Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 33: 5-6-2022: https://youtu.be/luld2zsO9mk

slides:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture3.pdf

Plan for the day: Lecture 3: May , 2022:

Topics:

Green’s Theorem

Stokes’ Theorem

Divergence Theorem

https://en.wikipedia.org/wiki/Green%27s_theorem

Theorem [edit]

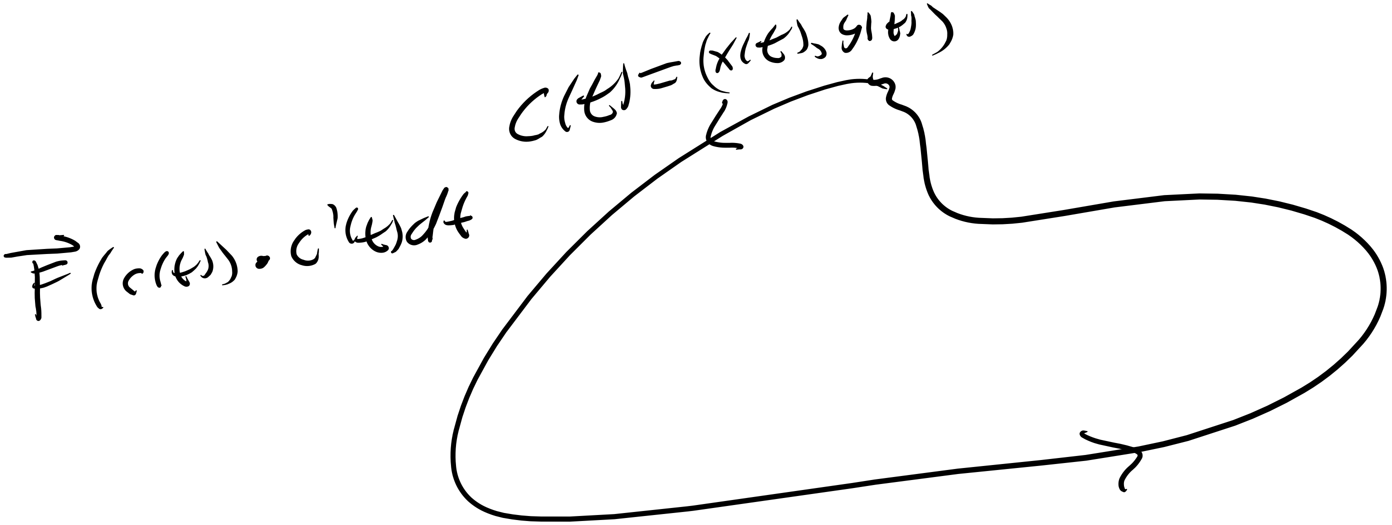
Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and having continuous partial derivatives there, then

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx\,dy$$

$$\int_C \vec{F}(c(t)) \cdot c'(t)\,dt = \iint_D (\nabla \times \vec{F})\,dx\,dy$$

where the path of integration along C is anticlockwise.[1][2]

In physics, Green's theorem finds many applications. One is solving two-dimensional flow integrals, stating that the sum of fluid outflowing from a volume is equal to the total outflow summed about an enclosing area. In plane geometry, and in particular, area surveying, Green's theorem can be used to determine the area and centroid of plane figures solely by integrating over the perimeter.



$$\vec{F}(x,y) = (L(x,y), M(x,y))$$

$$d\vec{s} = (dx, dy)$$

$$\vec{F} \cdot d\vec{s} = L\,dx + M\,dy$$

```
In[2]:= F[x_, y_] := {x / (x^2 + y^2)^(3/2), y / (x^2 + y^2)^(3/2)};
        Curl[F[x, y], {x, y}]

Out[3]= 0
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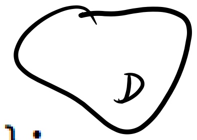
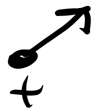
$$= \left(\frac{\cos\theta}{r^2}, \frac{\sin\theta}{r^2}\right)$$





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In[6]:= G[x_, y_] := {-y, x};
        Curl[G[x, y], {x, y}]

Out[7]= 2
```

$$\iint_D 2\,dxdy = 2\iint_D 1\,dxdy = 2 * \text{Area}(D)$$

number of primes ≤ 2 | binary form | range | perfect number? | more...

```
In[1]:= ?Curl
```

