MATH 209: DIFFERENTIAL EQUATIONS

- GOALS: build / solve models
  emphasize techniques
  heuristics / qualitative behavior complex systems
  ↳ trade-offs: more complicated model, more features captured, but harder to solve
  ↳ ex: Baseball paper

- GENERAL: Ask class pure / applied / topics
  describe exams / HW / computers

- EXAMPLES
  ↳ Newton: $F = ma \to F(t) = m\, d^2x/dt^2$
  Biology: Predator / Prey (humans / werewolves) (BRING ARTICLE)

$$\left.\begin{aligned} dx/dt &= \alpha_1 x - \alpha_2 x^2 - \alpha_3 xy \\ dy/dt &= \beta_1 y - \beta_2 y^2 - \beta_3 xy \end{aligned}\right\} \text{why reasonable?}$$

  Physics: Wave / Heat Eq: Fourier Series
  NumbTh / Phys / Buses: Random Matrix Theory (my research)

- "RESULTS"
  ↳ Bag of tricks / techniques for certain problems
  ↳ Sometimes existance / uniqueness Thm, hard to write down soln in general
  ↳ ex: orbit of planets: physically soln exists; math model is a simplification, thus may or may not have a soln LIBRARY TRIP
  ↳ Numerical techniques
  ↳ can approx soln in many cases
  ↳ danger: sometimes very sensitive to initial conds
  ↳ Chaos: Lorenz
  ↳ Newton's method + roots of poly: computer program
  ↳ pendulum
  ↳ sometimes numerics suggest answer
  ↳ $\sum n z^{-n}$, $\sum 1/n^2$

DIFFERENCE EQUATIONS (see also Section 2.9)

discrete version of differential eqs: time jumps discretely

Ex: Fibonacci Numbers (application to modeling bunny)

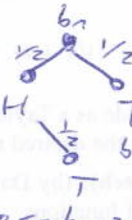
$a_{n+1} = a_n + a_{n-1}$

↳ DIVINE INSPIRATION: $a_n = r^n \to c_1 r_1^n + c_2 r_2^n$

↳ yields $a_{n-1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ (Binet's Formula)

↳ initial conditions: $a_0 = a_1 = 1$, Characteristic poly $r^2 - r - 1 = 0$

Ex: Double-plus-one in Roulette



↳ Red Casino 18/38, do 1/2 for ease

Say b_n is prob no 5 consec heads in n tosses

$b_n = \frac{1}{2} b_{n-1} + \frac{1}{4} b_{n-2} + \cdots + \frac{1}{32} b_{n-5}$

Initial conds easy

Get $b_{100} \approx$ so $1 - b_{100} \approx$

↳ Note: need roots of 5^{th} deg poly

↳ abstract alg and impossibilities

↳ Newton's Method / divide and conquer to approx sol'n

* Question: Is $b_{100} \gtrless 50\%$?

block argument

$[\cdots][\cdots]\cdots[\cdots]$

Ex: More involved specie population

whales: death 3 years, have one pair of calves at 2 yrs, 2 pairs at 1 yr

$$\vec{V}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} \qquad \vec{V}_{n+1} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \vec{V}_n \quad , \quad A = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Get $\vec{V}_{n+1} = A^{n+1} \vec{V}_0$

↳ see need to calculate high powers of A rapidly

↳ fast exponentiation (see Chap 1 my book): $x^{100} = x^{64} x^{32} x^4$

↳ Theoretically: eigenvalues, eigenvectors again

↳ not every matrix diagonalizable

↳ importance of largest eigenvalue

↳ See Section 7.2 for review of matrices

EXTRA CREDIT: $\vec{V}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ with pair George + Grace

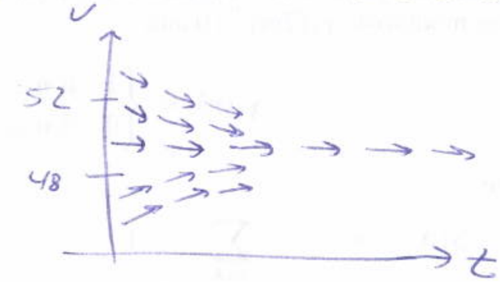
CHAPTER ONE: INTRODUCTION

SECTION 1.1: BASIC MATH MODELS / DIR FIELDS

Can often get qualitative feel of soln by inspection
↳ cannot overemphasize how important this is
↳ baseball example: A wins p%, B wins q% which of $\frac{p \pm pq}{p+q \pm 2pq}$ is Prob A beats B

Direction fields
↳ ex: $\frac{dv}{dt} = 9.8 - \frac{v}{5}$ (From F=ma with air resistance)

at each point in plane
draw arrow with slope
equal to dv/dt



SECTION 1.2: SOLNS TO SOME DIFF EQS

Consider $\frac{dy}{dt} = ay - b \Rightarrow \int \frac{dy}{ay-b} = \int dt \Rightarrow \ln|ay-b| = \frac{t}{a}$ then $at + C$

so $ay - b = \pm e^{c} e^{t\cancel{a}a}$

$\Rightarrow y(t) = \frac{b}{a} + c e^{at}$

Example from above: $a = -1/5$, $b = -9.8$
↳ as $t \to \infty$, approaches $b/a = 49$ exponentially fast

Can write general soln as

$$y(t) = b/a + (y_0 - b/a) e^{at}$$

SECTION 1.3: CLASSIFICATION OF DIFF EQS

- Ordinary Diff Eq (ODE): only depends on single indep variable (else a PDE)
- If need more than 1 eq: System of Diff Eq
  Predator - Prey: $dx/dt = ax - \alpha xy$ $dy/dt = -cy + \gamma xy$
- order: highest derivative appearing: $d^3y/dt = 3d^2y/dt^2 + ye^y$ has order 3
  ↳ always assume can write $y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$ ⊛
- linear if $F(t, y, y', \dots, y^{(n)})$ is linear in $y, y', \dots, y^{(n)}$: $a_0(t)y^{(n)} + \dots + a_n(t)y = g(t)$
- ϕ is a soln of ⊛ for $t \in (\alpha, \beta)$ if $\varphi^{(n)}(t) = f(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t))$ $\forall t \in (\alpha, \beta)$
  ↳ determining soln once know one exists can be hard
  ↳ Euclid: ∞ many primes; hard to find: OEIS: do we have all in this list?

