Math 150: Calculus III: Spring ‘22 (Williams)

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Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 31: 5-2-22: https://youtu.be/0f2OJ6AMvx0

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture31.pdf

Plan for the day: Lecture 31: May 2, 2022:

Topics: Difference Equations

- **Bode's Law**
- **Spherical Integration**
- **Fibonacci Numbers**
- **Generating Function for Fibonacci Numbers**
- **Application: Double plus one: Roulette and Fibonaccis**

https://en.wikipedia.org/wiki/Titius%E2%80%93Bode_law

Titius–Bode law

From Wikipedia, the free encyclopedia

The **Titius–Bode law** (sometimes termed just **Bode's law**) is a formulaic prediction of spacing between planets in any given solar system. The formula suggests that, extending outward, each planet should be approximately twice as far from the Sun as the one before. The hypothesis correctly anticipated the orbits of Ceres (in the asteroid belt) and Uranus, but failed as a predictor of Neptune's orbit. It is named after Johann Daniel Titius and Johann Elert Bode.

Later work by Blagg and Richardson significantly corrected the original formula, and made predictions that were subsequently validated by new discoveries and observations. It is these re-formulations that offer "the best phenomenological representations of distances with which to investigate the theoretical significance of Titius-Bode type Laws".[1]

Formulation [edit]

The law relates the semi-major axis a_n of each planet outward from the Sun in units such that the Earth's semi-major axis is equal to 10:

$$a = 4 + x$$

where $x = 0, 3, 6, 12, 24, 48, 96, 192, 384, 768 \ldots$ such that, with the exception of the first step, each value is twice the previous value. There is another representation of the formula:

