leq 5, \quad x \leq y \leq 2x + 3\}$.
45. Find the volume of the region bounded by $z = 40 - 10y$, $z = 0$, $y = 0$, $y = 4 - x^2$.

11.4. **Solutions.**

13. The semicircle can be described as the region

$$-2 \leq x \leq 2, 0 \leq y \leq \sqrt{4-x^2}.$$

We can evaluate this as follows:

$$\begin{aligned}\int\int_{\mathcal{D}}(x+y)dA &= \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}(x+y)dydx \\ &= \int_{-2}^{2} xy + \frac{1}{2}y^2\Big|^{\sqrt{4-x^2}} dx \\ &= \int_{-2}^{2}\left(x\sqrt{4-x^2}+\frac{1}{2}(4-x^2)\right)dx \\ &= \int_{-2}^{2}\left(x\sqrt{4-x^2}\right)dx + \int_{-2}^{2}\left(\frac{1}{2}(4-x^2)\right)dx.\end{aligned}$$

The first integral is zero because it is the integral of an odd function over a symmetric region, the second integral can be calculated as

$$\int_{-2}^{2} 2 - \frac{1}{2}x^2 dx = 2x - \frac{x^3}{6}\Big|_{-2}^{2} = \frac{16}{3}.$$

17. We calculate the double integral directly:

$$\begin{aligned}\int\int_{\mathcal{D}} x^3 y dA &= \int_0^5\int_x^{2x+3} x^3 y dy dx \\ &= \int_0^5 \frac{1}{2}x^3y^2\Big|_x^{2x+3}dx \\ &= \frac{1}{2}\int_0^5 x^3((2x+3)^2-x^2)dx \\ &= \frac{1}{2}\int_0^5 3x^5+12x^4+9x^3dx \\ &= \frac{1}{2}\left[\frac{1}{2}x^6+\frac{12}{5}x^5+\frac{9}{4}x^4\right]_0^5 \\ &= \frac{66875}{8}.\end{aligned}$$

45. This is the same as integrating the function $f(x,y) = 40 - 10y$ over the region

$$-2 \leq x \leq 2, \quad 0 \leq y \leq 4 - x^2.$$

Thus we have that

$$\begin{aligned}\int\int_{\mathcal{D}}(40-10y)dA &= \int_{-2}^{2}\int_0^{4-x^2}(40-10y)dydx \\ &= \int_{-2}^{2}(40y-5y^2)\Big|_0^{4-x^2}dx \\ &= \int_{-2}^{2}\left(40(4-x^2)-5(4-x^2)^2\right)dx \\ &= \int_{-2}^{2}(-5x^4+80)dx \\ &= \left[-x^5+80x\right]_{-2}^{2} = 256.\end{aligned}$$

11.5. **15.3: Triple Integrals – Problems.**

1. Integrate $f(x, y, z) = xz + yz^2$ over the region

$$0 \le x \le 2, \quad 2 \le y \le 4, \quad 0 \le z \le 4.$$

11. Integrate $f(x, y, z) = xyz$ over the region

$$0 \le z \le 1, \quad 0 \le y \le \sqrt{1 - x^2}, \quad 0 \le x \le 1.$$

33. Let $\mathcal{W}$ be the region bounded by $z = 1 - y^2$, $y = x^2$ and the plane $z = 0$. Calculate the volume of $\mathcal{W}$ as a triple integral.

11.6. **Solutions.**

1. We integrate directly:

$$\begin{aligned}
\int_0^4 \int_2^4 \int_0^2 (xz + yz^2) dx dy dz &= \int_0^4 \int_2^4 \left[\frac{1}{2}x^2 z + xyz^2\right]_0^2 dy dz \\
&= \int_0^4 \int_2^4 (2z + 2yz^2) dy dz \\
&= \int_0^4 \left[2yz + y^2 z^2\right]_2^4 dz \\
&= \int_0^4 (4z + 12z^2) dz \\
&= \left[2z^2 + 4z^3\right]_0^4 \\
&= 288.
\end{aligned}$$

11. We do another messy triple integral:

$$\begin{aligned}
\int\int\int_{\mathcal{W}} xyz dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^1 xyz dz dy dx & \\
&= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{xy}{2} dy dx \\
&= \int_0^1 \frac{xy^2}{4}\Big|_0^{\sqrt{1-x^2}} dx \\
&= \int_0^1 \frac{x(1-x^2)}{4} dx \\
&= \left[\frac{x^2}{8} - \frac{x^4}{16}\right]_0^1 = \frac{1}{16}.
\end{aligned}$$

33. $\mathcal{W}$ can be described by the following inequalities:

$$-1 \le x \le 1, \quad x^2 \le y \le 1, \quad 0 \le z \le 1 - y^2.$$

Thus we can integrate:

$$\begin{aligned}
V &= \int\int\int_{\mathcal{W}} 1 dV \\
&= \int_{-1}^1 \int_{x^2}^1 \int_0^{1-y^2} 1 dz dy dx \\
&= \int_{-1}^1 \int_{x^2}^1 (1 - y^2) dy dx \\
&= \int_{-1}^1 \frac{x^6}{3} - x^2 + \frac{2}{3} dx \\
&= \frac{16}{21}.
\end{aligned}$$

11.7. **15.4: Integration in Polar, Cylindrical, and Spherical Coordinates – Problems.**

7. Calculate the following integral by changing to polar coordinates:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} x^2+y^2 dydx.$$

27. Integrate $f(x,y,z)=x^2+y^2$ over the region $x^2+y^2\le 9, 0\le z\le 5$ by changing to cylindrical coordinates.
45. Integrate $f(x,y,z)=y$ over the region $x^2+y^2+z^2\le 1, x,y,z\le 0$ by changing to spherical coordinates.