Out[1]=

Symbol

Curl[{f_1, f_2}, {x_1, x_2}] gives the curl $\partial f_2/\partial x_1 - \partial f_1/\partial x_2$.

Curl[{f_1, f_2, f_3}, {x_1, x_2, x_3}] gives the curl

$(\partial f_3/\partial x_2 - \partial f_2/\partial x_3, \partial f_1/\partial x_3 - \partial f_3/\partial x_1, \partial f_2/\partial x_1 - \partial f_1/\partial x_2)$.

Curl[f, {x_1, …, x_n}] gives the curl of the n×n×…×n array f

with respect to the n-dimensional vector {x_1, …, x_n}.

Curl[f, x, *chart*] gives the curl in the coordinates *chart*.

https://en.wikipedia.org/wiki/Stokes%27_theorem

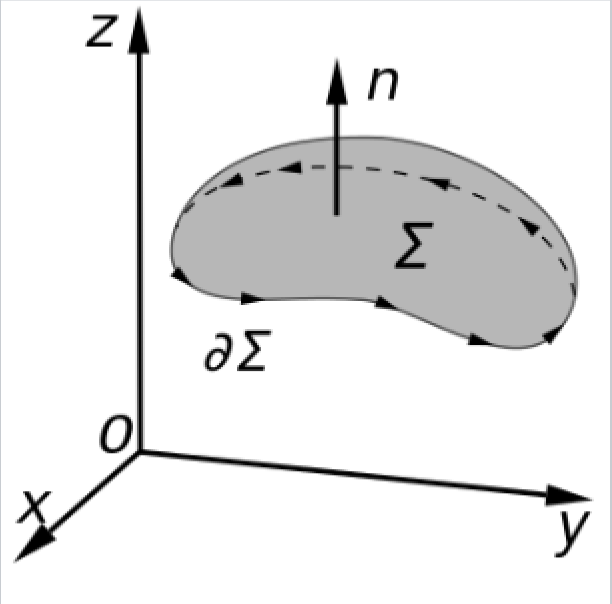
Theorem [edit]

Let Σ be a smooth oriented surface in $\mathbf{R}^3$ with boundary $\partial\Sigma$. If a vector field $\mathbf{F}(x,y,z) = (F_x(x,y,z), F_y(x,y,z), F_z(x,y,z))$ is defined and has continuous first order partial derivatives in a region containing Σ, then

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathrm{d}^2\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{F} \cdot \mathrm{d}\mathbf{\Gamma}.$$

More explicitly, the equality says that

$$\iint_\Sigma \left(\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathrm{d}y\,\mathrm{d}z + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathrm{d}z\,\mathrm{d}x + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y \right)$$
$$= \oint_{\partial\Sigma} \Big(F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z \Big).$$



An illustration of Stokes' theorem, with surface Σ, its boundary $\partial\Sigma$ and the normal vector n.