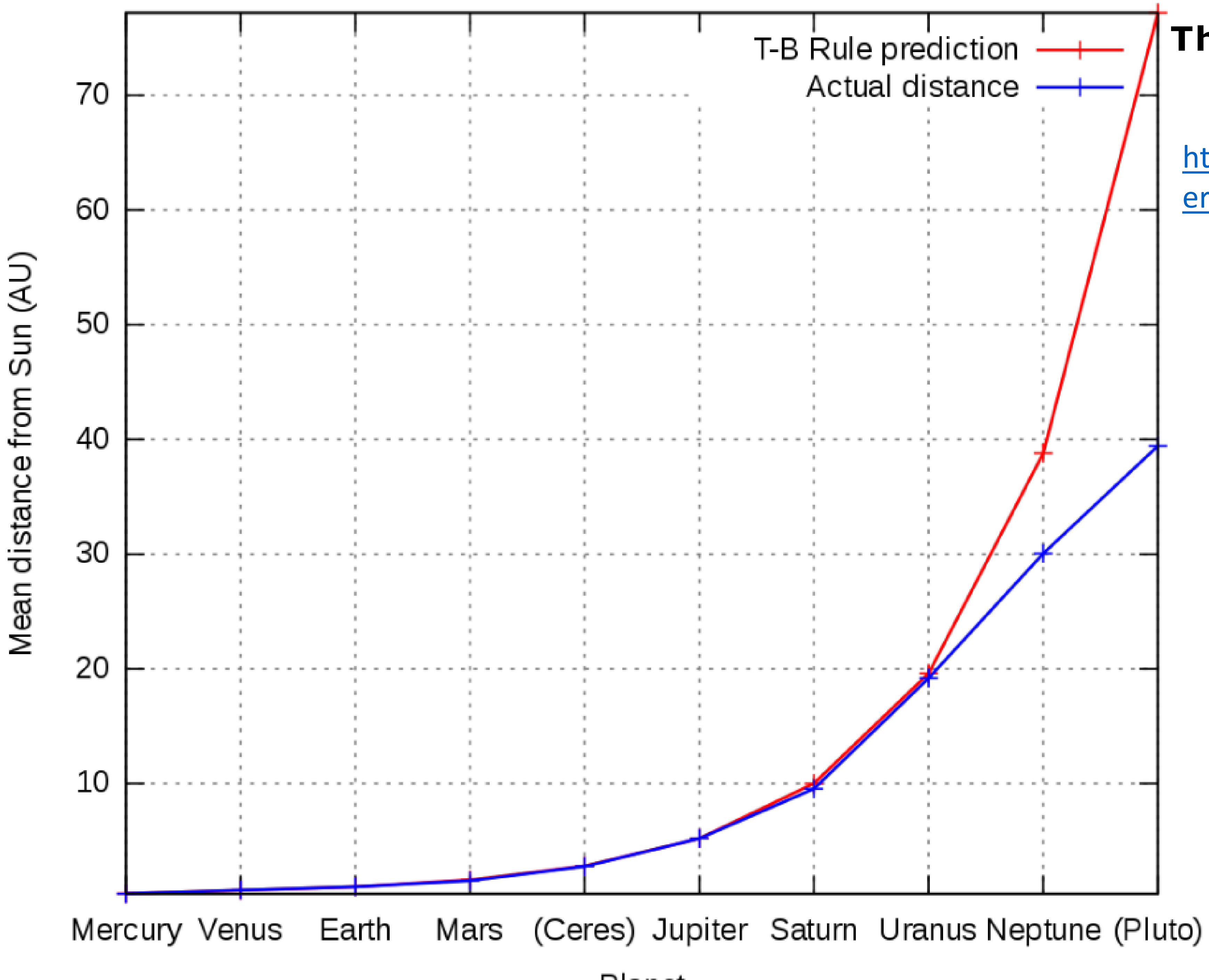
$$a = 4 + 3 \times 2^n$$

where $n = -\infty, 0, 1, 2, \ldots$. The resulting values can be divided by 10 to convert them into astronomical units (AU), resulting in the expression:

$$a = 0.4 + 0.3 \times 2^n \ .$$

For the far outer planets, beyond Saturn, each planet is predicted to be roughly twice as far from the Sun as the previous object. Whereas the Titius-Bode law predicts Saturn, Uranus, Neptune, and Pluto at about 10, 20, 39, and 77 AU, the actual values are closer to 10, 19, 30, 40 AU.[a]

This form of the law offered a good first guess; the re-formulations by Blagg and Richardson should be considered canonical.



The Planet That Wasn't

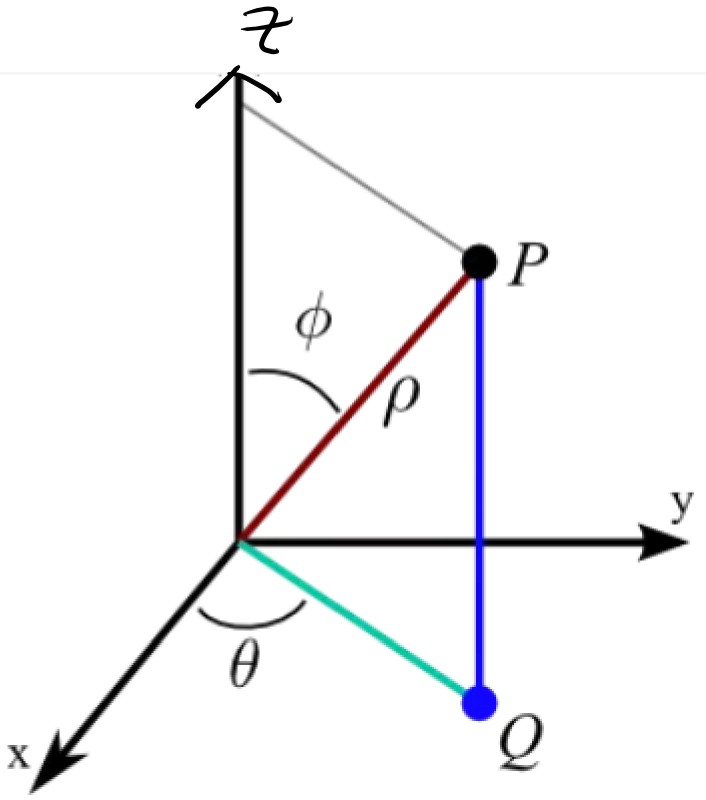
By Isaac Asimov

https://geobeck.tripod.com/frontier/planet.htm https://www.wikihow.com/Integrate-in-Spherical-Coordinates#:~:text=Integration%20in%20spherical%20coordinates%20is%20typically%20done%20when,which%20allows%20for%20easy%20factoring%20in%20most%20cases.