CHAPTER TWO: FIRST ORDER DIFF EQS

General first order eq

$dy/dt + p(t)\,y = g(t)$

idea: product rule: integrating factor $\mu(t)$

$\mu(t)\,y' + p(t)\,\mu(t)\,y = \mu(t)\,g(t)$

want LHS to be $(\mu(t)\,y)'$

$\Rightarrow \mu'(t) = p(t)\,\mu(t) \Rightarrow \mu(t) = \exp\left(\int p(t)\,dt\right) \cdot A$

w.l.o.g. take $A = 1$

Thus $(\mu(t)\,y)' = \mu(t)\,g(t)$

$\Rightarrow y = \dfrac{\int \mu(t)\,g(t)\,dt + c}{\mu(t)}$ $\leftarrow \mu(t) \neq 0$

or $y = \dfrac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)\,g(s)\,ds + c\right]$

↳ Thus explicit form for answer

↳ may not be able to do the two integrals

EX: $y' + (2/t)\,y = 4t \quad y(1) = 2$ (Example 3)

↳ get $\mu(t) = t^2$

$y = t^2 + 1/t^2$

Note soln only exists for $t > 0$

EX: $y' + (t/2)\,y = 1,\ y(0) = 1$ (Example 4)

↳ $\mu(t) = e^{t^2/4}$

$y = e^{-t^2/4}\left[\int_0^t e^{s^2/4}\,ds + c\right]$, $y(0) = 1 \to c = 1$

↳ cannot evaluate integral in closed form

↳ sadly, common

↳ erf function, $\int_{-\infty}^{\infty} e^{-u^2/2}\,du = \sqrt{2\pi}$

↳ Gaussian prob, very important, $(-\tfrac{1}{2})! = \sqrt{\pi}$

(*) Problem 38 IMPORTANT

SECTION 2.2: SEPARABLE EQS

Separable eq: $M(x) + N(y)\frac{dy}{dx} = 0$ or $M(x)dx + N(y)dy = 0$

↳ soln: $\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0$

↳ ex: $2 + e^{x+y}\frac{dy}{dx} = 0 \Rightarrow 2e^{-x}dx + e^{y}dy = 0$

$\Rightarrow -2e^{-x} + e^{y} + C = 0$

↳ can solve for y in terms of x

↳ not always possible

↳ Problem #30 very important

↳ ex: $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$ $\quad v = \frac{y}{x}$ so $xv = y$ or $y' = v + xv'$

↳ $v + xv' = 1 + v + v^2$

$xv' = 1 + v^2$

$$\int \frac{dv}{1+v^2} = \int \frac{dx}{x} \Rightarrow \arctan(v) = \ln|x| + C$$

(SKIP 2.3) Example 4 nice, but accurate? (sun exerts force). Also do Problem #12

SECTION 2.4: DIFF B/W LIN AND NON-LIN DIFF EQ

THM 2.4.1: p, g cont on $I = (\alpha, \beta)$, $t_0 \in I$, $\exists!$ fn $y = \varphi(t)$ st $\forall t \in I$
$y' + p(t)y = g(t)$ and $y(t_0) = y_0$ (y_0 arbitrary)

Proof: basically did @ integrating factors

In Section 2.8 we'll prove

Thm 2.4.2: f, $\partial f/\partial y$ cont in $(\alpha, \beta) \times (\gamma, \delta) \ni (t_0, y_0)$. Then $\exists$ interval $(t_0 - h, t_0 + h) \subset (\alpha, \beta)$ st $\exists!$ soln $y = \varphi(t)$ to $y' = f(t, y)$, $y(t_0) = y_0$

SUMMARY OF LIN EQS: $y' + p(t)y = g(t)$

1. Coeffs cont, $\exists$ gen soln (containing arb constant) including all solns of the diff eq; particular soln by choosing approp value
2. Explicit soln for $y(t)$ (though can involve two integrals)
3. Possible points of discont of soln can be found by locating the discont of coeffs (if coeff cont $\forall t$, soln exists and is cont $\forall t$)

↳ These statements typically false for nonlinear

SECTION 2.5: AUTONOMOUS EQS AND POP DYNAMICS

- Autonomous: do not explicitly depend on indep variable
  $dy/dt = f(y)$

Ex: Exponential Growth

$dy/dt = ry \Rightarrow y = y_0 e^{rt}$

↳ interest, half-life, pop growth

↳ unreasonable in many models in limit

↳ pop doesn't have unrestricted growth: limitations food, space...

Ex: logistic growth

$dy/dt = h(y)\, y$

Want $h(y)$ st $h(y) \approx r$ when y small, ↓ as y ↑, and neg if y large

↳ simplest is $h(y) = r(1 - y/K)$: Verhulst or logistic eq

↳ r is intrinsic growth rate

↳ soln by partial fractions: $\dfrac{dy}{(1 - y/K)\,y} = r\,dt$

so $\displaystyle\int \left(\frac{1}{y} + \frac{1/K}{1 - y/K} \right) dy = \int r\,dt$

$\ln|y| - \ln|1 - y/K| = rt + c \xRightarrow{alg} y = \dfrac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$