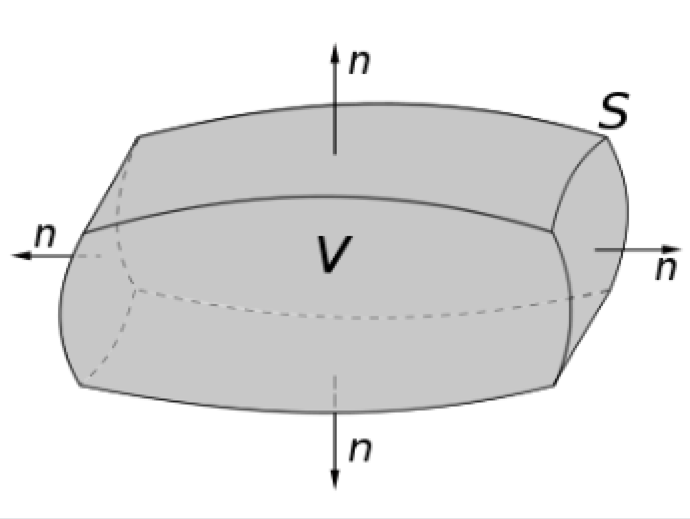
https://en.wikipedia.org/wiki/Divergence_theorem

Mathematical statement [edit]

Suppose V is a subset of $\mathbb{R}^n$ (in the case of $n = 3$, V represents a volume in three-dimensional space) which is compact and has a piecewise smooth boundary S (also indicated with $\partial V = S$). If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of V, then:[4][5]

$$\iiint_V (\nabla \cdot \mathbf{F})\, \mathrm{d}V = \oiint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, \mathrm{d}S.$$

The left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V. The closed manifold ∂V is oriented by outward-pointing normals, and $\hat{\mathbf{n}}$ is the outward pointing unit normal at each point on the boundary ∂V. ($\mathrm{d}\mathbf{S}$ may be used as a shorthand for $\mathbf{n}\mathrm{d}S$.) In terms of the intuitive description above, the left-hand side of the equation represents the total of the sources in the volume V, and the right-hand side represents the total flow across the boundary S.



A region V bounded by the surface $S = \partial V$ with the surface normal n

https://en.wikipedia.org/wiki/Generalized_Stokes_theorem

The generalized Stokes theorem reads:

Theorem (*Stokes–Cartan*) — If ω is a smooth $(n-1)$-form with compact support on smooth n-dimensional manifold-with-boundary Ω, $\partial\Omega$ denotes the boundary of Ω given the induced orientation, and $i : \partial\Omega \hookrightarrow \Omega$ is the inclusion map, then

$$\int_\Omega d\omega = \int_{\partial\Omega} i^*\omega.$$

FTC: $\int_{[a,b]} F' = \int_{[a,b]} dF = F(b) - F(a) = F\Big|_a^b = F\Big|_{\partial[a,b]}$

boundary

$$\int_a^a F' = 0$$

Boring

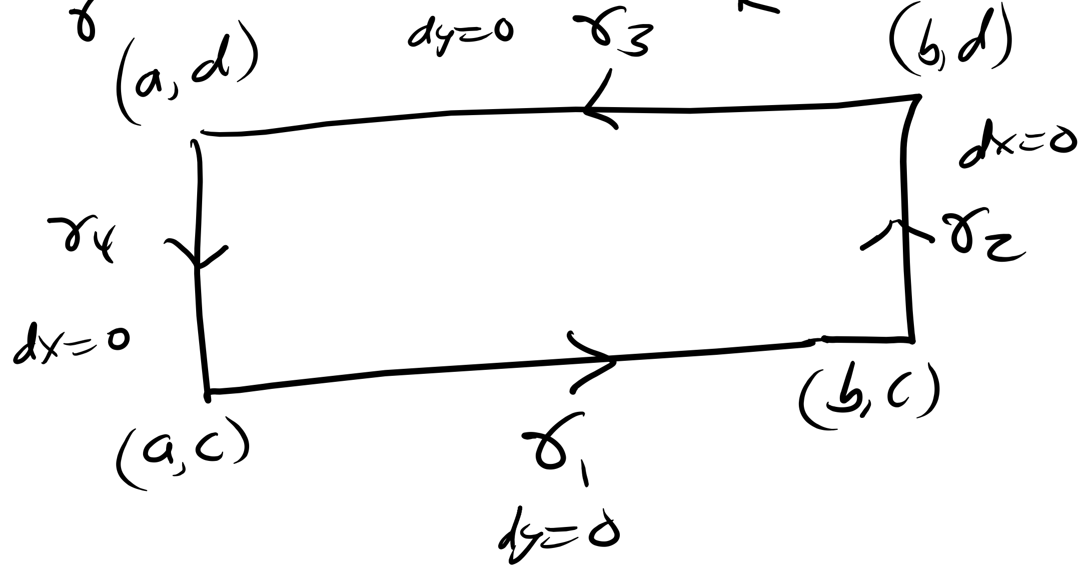
Name	Differential form	Integral form (using three-dimensional Stokes theorem plus relativistic invariance, $\int \frac{\partial}{\partial t} \dots \to \frac{d}{dt} \int \dots$)
Maxwell–Faraday equation Faraday's law of induction:	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{A} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$ (with C and S not necessarily stationary)
Ampère's law (with Maxwell's extension):	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \nabla \times \mathbf{H} \cdot d\mathbf{A} = \iint_S \mathbf{J} \cdot d\mathbf{A} + \iint_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{A}$ (with C and S not necessarily stationary)