$F(x, y, z)$

$\Updownarrow$

$F(\rho \sin\phi \cos\theta, \ldots)$



1 **Recall the coordinate conversions.** Coordinate conversions exist from Cartesian to spherical and from cylindrical to spherical. Below is a list of conversions from Cartesian to spherical. Above is a diagram with point P described in spherical coordinates.

- $$x = \rho \sin\phi \cos\theta$$
  $$y = \rho \sin\phi \sin\theta$$
  $$z = \rho \cos\phi$$
  $$\rho^2 = x^2 + y^2 + z^2$$

2 **Set up the coordinate-independent integral.** We are dealing with volume integrals in three dimensions, so we will use a volume differential $\mathrm{d}V$ and integrate over a volume V.

- $\int_V \mathrm{d}V$
- Most of the time, you will have an expression in the integrand. If so, make sure that it is in spherical coordinates.

3 **Set up the volume element.**

- $\mathrm{d}V = \rho^2 \sin\phi \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta$
- Those familiar with polar coordinates will understand that the area element $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$. This extra r stems from the fact that the side of the differential polar rectangle facing the angle has a side length of $r\mathrm{d}\theta$ to scale to units of distance. A similar thing is occurring here in spherical coordinates.

$d\rho * \rho \sin\phi d\phi * \rho d\theta$

4 **Set up the boundaries.** Choose a coordinate system that allows for the easiest integration.

- Notice that ϕ has a range of $[0, \pi]$, **not** $[0, 2\pi]$. This is because θ already has a range of $[0, 2\pi]$, so the range of ϕ ensures that we don't integrate over a volume twice.

5 **Integrate.** Once everything is set up in spherical coordinates, simply integrate using any means possible and evaluate.

https://legacy-www.math.harvard.edu/~knill/teaching/summer2021/handouts/lecture18.pdf

Problem 18.4: Integrate the function

$$f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$$

over the solid which lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, which is in the first octant ~~and which is above the cone $x^2 + y^2 = z^2$~~.

$\theta \in [0, 2\pi]$ $\phi \in [0, \pi] \to 2\pi^2$

vs $\theta \in [0, \pi/2]$ $\phi \in [0, \pi/2] \to \pi^2/4$ 1/8th

$$\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{R} e^{\rho^3} * \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{\theta=0}^{\pi/2} d\theta \int_{\phi=0}^{\pi/2} \sin\phi \, d\phi \, \frac{1}{3} \int_{\rho=0}^{R} e^{\rho^3} 3\rho^2 \, d\rho$$

$$= \frac{\pi}{2} * \left[-\cos\phi\right]_0^{\pi/2} \frac{1}{3} e^{\rho^3} \Big|_0^R = \frac{\pi}{6} e^{R^3}$$