↳ $y_0 \neq 0$: $y \to K$: asymptotically stable soln

$y_0 = 0$: $y \equiv 0$: asymptotically unstable soln

K saturation level or carrying capacity

↳ Equilibrium solns $0, K$ from $dy/dt = 0$

Bifurcation Points

↳ See exercise 25

↳ article: Do Dogs Know Calc II and AM/GM

SECTION 2.6: EXACT EQS AND INTEGRATING FACTORS

THM 2.6.1: M, N, M_y, N_x cont in $\underbrace{(\alpha,\beta)\times(\gamma,\delta)}_{=R}$. Then

$M(x,y)+N(x,y)\,y'=0$ is an exact diff eq

iff $M_y = N_x$ $\forall (x,y)\in R$. That is, $\exists$ a $\psi = \psi(x,y)$

st $\psi_x = M$ and $\psi_y = N$ and soln to diff eq is $\psi(x,y)=C$

Proof: $\Rightarrow$ Assume $\exists\ \psi$ st $\psi_x = M$ and $\psi_y = N$

Then $M_y = \psi_{xy}$ and $N_x = \psi_{yx}$ and $\psi_{xy}=\psi_{yx}$ as M_y, N_x cont

Thus $M + N y' = \dfrac{\partial \psi}{\partial x} + \dfrac{\partial \psi}{\partial y}\dfrac{dy}{dx} = \dfrac{d}{dx}\left[\psi(x,\varphi(x))\right]$ with $y=\varphi(x)$

$\Leftarrow$ Assume $M_y = N_x$

Must find ψ st $\psi_x = M$ and $\psi_y = N$

Choose any $Q(x,y)$ st $Q_x = M$

$\hookrightarrow$ ex: $Q(x,y) = \int_{x_0}^{x} M(s,y)\,ds$

Set $\psi(x,y) = Q(x,y) + h(y)$

$\Rightarrow \psi_y = Q_y(x,y) + h'(y)$ must equal $N(x,y)$

$\Rightarrow h'(y) = N(x,y) - \dfrac{\partial Q}{\partial y}$: is this just a fn of y?

$\dfrac{\partial}{\partial x}$ yields $0 = N_x - \dfrac{\partial}{\partial x}\dfrac{\partial Q}{\partial y}$

$= N_x - \dfrac{\partial}{\partial y}\dfrac{\partial Q}{\partial x}$ but $\dfrac{\partial Q}{\partial x} = M$

$= N_x - M_y = 0$ by assump

Thus $N(x,y) - \dfrac{\partial Q}{\partial y}$ is a fn just of y, can find $h(y)$ by $\int$

Ex: $2x + y^2 + 2xy\,y' = 0$

$\psi_x = 2x + y^2$ $\quad \psi_y = 2xy$ $\quad \psi(x,y) = x^2 + xy^2 = c$

alt: $\psi_x = 2x + y^2 \Rightarrow \psi = \int \psi_x\,dx = x^2 + xy^2 + f_1(y)$

$\psi_y = 2xy \Rightarrow \psi = \int \psi_y\,dy = xy^2 + f_2(x)$

$\Rightarrow x^2 + f_1(y) = f_2(x)$

take $f_1(y) = 0$

$f_2(x) = x^2$

Note: integrating factors, text above and example 4, Problem #23

Newton ' s Method to Compute Sqrt[3]

```
iter[x_] := (1 / 2) (x + (3 / x));
temp = 2;
Print["First four iterates, first guess is x0 = 2."];
For[i = 1, i ≤ 4, i++, {
  temp = iter[temp];
  Print[temp];}]
temp = 2;
Print["Numerical approximations to first four iterates."];
For[i = 1, i ≤ 4, i++, {
   temp = iter[temp];
   Print[SetAccuracy[N[1.0 temp, 25], 25]]; }];
Print[SetAccuracy[Sqrt[3.], 25]];
```

```
First four iterates, first guess is x0 = 2.
```

$\frac{7}{4}$

$\frac{97}{56}$

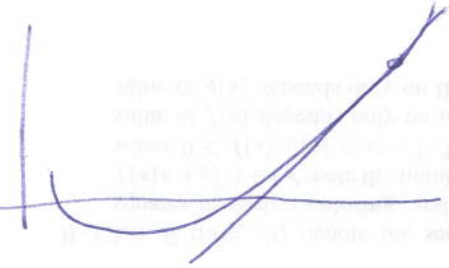
$\frac{18817}{10864}$

$\frac{708158977}{408855776}$

```
Numerical approximations to first four iterates.

1.750000000000000000000000

1.732142857142857206298459

1.732050810014727604269069

1.732050807568877193176604

1.732050807568877193176604
```

SECTION 2.7: NUMERICAL APPROX: EULER'S METHOD

Tangent line + Newton's Method



approx complicated fn @ simpler one

$f(x) = x^2$

$X_{n+1} = \frac{1}{2}\left(X_n + \frac{\alpha}{X_n}\right)$

$\boxed{\text{Compare with Divide and Conquer} \quad \text{Assume more, get more}}$

$\alpha = \sqrt{3}$

$X_1 =$

$X_2 =$

$X_3 =$

Other: Simpson's Rule ...

Euler's Method: $dy/dt = f(t,y)$ and $y(t_0) = y_0$

↳ Approx soln $y = \varphi(t)$ near t_0

tangent line: $y = y_0 + f(t_0, y_0)(t - t_0)$

↳ continue: $y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$

↳ often all step sizes are equal, denote by h: $y_{n+1} = y_n + f(t_n, y_n)\,h$

↳ Question: how accurate?

Question: how does error depend on h?

