Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 29: 4-27-2022: https://youtu.be/cTPKck5msBI

slides:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture2.pdf

Plan for the day: Lecture 29: April 27, 2022:

Topics:

Multivariable Taylor Series

Convergence of Taylor Series

Applications to Max/Min

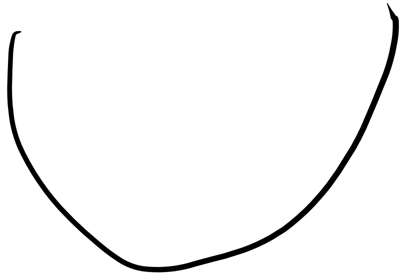
$$(Tf)(x_1, \ldots, x_n) = f(0, 0, \ldots, 0)$$

$$+ (\nabla f)(0, \ldots, 0) \cdot (x_1, \ldots, x_n)$$

$$+ \frac{1}{2} (x_1 \ \ldots \ x_n) \, (Hf)(0, \ldots, 0) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$+ \text{Tensors} + \cdots$$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

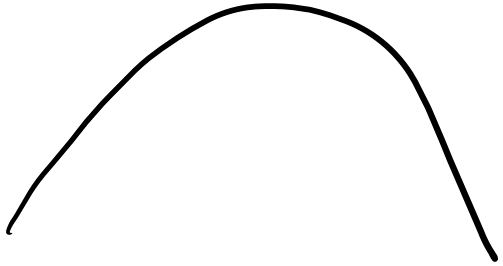


$y = x^2$

$f'(x) = 2x$

$f''(x) = 2$

min



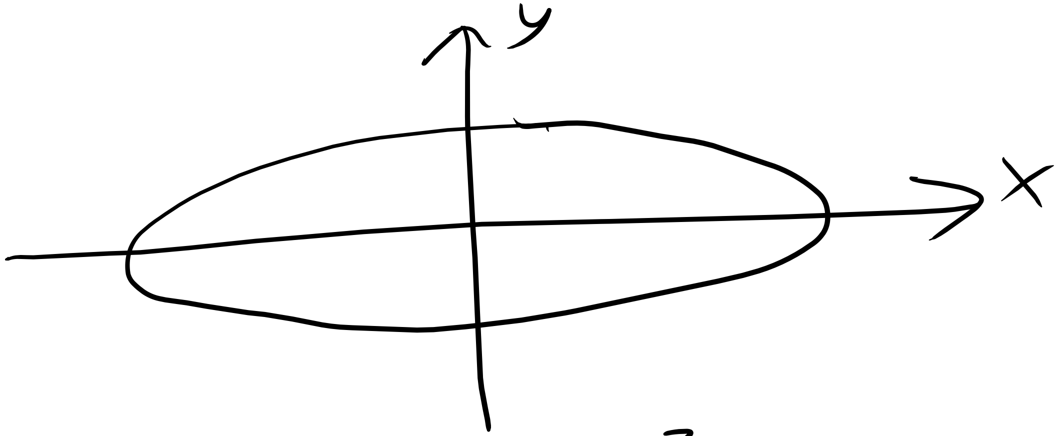
$y = -x^2$

$f'(x) = -2x$

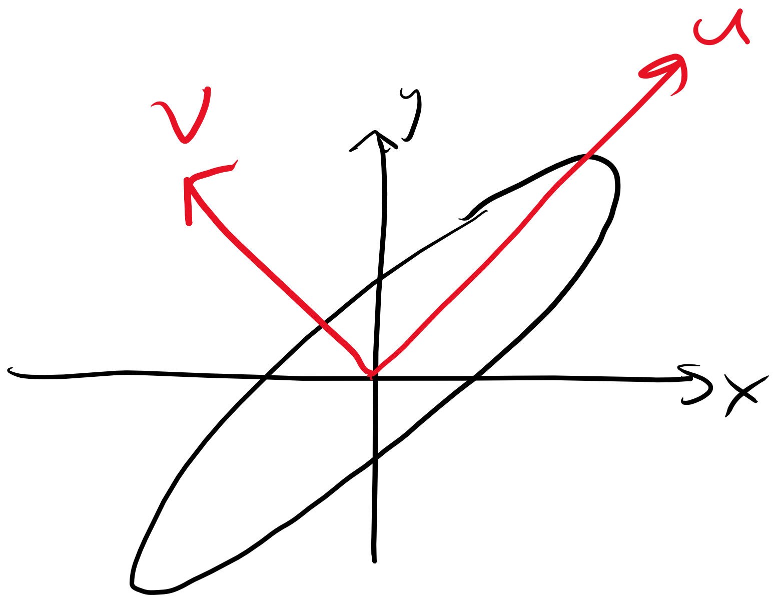
$f''(x) = -2$

max

Linear Algebra



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

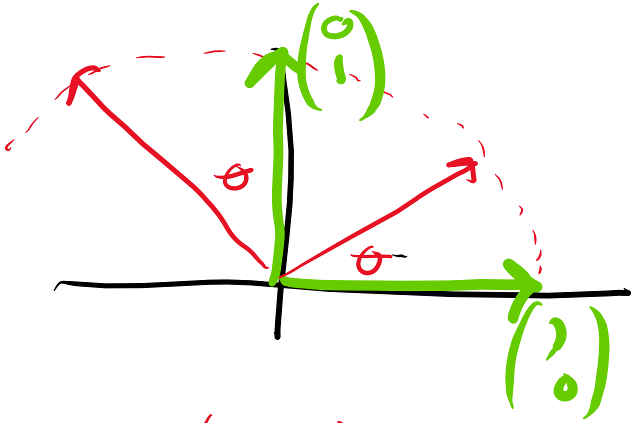


$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

Convert $u = u(x, y)$

$v = (x, y)$

$$Q(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



$$Q(\theta)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$$Q(\theta)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

If A is real symmetric there is an invertible rotation matrix Q such that

$$Q\,A\,Q^T = D \quad \text{diagonal}$$

or $\quad A = Q^T D\, Q$ $\qquad$ For us, take $A = Hf$

$$\frac{1}{2}\vec{x}^{\,T}(Hf)\,\vec{x} = \frac{1}{2}\vec{x}^{\,T}Q^T D\, Q\,\vec{x}$$

$$= \frac{1}{2}\left(Q\vec{x}\right)^T D\left(Q\vec{x}\right)$$