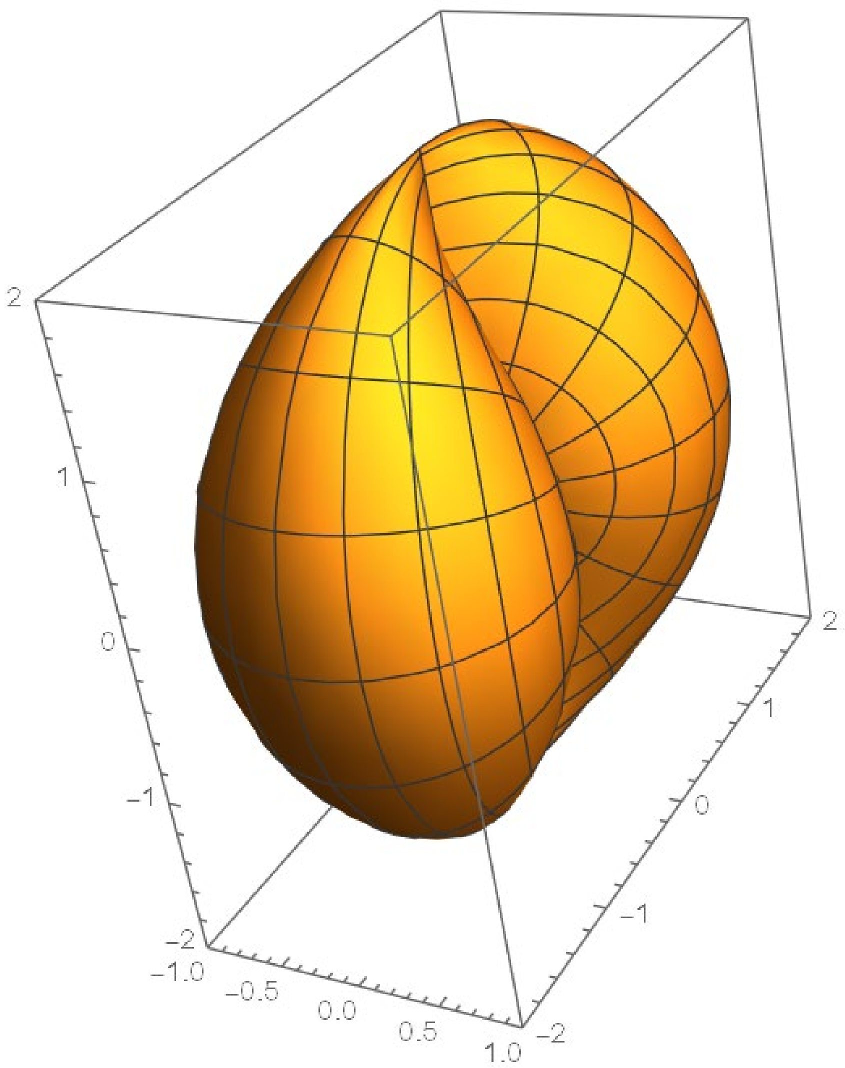
Problem 18.3: A solid is described in spherical coordinates by the inequality $\rho \leq 2\sin(\phi)$. Find its volume.

SphericalPlot3D[r, θ, ϕ] generates a 3D plot with a spherical radius r as a function of spherical coordinates θ and ϕ.

SphericalPlot3D[r, {θ, θ_{min}, θ_{max}}, {ϕ, ϕ_{min}, ϕ_{max}}] generates a 3D spherical plot over the specified ranges of spherical coordinates.

SphericalPlot3D[{r_1, r_2, …}, {θ, θ_{min}, θ_{max}}, {ϕ, ϕ_{min}, ϕ_{max}}] generates a 3D spherical plot with multiple surfaces.

```
SphericalPlot3D[2 Sin[phi], {theta, 0, 2 Pi}, {phi, 0, Pi}]
```



Exercise 1.1 (Recurrence Relations). *Let $\alpha_0, \dots, \alpha_{k-1}$ be fixed integers and consider the recurrence relation of order k*

$$x_{n+k} \;=\; \alpha_{k-1}x_{n+k-1} + \alpha_{k-2}x_{n+k-2} + \cdots + \alpha_1 x_{n+1} + \alpha_0 x_n. \tag{1.1}$$

Show once k values of x_m are specified, all values of x_n are determined. Let

$$f(r) \;=\; r^k - \alpha_{k-1}r^{k-1} - \cdots - \alpha_0; \tag{1.2}$$

we call this the characteristic polynomial of the recurrence relation. Show if $f(\rho) = 0$ then $x_n = c\rho^n$ satisfies the recurrence relation for any $c \in \mathbb{C}$.

Exercise 1.2. *Notation as in the previous problem, if $f(r)$ has k distinct roots $r_1, \dots, r_k$, show that any solution of the recurrence equation can be represented as*

$$x_n \;=\; c_1 r_1^n + \cdots + c_k r_k^n \tag{1.3}$$

for some $c_i \in \mathbb{C}$. The Initial Value Problem is when k values of x_n are specified; using linear algebra, this determines the values of $c_1, \dots, c_k$. Investigate the cases where the characteristic polynomial has repeated roots. For more on recursive relations, see [GKP], *§7.3.*