↳ Can get some sense by looking at cases when know answer

SECTION 2.8: EXISTENCE AND UNIQUENESS THM

THM 2.8.1: f, $\partial f/\partial y$ cont in rectangle $R: |t| \le a,\ |y| \le b$. Then $\exists$ interval $|t| \le h \le a$ s.t. $\exists !$ soln $y = \varphi(t)$ to $y' = f(t,y)$, $y(0) = 0$

↳ Note: wlog reduce to this case

Proof: Picard's method of successive approx / iterative method

Try $\varphi_0(t) \equiv 0$ and construct seq s.t. soln satisfies $\varphi(t) = \int_0^t f(s, \varphi(s))\,ds$

$\varphi_{n+1}(t) = \int_0^t f(s, \varphi_n(s))\,ds$,

Yields seq $\{\varphi_n\}_{n=1}^{\infty}$

↳ if $\varphi_{k+1} = \varphi_k$ then seq terminates and have soln

else want to show $\lim_{n\to\infty} \varphi_n$ soln

SECTION 2.8 (CONT): EXISTENCE + UNIQUENESS THM

Proof depends on 4 items:

↳ ① Does φ_n exist for all n?

② Does seq conv (in what sense: pointwise?)

③ What are prop of limit fn? Does it satisfy integral eq, the initial value prob!

④ Is this only soln?

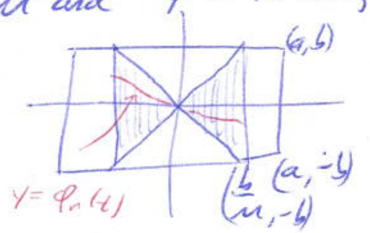
Notes on ①

↳ Danger is at some point the graph $y = \varphi_k(t)$ is outside rectangle R

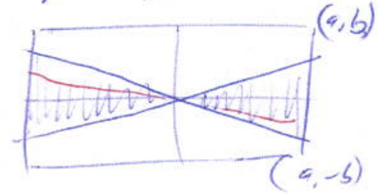
↳ Can fix this by restricting t to $|t| \le h = \min(a, b/M)$

↳ as $|f(t,y)| \le M$ and $y' = f(t,y)$, slope of $y = \varphi_i(t)$ at most M

def restrict.



or



Consider $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$= \sum \frac{1}{n} - \frac{1}{n+1}$

Notes on ②

↳ Write $\varphi_n(t) = \varphi_1(t) + \underbrace{\sum_{k=1}^{n-1} \left(\varphi_{k+1}(t) - \varphi_k(t)\right)}_{\text{need to show converges: Problems 15-18}}$

↳ More generally (ie, more advanced) Cauchy sequence:

$$\left|\varphi_{\ell+1}(t) - \varphi_{k+1}(t)\right| \le \left|\int_0^t \left(f(w, \varphi_\ell(w)) - f(w, \varphi_k(w))\right) dw\right|$$

$$\le \int_0^t \underbrace{\left|\frac{\partial f}{\partial y}(w, w^*)\right|}_{\text{bounded by assump, say by } K} \cdot \left|\varphi_\ell(w) - \varphi_k(w)\right| dw$$

so $\max_{|t| \le h} \left|\varphi_{\ell+1}(t) - \varphi_{k+1}(t)\right| \le |h| K \cdot \max_{|t| \le h} \left|\varphi_\ell(t) - \varphi_k(t)\right|$

if $|h| < 1/2K$, see these terms dominated by geo series

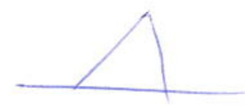
Thus $\varphi = \lim_{n \to \infty} \varphi_n$ exists (see Rudin, Real Analysis, for more details)

SECTION 2.8 (CONT): EXISTENCE + UNIQUENESS THM

Notes on ③

↳ φ need not be cont

Ex: $\psi = \lim \psi_n$ where $\psi_n(x) = \begin{cases} 1 - n|x| & \text{if } |x| \le \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$



↳ $\psi(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$ (see also Problem 13)

(Advanced: φ is cont for $|t| \le h$)

Have $\varphi_{n+1}(t) = \int_0^t f(s, \varphi_n(s))\, ds$

$$\varphi(t) = \lim_{n\to\infty} \varphi_{n+1}(t) = \lim_{n\to\infty} \int_0^t f(s, \varphi_n(s))\, ds$$

not always legal (*)

$$= \int_0^t \lim_{n\to\infty} f(s, \varphi_n(s))\, ds$$

ok: equiv to f is cont in second variable

$$= \int_0^t f(s, \lim_{n\to\infty} \varphi_n(s))\, ds$$

$$= \int_0^t f(s, \varphi(s))\, ds \Rightarrow \varphi \text{ solves initial diffeq}$$

danger on interchanges: $\sum_n \sum_m a_{nm} \ne \sum_m \sum_n a_{nm}$

↳ ex:

$$\begin{matrix} \cdot & \cdot & -1 & +1 \\ \cdot & -1 & +1 & \cdot \\ -1 & +1 & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot \end{matrix}$$

↑↑ then ⇉ gives 0

⇉ then ↑↑ gives +1

Notes on ④

↳ soln is unique: easy for $|t|$ suff small, say $|t| \le h \le 1/2K$

say φ, ψ both solve $\Rightarrow |\varphi(t) - \psi(t)| \le hK\,|\varphi(t) - \psi(t)|$

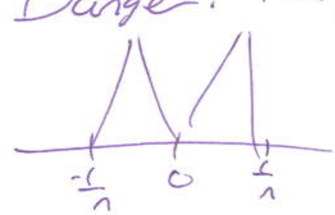
Example: $y' = 2t(1+y)$ $y(0) = 0$

$\varphi_0(t) = 0$, $\varphi_1(t) = \int_0^t 2s(1 + \varphi_0(s))\, ds = t^2$

↳ see $\varphi_n(t) = t^2 + \frac{t^4}{2!} + \cdots + \frac{t^{2n}}{n!}$

See $\lim_{n\to\infty} \varphi_n(t) = e^{t^2} - 1$

(*) Danger: $\lim \int \ne \int \lim$



Chapter 3: Second Order Linear Equations

Section 3.1: Homogeneous Eqs @ Constant Coeffs

2nd Order DiffEq: $y'' = f(t, y, y')$

↳ linear if $f(t,y,y') = g(t) - p(t)y' + q(t)y$

↳ rewrite as $y'' + p(t)y' + q(t)y = g(t)$

↳ sometimes see $P(t)y'' + Q(t)y' + R(t)y = G(t)$

↳ called homogeneous if $g(t)$ or $G(t)$ is zero (else non-homogeneous)

↳ study special case where functions are constant:

↳ $ay'' + by' + cy = 0$

↳ Vector Space of Solns

Review of VSpace: field of scalars F, vectors $\vec{v}$ st

$\forall a_i \in F, \forall \vec{v}_i \in V \quad \sum_{i=1}^{n} a_i \vec{v}_i \in V$

add, scalar mult nice, have zero, ...