if all $d_k > 0$: min

if all $d_k < 0$: max

if mixed: saddle point

$Q\vec{x} = \vec{v}$ above is just $\displaystyle\sum_{k=1}^{n}\frac{1}{2}d_k\,v_k^2$

Convergence

$$(T_N f)(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N)}(0)}{N!}x^N$$

$$(E_N f)(x) = f(x) - (T_N f)(x)$$

$$= \underbrace{f(x) - f(0)}_{MVT} - \left[f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N)}(0)}{N!}x^N \right]$$

$$= \left[f'(c_0)(x-0) - f'(0)\,x \right] - \left[\frac{f''(0)}{2!}x^2 + \cdots \right]$$

$$= \left[\underbrace{f'(c_0) - f'(0)}_{MVT} \right] x - \left[\frac{f''(0)}{2!}x^2 + \cdots \right]$$

$$= f''(c_1)(c_0 - 0)\,x - \left[\frac{f''(0)}{2!}x^2 + \cdots \right]$$

Convergence

$$(T_n f)(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$g(x) = f(x) - (T_n f)(x)$$

$$(E_n f)(x) = f(x) - (T_n f)(x)$$

$$g(0) = 0$$

$$g'(0) = 0 \quad \cdots \quad g^{(n)}(0) = 0$$

$$(E_n f)(x) = g(x) - g(0) \quad \text{(did nothing)}$$

$$\underset{MVT}{=} g'(c_0)(x-0) \qquad c_0 \text{ b/w } 0 \text{ and } x$$

$$= g'(c_0)\,x$$

$$= \left[g'(c_0) - g'(0)\right] x$$

$$\underset{MVT}{=} g''(c_1)(c_0 - 0)\,x \qquad c_1 \text{ is b/w } 0 \text{ and } c_0$$

$$= \left[g''(c_1) - g''(0)\right] c_0 x \underset{MVT}{=} g'''(c_2)\, c_1 c_0 x$$

Convergence

$$(T_N f)(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N)}(0)}{N!}x^N$$

$$(E_N f)(x) = f(x) - (T_N f)(x)$$

Continue:

$$(E_N f)(x) = g^{(N+1)}(c_N) \cdot c_{N-1} c_{N-2} \cdots c_1 c_0 x$$

$$|(E_N f)(x)| \leq \max_{c_N \leq |x|} \left| f^{N+1}(c_N) \right| * |x|^{N+1}$$

$$\text{MVT: } f(b) = f(a) + \underline{f'(c)}(b-a)$$