The above listed subset of Maxwell's equations are valid for electromagnetic fields expressed in SI units. In other systems of units, such as CGS or Gaussian units, the scaling factors for the terms differ. For example, in Gaussian units, Faraday's law of induction and Ampère's law take the forms:[17][18]

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\,,$$
$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}\,,$$

respectively, where c is the speed of light in vacuum.

$$\int_\gamma P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \text{ : Green's Thm}$$



$$\oint_\gamma \vec{F} \cdot d\vec{s} = \int F(c(t)) \cdot c'(t)\,dt$$

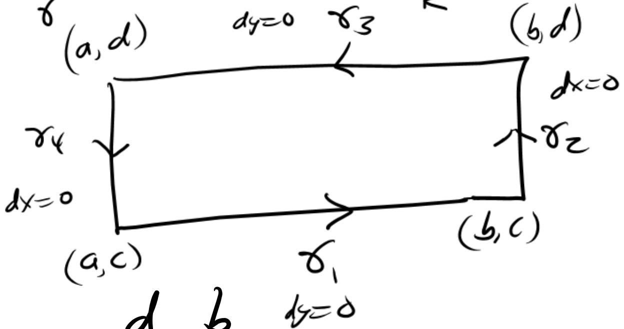
$$F(c(t)) = (P(c(t)), Q(c(t)))$$

$$\oint_\gamma P = \int P(c(t)) \cdot x'(t)\,dt \qquad \frac{dx}{dt}dt = dx$$

$$\oint_\gamma Q = \int Q(c(t)) \cdot y'(t)\,dt \qquad \frac{dy}{dt}dt = dy$$

$$\int_\gamma P\,dx = \int_{\gamma_1} P\,dx + \int_{\gamma_3} P\,dx$$

$$\int_\gamma P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad \text{: Green's Thm}$$