Key Fact: solns to lin diff eq form a vector space

↳ if y_1, y_2 solve so does $c_1 y_1(t) + c_2 y_2(t)$

↳ choose c_i to satisfy initial conds

Ex: $y'' - y = 0$, $y(0) = 2$ $y'(0) = -1$ (note need two conds)

↳ easy to see $y_1 = e^t$ and $y_2 = e^{-t}$ solns.

infinite family $y(t) = c_1 e^t + c_2 e^{-t}$

initial conds: $y(0) = c_1 + c_2 = 2 \quad y'(0) = c_1 - c_2 = -1$

↳ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}$

↳ Note: matrix above is invertible: coincidence?

General Case: $ay'' + by' + cy = 0$, $y(0) = y_0$ $y'(0) = y_0'$

↳ Similar to Recurrence relations guess $y(t) = e^{rt}$

yields characteristic eq $(ar^2 + br + c)e^{rt} = 0$

↳ solve $ar^2 + br + c = 0$ for r_1, r_2

↳ solve when $r_1 \neq r_2$ and real

SECTION 3.2: FUND SOLNS OF LIN HOMOGENEOUS EQS

<u>Notation</u>: p, q cont fns on an open interval $I = (\alpha, \beta)$

ϕ twice diff on I, define the differential operator

$L[\phi]$ by $L[\phi] = \phi'' + p\phi' + q\phi$

↳ Note: $L[c_1\phi_1 + c_2\phi_2] = c_1 L[\phi_1] + c_2 L[\phi_2]$

THM 3.2.1: FUND THM OF 2ND ORDER LIN DIFF EQS

Initial Value Problem $y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_0'$

with p, q, g cont on open $I \ni t_0$. Then $\exists!$ soln $y = \phi(t)$ and soln exists $\forall t \in I$

↳ <u>Remarks</u>: Similar to 1st order lin diff eqs but can not write down an explicit soln thru integrating factors.

↳ must use more general methods

↳ proof beyond scope of class

↳ in many cases can find soln

PRINCIPLE OF SUPERPOSITION

$L[y] = y'' + p(t)y' + q(t)$. If $L[y_i] = 0$ Then $L[c_1 y_1 + c_2 y_2] = 0$

<u>Proof</u>: L is a linear operator: brute force calculation

WRONSKIANS

<u>Question</u>: If have two solns, can we generate all? Ie, all solns of $L[y] = 0$ with a given initial cond?

By Thm 3.2.1, know unique soln with given initial conds so if can show a lin comb $c_1 y_1(t) + c_2 y_2(t)$ has the right initial conds, done

$\Rightarrow \quad c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \qquad c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$

Lin Alg: $c_1 = \begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix} / W \qquad c_2 = \begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix} / W$

where $W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$ is the WRONSKIAN DETERMINANT

(sometimes write $W(y_1, y_2)(t_0)$)

↳ If $W \neq 0$, can find c_1 and c_2!

THM 3.2.3: y_1, y_2 solve $L[y] := y'' + p(t)y' + q(t)y = 0$ and $W(y_1, y_2)(t_0) \neq 0$ where $y(t_0) = y_0$, $y'(t_0) = y_0'$. Then $\exists\, c_1, c_2$ st $y = c_1 y_1 + c_2 y_2$

THM 3.2.4: ALL SOLN'S ARE OF FORM $c_1 y_1(t) + c_2 y_2(t)$.

Call $c_1 y_1(t) + c_2 y_2(t)$ the general soln when c_1, c_2 arbitrary; the solns y_1 and y_2 with non-zero Wronskian are called a fundamental set of solns.

$\hookrightarrow$ a fund set of solns always exists

THM 3.2.5: Choose solns to IVP st $y_1(t_0) = 1$, $y_1'(t_0) = 0$ and $y_2(t_0) = 0$, $y_2'(t_0) = 1$. Then y_1, y_2 are a fund set of soln.

Proof: existence of y_1, y_2 from Thm 3.2.1 ($\exists!$ soln)

Simple exercise to see $W(y_1, y_2)(t_0) \neq 0$

Example: Problem #22

$y'' + y' - 2y = 0$ $\quad t_0 = 0$ Find fund set of solns

$\hookrightarrow (r^2 + r - 2)e^{rt} = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow r = -2, 1$

so solns lin combs of e^{-2t}, e^{+t} but these not fund set of solns

Find a_1, a_2 st $y_1(t) = a_1 e^{-2t} + a_2 e^{t}$ has $y_1(0) = 1$, $y_1'(0) = 0$

$\hookrightarrow$ Thus $a_1 + a_2 = 1$, $-2a_1 + a_2 = 0$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ +2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

so $y_1(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{t}$

Similarly $\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$

so $y_2(t) = -\frac{1}{3} e^{-2t} + \frac{1}{3} e^{t}$

Note $W(e^{-2t}, e^{t})(0) = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3 \neq 0$: Advantages / disadvantages to fund solns

SECTION 3.3: LINEAR INDEP AND THE WRONSKIAN

Linear indep in a vector space

↳ $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = 0 \iff$ all $a_i = 0$ then Lin Indep
  ↳ if can do with some $a_i \neq 0$, lin dep