11.8. **Solutions.**

7. In polar coordinates the region is $0\le\theta\le\pi, 0\le r\le 2$ and the function is $f(x,y)=r^2$. Using the change of variables formula we thus have that

$$\begin{aligned}\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} x^2+y^2 dydx &= \int_0^\pi\int_0^2 r^2\cdot r dr d\theta\\ &= \int_0^\pi 4d\theta = 4\pi.\end{aligned}$$

27. In cylindrical coordinates we have that $0\le\theta\le 2\pi, 0\le r\le 3, 0\le z\le 5$, so the integral is (after the change of variables formula)

$$\begin{aligned}\int_0^{2\pi}\int_0^3\int_0^5 r^2\cdot r dz dr d\theta &= \left(\int_0^{2\pi} 1d\theta\right)\left(\int_0^3 r^3 dr\right)\left(\int_0^5 1dz\right)\\ &= 2\pi\cdot\frac{3^4}{4}\cdot 5\\ &\approx 636.17\end{aligned}$$

45. We can parameterize in spherical coordinates by $\pi\le\theta\le 3\pi/2$, $\pi/2\le\phi\le\pi$, $0\le\rho\le 1$. The function is equal to $f(x,y,z)=y=\rho\sin\theta\sin\phi$ in spherical coordinates, so after a change of variables the integral is

$$\begin{aligned}\int\int\int_{\mathcal{W}} ydV &= \int_\pi^{3\pi/2}\int_{\pi/2}^{\pi}\int_0^1(\rho\sin\theta\sin\phi)\rho^2\sin\phi d\rho d\phi d\theta\\ &= \left(\int_\pi^{3\pi/2}\sin\theta d\theta\right)\left(\int_{\pi/2}^{\pi}\sin^2\phi d\phi\right)\left(\int_0^1\rho^3 d\rho\right).\end{aligned}$$

We can calculate these three integrals separately to find that the answer is $-\pi/16$.

11.9. **15.6: Change of Variables – Problems.**

7. Let $G(u,v)=(2u+v,5u+3v)$ be a map from the uv-plane to the xy-plane. Describe the image of the line $v=4u$ under G.
13. Calculate the Jacobian of $G(u,v)=(3u+4v,u-2v)$.
17. Calculate the Jacobian of $G(r,\theta)=(r\cos\theta,r\sin\theta)$.
35. Calculate

$$\int\int_{\mathcal{D}} e^{9x^2+4y^2}dxdy,$$

where $\mathcal{D}$ is the interior of the ellipse

$$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2\le 1.$$

11.10. **Solutions.**

7. Take two points on the line $v=4u$, for instance $(1,4),(0,0)$. These are mapped to $(6,17)$, and $(0,0)$ by G, so the image of G will be the line passing through these two points. We can calculate to find that this line is $y=\frac{17}{6}x$.
13. This is a linear mapping so the Jacobian can be calculated as follows:

$$\mathrm{Jac}(G)=\begin{vmatrix}3 & 4\\ 1 & -2\end{vmatrix}=3\cdot(-2)-1\cdot 4=-10.$$

17. We have that $x = r\cos\theta$ and $y = r\sin\theta$, so
$$\mathrm{Jac}(G) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

35. We change variables by setting $x = 2u$, $y = 3v$. This mapping is linear and we can find that its Jacobian is 6. The ellipse under this map is mapped to the unit circle (which we denote by $\mathcal{B}$, so our integral becomes
$$\int\int_{\mathcal{D}} e^{9x^2+4y^2}\,dxdy = \int\int_{\mathcal{B}} e^{36(x^2+y^2)} \cdot 6dudv.$$

Now, we switch to polar coordinates by seting $u = r\cos\theta$, $v = r\sin\theta$, which after doing another change of variables gives us
$$\begin{aligned} \int\int_{\mathcal{B}} e^{36(x^2+y^2)} \cdot 6dudv &= \int_0^{2\pi}\int_0^1 6e^{36r^2} \cdot rdrd\theta \\ &= 6\left(\int_0^{2\pi} 1d\theta\right)\left(\int_0^1 e^{36r^2} rdr\right) \\ &= 12\pi \cdot \frac{1}{72} e^{36r^2}\Big|_0^1 \\ &= \frac{12\pi(e^{36}-1)}{72} = \frac{\pi(e^{36}-1)}{6}. \end{aligned}$$