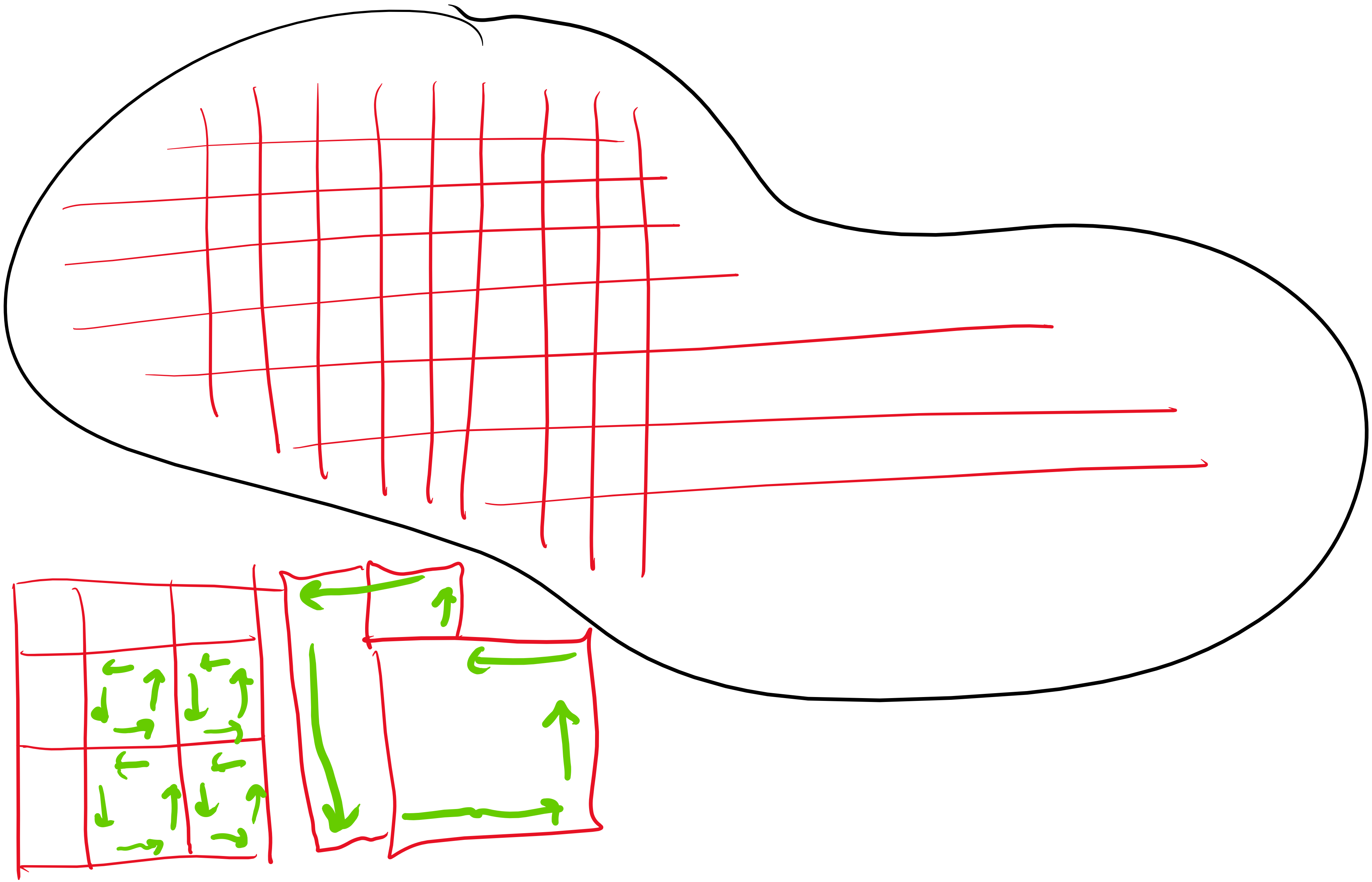


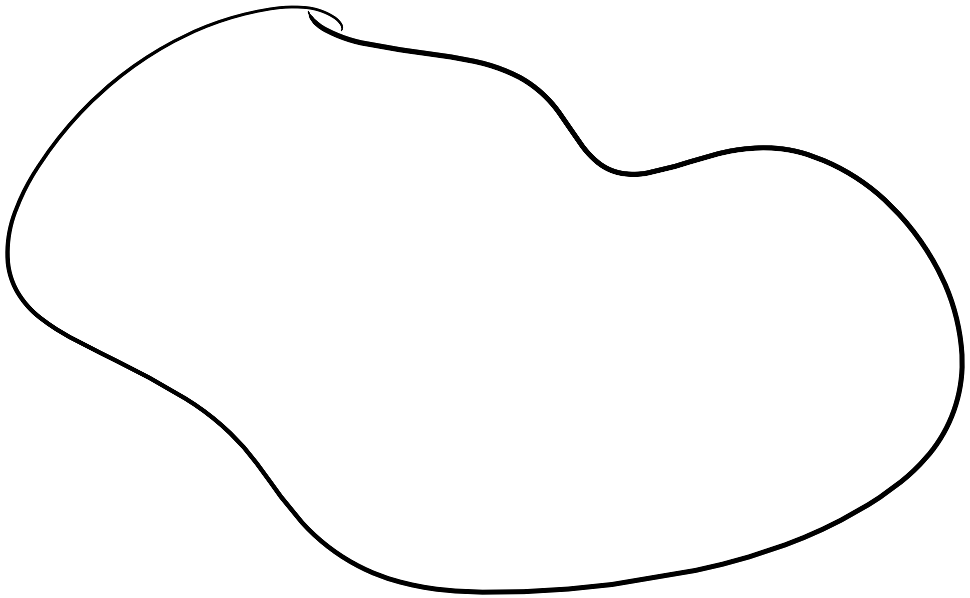
$$\int_{y=c}^{d}\int_{x=a}^{b} \frac{\partial Q}{\partial x}\,dx\,dy \qquad \iint \frac{\partial P}{\partial y}\,dy\,dx \ \text{(Fubini)}$$