↳ ex: any $n+1$ vectors in $\mathbb{R}^n$ are Lin Dep
  "with prob 1" any n vectors in $\mathbb{R}^n$ are Lin Indep

↳ Can test for Lin Indep in $\mathbb{R}^n$ by det: $\begin{vmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{vmatrix} = 0 \iff$ Lin Dep

↳ in $\mathbb{R}^2$, two vectors LD iff parallel, LD if det is zero

Generalization to functions

↳ For now just do pairs of fns
  say Lin Indep if $\exists k_1, k_2$ not both zero st $k_1 f(t) + k_2 g(t) = 0 \ \forall t$

  ↳ note: lot of concepts can be generalized to Vector Spaces of fns
    ↳ ex: $\vec{v}\cdot\vec{w} \Rightarrow \int f(t)\, g(t)\, dt$

↳ exs: $\sin t$, $\cos(t - \pi/2)$ Lin Dep: $k_1 = 1, k_2 = -1$
  e^t, e^{2t} Lin Indep

THM 3.3.1: f, g diff on I, $W(f,g)(t_0) \neq 0$ then f, g Lin Indep, and if $W(f,g)(t) = 0 \ \forall t \in I$ then Lin Dep
  $\Rightarrow W \neq 0$ for some t_0

Proof: Assume $k_1 f(t) + k_2 g(t) = 0 \quad \forall t \in I$.
  Then $k_1 f'(t) + k_2 g'(t) = 0$
  so $k_1 = k_2 = 0$ as $W \neq 0$ so can solve and thus Lin Indep

$\Leftarrow$ ~~Ass~~ Assume f, g Lin Dep but $W(f,g)(t)$ not $\equiv 0$ in I
  Then $\exists t_0 \in I$ st $W \neq 0$ and thus f, g indep, Contra

CAVEAT: f, g can be Lin Indep even though $W(f,g)(t) \equiv 0$
  ↳ see Problem #28

SECTION 3.3 (CONT): LIN INDEP AND THE WRONSKIAN

Thm 3.3.2: ABEL'S THM: y_1, y_2 solns to $y'' + p(t)y' + q(t)y = 0$, p, q cont on open I, then Wronskian is
$$W(y_1, y_2)(t) = c \exp\left(-\int p(t)\,dt\right), \quad c = c(y_1, y_2)$$
and c indep of t. Thus either identically zero or never zero.

Proof (algebra)

$y_1'' + py_1' + qy_1 = 0 \leftarrow$ mult by $-y_2$

$y_2'' + py_2' + qy_2 = 0 \leftarrow$ mult by y_1

$\Rightarrow (y_1 y_2'' - y_1'' y_2) + p(y_1 y_2' - y_1' y_2) = 0$

$\Rightarrow W' + pW = 0 \Rightarrow W = c\exp\left(-\int p(t)\,dt\right)$

Note: can get Wronskian upto mult constant w/o finding y_1 & y_2, but of course biggest application requires knowing if $c \neq 0$!

THM 3.3.3: y_1, y_2 solve $y'' + py' + qy = 0$ with p, q cont on open I.
Then y_1, y_2 ~~lin indep~~ LIN DEP on I iff $W(y_1, y_2)(t) = 0$ $\forall t \in I$.
Alternatively, y_1, y_2 LIN INDEP iff W never zero on I

SECTION 3.4: COMPLEX ROOTS OF THE CHARACTERISTIC EQ

Defn: $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = \lim_{m \to \infty}\left(1 + \frac{t}{m}\right)^m$

↳ notes: must show $e^t e^u = e^{t+u}$!

$$e^{it} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i\sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!} = \cos t + i \sin t$$

$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}(\cos \mu t + i \sin \mu t)$$

Can now solve $ay'' + by' + cy = 0$

Often want real valued solns, especially if $a, b, c \in \mathbb{R}$

↳ can do by taking appropriate lin comb of solns

↳ roots of poly (as deg 2 with real coeff) occur in $\mathbb{C}$-conjugate pairs

↳ if roots $ar^2 + br + c$ are $\lambda \pm i\mu$, solns $c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t} \sin \mu t$

Ex: $y'' + y' + y = 0$

SECTION 3.5: REPEATED ROOTS, REDUCTION OF ORDER

Recurrence Relation: $a_{n+1} - 2a \cdot a_n + a^2 a_{n-1} = 0$

this isn't in the book

↳ charpoly $r^2 - 2ar + a^2 = 0 \implies r = a, a$

↳ solns a^n, $n \cdot a^n$

↳ can find by $\lim_{r_2 \to r_1 = r} \frac{r_2^n - r_1^n}{r_2 - r_1}$: evaluate @ L'Hopital, get $n r^{n-1}$ or $n r^n$

What do we do for say $y'' + 4y' + 4y = 0$? Roots $r = -2, -2$.

Try $\lim_{r_2 \to r_1 = r} \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} = t e^{rt}$ by L'Hopital

Alternate Method

Know $y(t) = C e^{-2t}$ soln: replace C with $v(t)$

$y(t) = v(t) e^{-2t}$

$y'(t) = v'(t) e^{-2t} - 2v(t) e^{-2t}$

$y''(t) = \underline{\quad}$

algebra: $[v''(t) - 4v'(t) + 4v(t) + 4v'(t) - 8v(t) + 4v(t)] e^{-2t} = 0$

$\implies v''(t) = 0 \implies v(t) = \alpha t + \beta$

↳ Can prove works more generally

↳ Called Reduction of Order

↳ Know $y_1(t)$ solves $y'' + p(t) y' + q(t) = 0$

Find $y_2(t) = v(t) y_1(t)$

↳ algebra: $y_1(t) v''(t) + (2y_1'(t) + p(t) y_1(t)) v'(t) = 0$

↳ actually a first order eq for $v'(t)$!