Exercise 1.3. *Solve the Fibonacci recurrence relation $F_{n+2} = F_{n+1} + F_n$, given $F_0 = F_1 = 1$. Show F_n grows exponentially, i.e., F_n is of size r^n for some $r > 1$. What is r? Let $r_n = \frac{F_{n+1}}{F_n}$. Show that the even terms r_{2m} are increasing and the odd terms r_{2m+1} are decreasing. Investigate* $\lim_{n\to\infty} r_n$ *for the Fibonacci numbers. Show r_n converges to the golden mean, $\frac{1+\sqrt{5}}{2}$. See* [PS2] *for a continued fraction involving Fibonacci numbers.*

Exercise 1.4 (Binet's Formula). *For F_n as in the previous exercise, prove*

$$F_{n-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \qquad (1.4)$$

This formula should be surprising at first: F_n is an integer, but the expression on the right involves irrational numbers and division by 2.

Exercise 1.5. *Notation as in the previous problem, more generally for which positive integers m is*

$$\frac{1}{\sqrt{m}}\left[\left(\frac{1+\sqrt{m}}{2}\right)^n - \left(\frac{1-\sqrt{m}}{2}\right)^n\right] \qquad (1.5)$$

an integer for any positive integer n?

Exercise(h) **1.6** (Zeckendorf's Theorem). Consider the set of distinct Fibonacci numbers: $\{1, 2, 3, 5, 8, 13, \dots\}$. Show every positive integer can be written uniquely as a sum of distinct Fibonacci numbers where we do not allow two consecutive Fibonacci numbers to occur in the decomposition. Equivalently, for any n there are choices of $\epsilon_i(n) \in \{0, 1\}$ such that

$$n \ = \ \sum_{i=2}^{\ell(n)} \epsilon_i(n) F_i, \quad \epsilon_i(n)\epsilon_{i+1}(n) = 0 \text{ for } i \in \{2, \dots, \ell(n) - 1\}. \tag{1.6}$$

Does a similar result hold for all recurrence relations? If not, can you find another recurrence relation where such a result holds?

Exercise(hr) **1.7.** Assume all the roots of the characteristic polynomial are distinct, and let λ_1 be the largest root in absolute value. Show for almost all initial conditions that the coefficient of λ_1 is non-zero.

Exercise(hr) **1.8.** Consider 100 tosses of a fair coin. What is the probability that at least three consecutive tosses are heads? What about at least five consecutive tosses? More generally, for a fixed k what can you say about the probability of getting at least k consecutive heads in N tosses as $N \to \infty$?

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;

$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$

Cookie Monster Meets the Fibonacci Numbers. Mmmmmm -- Theorems!: http://youtu.be/5e6HsfxqVSE
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/CookiesToCLTtoGaps_Yale2014.pdf

$F_0 = 0, F_1 = 1, F_2 = 1,$

$0, 1, 1, 2, 3, 5, 8, 13, \ldots$

$F_{n+1} = F_n + F_{n-1}$ $F_0 = 0, F_1 = 1$

$F_{n+1} \le 2F_n$ so F_n grows slower than 2^n

$F_{n+1} \ge 2F_{n-1}$ at least double every 2 indices, so faster than $\sqrt{2}^n$

Get $n \ge 2$: $\sqrt{2}^n \le F_n \le 2^n$ Guess (Divine Inspiration) $F_n = r^n$

$r^{n+1} = r^n + r^{n-1}$ characteristic polynomial

$r^{n-1}(r^2 - r - 1) = 0$ so $r = 0$ (boring) or $r = \frac{1 \pm \sqrt{5}}{2}$ quadratic formula

General Soln: $F_n = c_1 r_1^n + c_2 r_2^n$

$F_0 = 0 = c_1 + c_2 \implies c_2 = -c_1$

$F_1 = 1 = c_1 r_1 + c_2 r_2 \implies c_1(r_1 - r_2) = 1$ or $c_1 = \frac{1}{r_1 - r_2} = \frac{1}{\sqrt{5}}$

$$F_n = \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\text{Golden Mean}}^n - \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{\text{less than 1 in abs value}}^n$$

```
lower = 0;
upper = 1;
max = 10000000;
Timing[For[n = 2, n <= max, n++,
  {
   new = lower + upper;
   lower = upper;
   upper = new;
   }];]
Print[upper];
{356.063,Null}
```

```
Log[10.,Fibonacci[1000000]]
Log[10.,Fibonacci[500000]]
208987.
104493.
```

Estimate on how many digit operations base 10 to get to the millionth Fibonacci number. The 500,000th has 104,493 digits, so have at least
100000 * 500000 = 50,000,000,000.