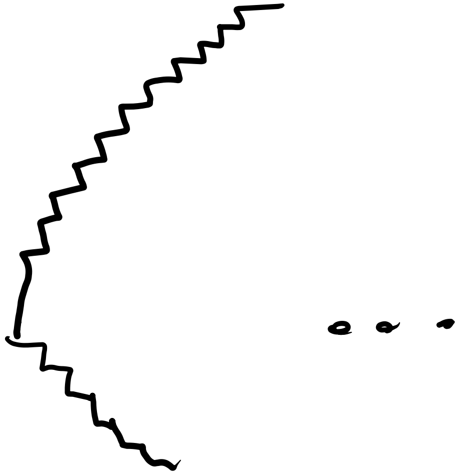
$$= \int_{y=c}^{d} \Big[Q(x,y) + h(y)\Big]_a^b dy = \int_{y=c}^{d} \Big[Q(b,y) - Q(a,y)\Big] dy$$

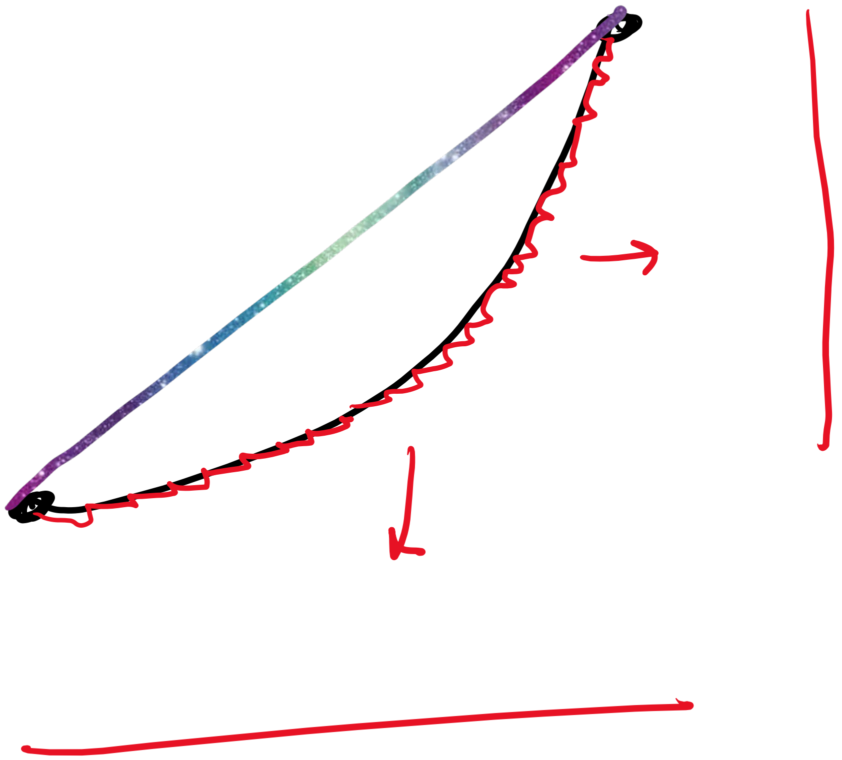
$$= \int_{y=c}^{d} Q(b,y)\,dy + \int_{y=d}^{c} Q(a,y)\,dy$$