↳ can solve!

SUMMARY OF 2ND ORDER LIN HOMOGENEOUS EQS @ CONST COEFF

$r_1, r_2 \in \mathbb{R}$ and distinct: soln $c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$r_1 = r_2 \in \mathbb{R}$: soln $c_1 e^{rt} + c_2 t e^{rt}$

$r_1 = \lambda + i\mu = \overline{r_2}$: soln $c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$

Good Problems: #20, #21, #22

SECTION 3.6: NON-HOMOGENEOUS EQ: METHOD UNDETERMINED COEFF

THM 3.6.1: Y_1, Y_2 solns to $y''+py'+qy=g$ (non-homog eq). Then Y_1-Y_2 solves $y''+py'+qy=0$ (homog eq).
If y_1, y_2 fund soln to homog eq Then $Y_1(t)-Y_2(t)=c_1y_1(t)+c_2y_2(t)$.

Thm 3.6.2: General soln to non-homog is $c_1y_1(t)+c_2y_2(t)+Y(t)$ where
Y is any soln to the non-homog eq.

Thus to solve a non-homog
↳ ① find the soln to the homog eq
② find any soln to the non-homog eq

Method of Undetermined Coeff

↳ Guess form of $Y(t)$ but with free coeff
↳ "Divine Inspiration": special cases (assume constant coeff)

GUESSES: $ay''+by'+cy=g$

if: $g(t)=\underbrace{a_0t^n+\dots+a_n}_{P_n(t)}$ guess $Y(t)=\underbrace{t^s(A_0t^n+\dots+A_n)}_{P_{n,s}(t)}$

$g(t)=P_n(t)e^{\alpha t}$ $\quad Y(t)=P_{n,s}(t)e^{\alpha t}$

$g(t)=P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$ $\quad Y(t)=P_{n,s}(t)\cos\beta t+Q_{n,s}(t)\sin\beta t$
where $Q_{n,s}(t)=t^s(B_0t^n+\dots+B_n)$

Proofs that work tedious algebra

Ex 1: $y''-3y'-4y=3e^{2t}$: Try $Y(t)=Ae^{2t}$

Ex 2: $y''-3y'-4y=2\sin t$: Try $Y(t)=A\sin t+B\cos t$
↳ need sin and cos as derivatives flop

Ex 3: $y''-3y'-4y=-8e^t\cos 2t$: Try $Y(t)=Ae^t\cos 2t+Be^t\sin 2t$

SECTION 3.7: VARIATION OF PARAMETERS

General method! takes as input lin indep soln of homogeneous eq

Thm 3.7.1: p, q, g cont on open I, y_1 and y_2 Lin Indep solns to homog eq $y'' + py' + qy = 0$, then a particular soln to non-homog is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\,ds$$

Notes: in general this is hard, as involves integrals of integrals!
↳ does, however, give an explicit soln

Proof: Try $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$

$$Y'(t) = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$\Rightarrow Y'(t) = u_1 y_1' + u_2 y_2'$$

$$Y''(t) = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

make life easy by, having $u_1' y_1 + u_2' y_2 = 0$

Many u_i work: freedom to choose

Substituting:

$$u_1(t)\left[y_1'' + py_1' + qy_1\right] + u_2(t)\left[y_2'' + py_2' + qy_2\right] + u_1' y_1' + u_2' y_2' = g(t)$$

or $u_1' y_1' + u_2' y_2' = g(t)$ and $u_1' y_1 + u_2' y_2 = 0$

Thus $$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

Yields $u_1'(t) = -\dfrac{y_2(t)g(t)}{W(y_1,y_2)(t)}$ $\qquad u_2'(t) = \dfrac{y_1(t)g(t)}{W(y_1,y_2)(t)}$

integrate: $u_1(t) = -\displaystyle\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt + c_1 \cdots$

See book for example
Problem #27 interesting

CHAPTER 4: HIGHER ORDER LINEAR EQUATIONS

SECTION 4.1: GENERAL THEORY OF n^{TH} ORDER EQS

nth order lin diff eq: $P_0(t)\,y^{(n)}(t) + \cdots + P_{n-1}(t)\,y'(t) + P_n(t)\,y(t) = G(t)$

↳ if $P_0(t) \neq 0$: $y^{(n)}(t) + \cdots + P_n(t)\,y(t) = g(t)$

Results very similar to 2nd order diff eq, proofs similar (more Lin Alg)

THM 4.1.1: $P_1, \dots, P_n, g$ cont on open I, $\exists!$ soln $y = \varphi(t)$ to $y^{(n)}(t) + \cdots + P_n(t)\,y = g(t)$ satisfying initial conds $y(t_0) = y_0, \dots, y^{(n-1)}(t_0) = y_0^{(n-1)}$

Wronskian: $W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & & y_n^{(n-1)} \end{vmatrix}$

THM 4.1.2: $P_1, \dots, P_n$ cont on open I and y_ℓ solves $y^{(n)}(t) + \cdots + P_n(t)\,y(t) = 0$ for $1 \le \ell \le n$ Then if $W(y_1, \dots, y_n)(t_0) \neq 0$ for some $t_0 \in I$ then any soln of diff eq is a lin comb of $y_1, \dots, y_n$

SOME (HOPEFULLY INTERESTING) QUESTIONS

MATH 209 (2009): INSTRUCTOR: STEVEN MILLER

Question 1 : Consider $\sum_{n=0}^{\infty} a_n x^n$; must this converge for some $x \neq 0$?

- Yes
- No
- Unknown (open question).

Question 2 : Consider the Taylor series for f and g at the $x = 0$. If the two series are equal, must $f(x) = g(x)$ for some $x \neq 0$?

- Yes
- No
- Unknown (open question).

Question 3 : Let a_n be *any* sequence of real numbers. Is there always an infinitely differentiable function such that the n^{th} derivative at $x = 0$ equals a_n?

