Proposition: $F_{n+1}^2 + F_n^2 = F_{2n+1}$.
Proof: We proceed inductively. For $n = 1$ we see:

$$F_2^2 + F_1^2 = 1^2 + 1^2 = 2 = F_3. \tag{1}$$

Assume Equation 1 holds for all n up to some integer k, meaning

$$F_{k+1}^2 + F_k^2 = F_{2k+1}. \tag{2}$$

We show the $k+1$ case follows. Consider

$$\begin{aligned} F_{k+2}^2 + F_{k+1}^2 &= (F_{k+1} + F_k)^2 + F_{k+1}^2 &(3)\\ &= F_{k+1}^2 + 2F_{k+1}F_k + F_k^2 + F_{k+1}^2 &(4)\end{aligned}$$

By our inductive hypothesis we can group F_{k+1}^2 and F_k^2 as F_{2k+1}. Let $S_k = F_{k+1}^2 + 2F_{k+1}F_k$. If $S_k = F_{2k+2}$, we're done. Let's continue to expand S_k. Substituting $F_{k+1} = F_k + F_{k-1}$, we see:

$$\begin{aligned} S_k &= F_{k+1}^2 + 2F_{k+1}F_k = F_k^2 + 2F_kF_{k-1} + F_{k-1}^2 + 2F_k^2 + 2F_kF_{k-1} &(5)\\ &= F_{2k-1} + 2(F_k^2 + F_kF_{k-1}) = F_{2k-1} + 2S_{k-1}. &(6)\end{aligned}$$

We can analogously expand S_{k-1} and we find $S_{k-1} = F_{2k-3} + 2S_{k-2}$. We can continue this process all the way down to $S_1 = F_2^2 + 2F_2F_1 = 3$. Therefore:

$$S_k = F_{2k-1} + 2F_{2k-3} + 4F_{2k-5} + ... + 2^{k-1} \cdot 3 \tag{7}$$

Since our goal is to show $S_k = F_{2k+2}$, let's use induction again.

Claim: $S_n = F_{2n+2}$.

Proof: With $n = 2$ we see $F_3 + 2 * 3 = 8 = F_6$. Assume our claim holds up to some integer k. Then when $n = k+1$ we see:

$$S_{k+1} = F_{2k+1} + 2F_{2k-1} + ... + 2^k \cdot 3 = F_{2k+1} + 2(F_{2k+2}) = F_{2k+4}, \tag{8}$$

and our claim is proven. Thus $S_n = F_{2n+2}$, meaning $F_{n+1}^2 + F_n^2 = F_{2n+1}$.