$$= \int_{\gamma_2} Q\,dy + \int_{\gamma_4} Q\,dy = \int_\gamma Q\,dy \quad \text{as} \int_{\gamma_1} Q\,dy = \int_{\gamma_3} Q\,dy = 0$$









$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

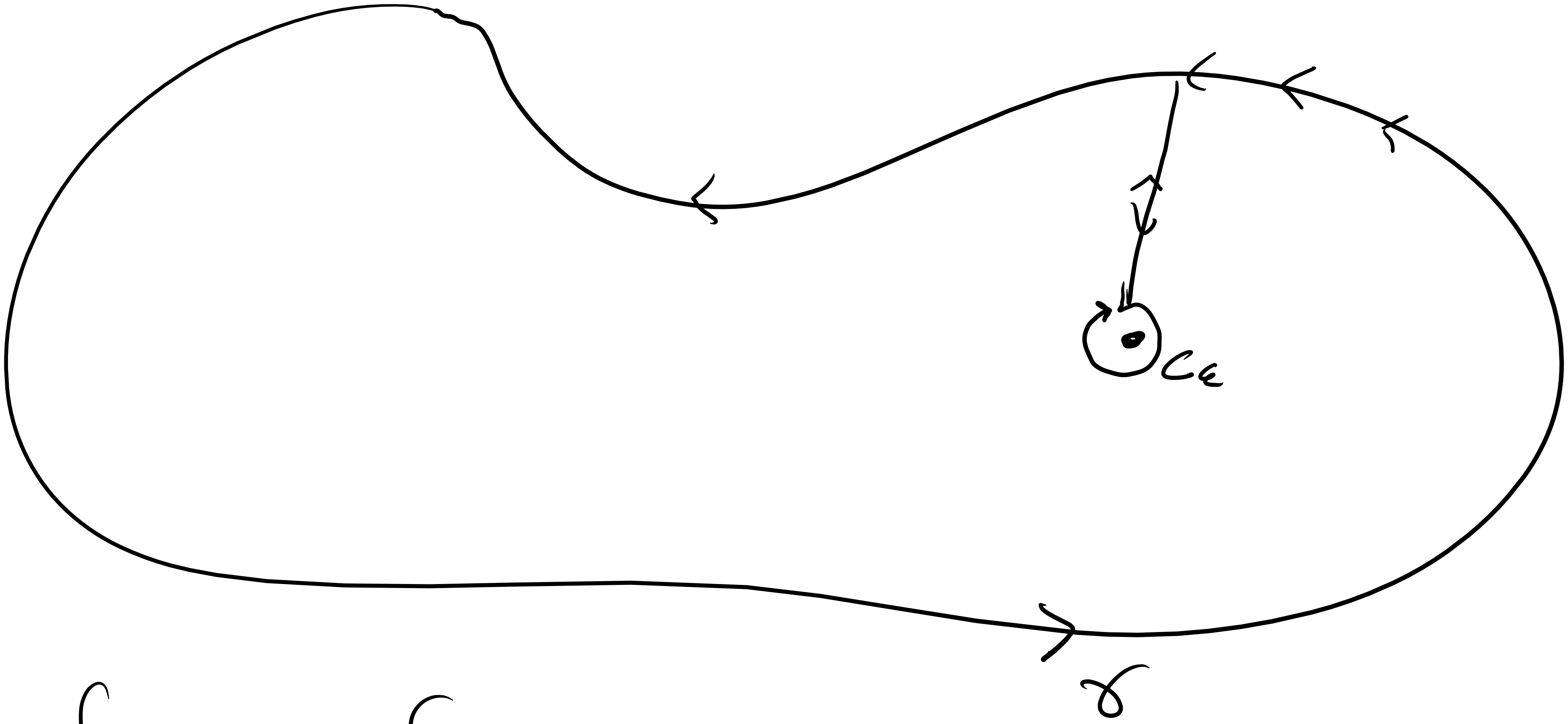
$$\sqrt{3^2 + 4^2} = 5$$

$$\neq 3 + 4$$