How many seconds in a year?
3600*24*365.25*4 = $1.2623*10^8$ or approximately 100,000,000.

So if do 100 digits a second get to 10,000,000,000.

We're off by AT LEAST a factor of 5, and this is doing 100 digits a second!

Binet's Formula

$$F_1 = F_2 = 1; \; F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

As $|F_n| \le 2^n$
converges if $|x| < 1/2$

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$(1) \Rightarrow \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1}$$

$g(x) - F_1 x - F_2 x^2$

$x^n * x$ pull out x

$x^{n-1} * x^2$ pull out x^2

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

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$$\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2}$$

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

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$$\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2}$$

$$\Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2\sum_{n\geq 1} F_n x^n$$

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$\begin{aligned}
(1) \quad &\Rightarrow \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2\sum_{n\geq 1} F_n x^n \\
&\Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x)
\end{aligned}$$

Binet's Formula

$$F_1 = F_2 = 1; \; F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

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$$\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2}$$

$$\Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2 \sum_{n\geq 1} F_n x^n$$

$$\Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x)$$

$$\Rightarrow g(x) = x/(1-x-x^2).$$

- Generating function: $g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2} = \frac{x}{1-r}$

with $r = x + x^2$

$$\frac{x}{1-r} = x\left[1 + r + r^2 + r^3 + \cdots\right]$$

$$= x\left[1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \cdots\right]$$

$$= x\left[1 + (x+x^2) + (x^2+2x^3+x^4) + (x^3+3x^4+\cdots) + \cdots\right]$$

- Generating function: $g(x) = \sum_{n>0} \boldsymbol{F}_n x^n = \frac{x}{1-x-x^2}$.
- Partial fraction expansion:

$$\Rightarrow g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x}\right).$$

- Generating function: $g(x) = \sum_{n>0} \boldsymbol{F}_n x^n = \frac{x}{1-x-x^2}$.
- Partial fraction expansion:

$$\Rightarrow \; g(x) \; = \; \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \left(\frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x} \right).$$

Coefficient of x^n (power series expansion):

$$\boldsymbol{F}_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{-1+\sqrt{5}}{2} \right)^n \right]$$ - Binet's Formula!

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).

Video for Double plus One

We consider the following simplified model for the number of pairs of whales alive at a given moment in time. We make the following simplifying assumptions:

(1) Time moves in discrete steps of 1 year.
(2) The number of whale pairs that are 0, 1, 2 and 3 years old in year n are denoted by a_n, b_n, c_n and d_n respectively; all whales die when they turn 4.
(3) If a whale pair is 1 year old it gives birth to two new pairs of whales, if a whale pair is 2 years old it gives birth to one new pair of whales, and no other pair of whales give birth.

zero $a_{n+1} = 0 \cdot a_n + 2b_n + 1 \cdot c_n + 0 \cdot d_n$

one $b_{n+1} = 1 \cdot a_n + 0 \cdot b_n + 0 \cdot c_n + 0 \cdot d_n$

two $c_{n+1} = 0 \cdot a_n + 1 \cdot b_n + 0 \cdot c_n + 0 \cdot d_n$

three $d_{n+1} = 0 + a_n + 0 \cdot b_n \quad 1 \cdot c_n + 0 \cdot d_n$

Letting

$$v_n = \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix}, \tag{1}$$

$v_n = A v_{n-1}$
so $v_{n+1} = A v_n$
$= A A v_{n-1}$
$= A^2 v_{n-1}$

we see that

$$v_{n+1} = A v_n, \tag{2}$$

where

$$A = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

Thus

$$v_{n+1} = A^{n+1} v_0, \tag{4}$$

where v_0 is the initial populations at time 0. As discussed before, it is one thing to write down a solution and another to have be able to numerically work with it. This matrix is fortunately easily diagonalizable.