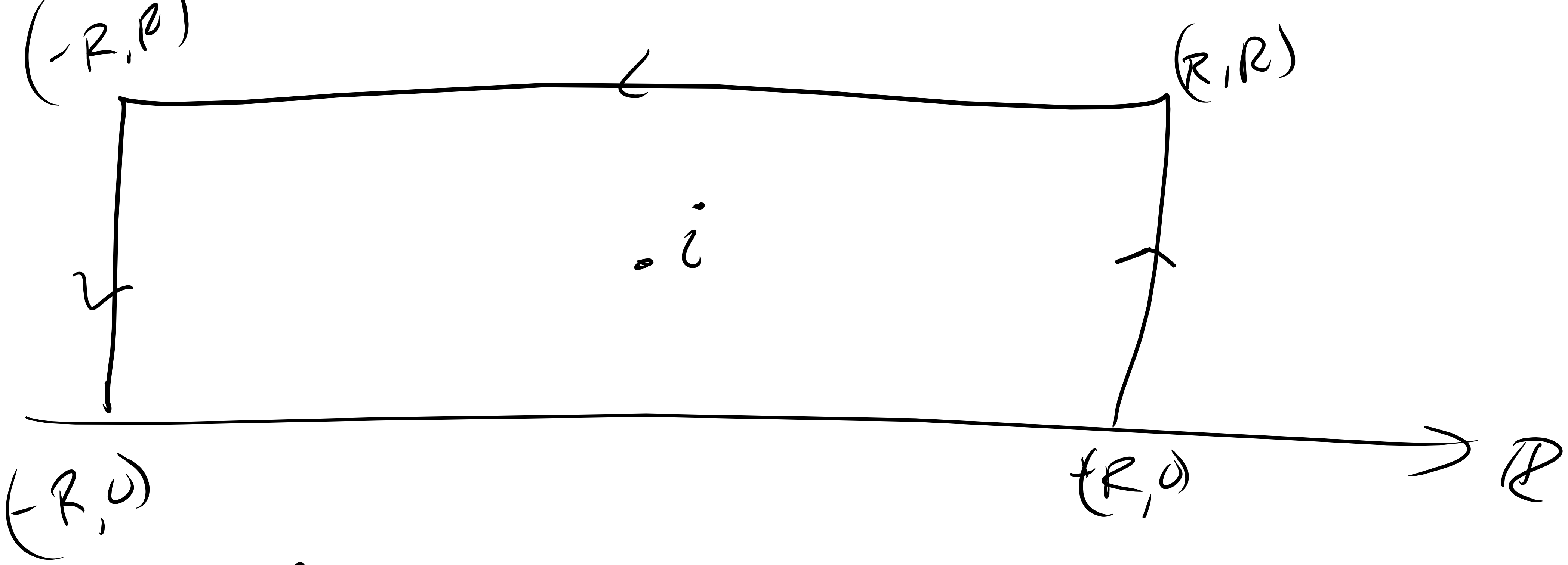
we did

$$ds = dx + dy$$

or $\Delta s = \Delta x + \Delta y$



$$\oint_\gamma f = \int_{c_\varepsilon} f$$



$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \arctan(\infty) - \arctan(-\infty) = \pi$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx$$