- Yes
- No
- Unknown (open question).

Question 4 : Let $f(x)$ be a continuous function. Must $f(x)$ be differentiable for at least one x?

(1) Yes
(2) No
(3) Unknown (open question).

Date: March 18, 2009.

SECTION 5.2: SERIES SOLN NEAR AN ORDINARY POINT, PART 1

Study $P(x)y'' + Q(x)y' + R(x)y = 0$

__Ordinary point__ where $P(x) \neq 0$: ex: Legendre $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

then all points but ± 1 ordinary; otherwise a __singular point__

Method: Guess soln $\sum a_n (x-x_0)^n$ or $\sum a_n (x-x_0)^{n+r}$ and try to solve

__Ex1__: $y'' + y = 0$: Yields cosine and sine Taylor Series

__Ex2__: Airy Eq: $y'' - xy = 0$ (all points ordinary)

↳ get $y'' = \sum (n+2)(n+1) a_{n+2} x^n$

Substitute: $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} a_n x^n = 0$

↳ now shift sums and find __recurrence relation__

$$\hookrightarrow 2 \cdot 1 \cdot a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left((n+2)(n+1) a_{n+2} - a_{n-1} \right) x^n = 0$$

Thus $a_2 = 0$ and $a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}$

$\Rightarrow a_2 = a_5 = a_8 = \cdots = 0$

a_3, a_4 free and determine rest

$$\Rightarrow a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1) 3n} = a_0 \cdot C_{3n}$$

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots 3n \cdot (3n+1)} = a_1 \cdot C_{3n+1}$$

Get two solns, two free params!

$$y(x) = a_0 \sum C_{3n} x^{3n} + a_1 \sum C_{3n+1} x^{3n+1}$$

CHAPTER 7: SYSTEMS OF FIRST ORDER LINEAR EQS

Defn of system of first order differential eqs

$$x_1'(t) = F_1(t, x_1, x_2, \dots, x_n)$$
$$\vdots$$
$$x_i'(t) = F_i(t, x_1, x_2, \dots, x_n)$$
$$\vdots$$
$$x_n'(t) = F_n(t, x_1, x_2, \dots, x_n)$$

Linear if $F_i(t, x_1, \dots, x_n) = a_{i1}(t)\, x_1(t) + \cdots + a_{in}(t)\, x_n(t) + g_i(t)$

↳ often write (use arrows as can't do bold well on blackboard)

$$\vec{x}' = A\vec{x} + \vec{g}(t), \text{ with } \vec{x} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{ij}(t) \end{pmatrix}$$

Note: book often uses $P_{ij}(t)$ for $a_{ij}(t)$

↳ why care?

↳ ① Theory just ordinary first order (integrating factors) plus linear algebra (mostly eigenvalues (vectors)), VERY solvable

↳ ② VERY applicable: can rewrite many eqs as a system.

↳ example (pg 357): $u''(t) + \frac{1}{8}u'(t) + u(t) = 0$

↳ let $x_1(t) = u(t)$, $x_2(t) = u'(t)$ (so $x_1'(t) = u'(t) = x_2(t)$)

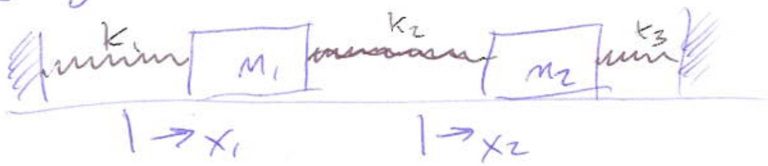
Note $u''(t) = x_2'(t)$. Thus we find

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = -x_1(t) - \frac{1}{8}x_2(t)$$

So
$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1/8 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

SECTION 7.1: INTRODUCTION

Example: Pgs 405 – 406



<u>Physics</u>: $m_1 x_1'' = -(k_1+k_2)x_1 + k_2 x_2$

$m_2 x_2'' = k_2 x_1 - (k_2+k_3)x_2$

Set $y_1 = x_1 \quad y_3 = x_1' \quad (\text{so } y_1' = y_3) \quad (\text{need } y_3',\ y_4')$

$y_2 = x_2 \quad y_4 = x_2' \quad (\text{so } y_2' = y_4)$

Find $y_3' = -\dfrac{k_1+k_2}{m_1} y_1 + \dfrac{k_2}{m_1} y_2$

$y_4' = \dfrac{k_2}{m_2} y_1 - \dfrac{k_2+k_3}{m_2} y_2$

or
$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

<u>THM 7.1.1</u>: F_i and $\partial F_i/\partial x_j$ cont in region R defined by $t \in [\alpha, \beta]$ and $x_j \in [\alpha_j, \beta_j]$, $(t_0, x_1^0, \dots, x_n^0) \in R$. Then for some $h > 0$ There is a soln $x_i(t) = \varphi_i(t)$ to the initial value problem and the soln is unique.

<u>THM 7.1.2</u>: Assume in the system of linear equations each $P_{ij}(t)$ and each $g_j(t)$ cont on open interval (α, β). Then there is a unique soln to the initial value problem which exists throughout (α, β).

7.2. Review of Matrices and 7.3 Alg Eqs, Lin Indep, Evalues...

↳ talk about cA, $A+B$, $A \cdot B$, $A\vec{v}$, $A(c_1\vec{v}_1 + c_2\vec{v}_2)$: linear

↳ talk about $\det(A)$: volume parallelpiped, independence

↳ talk about lin independent / dependent

↳ talk about Vector Space of Solns

↳ Won't get into invertible, Gaussian Elimination, Gauss Jordan

↳ Matrix operations:

$$A = \left(a_{ij}(t) \right) \quad \text{Then} \quad \frac{dA}{dt} = \left(\frac{da_{ij}}{dt} \right) \quad \text{and} \quad \int_a^b A(t)\,dt = \left(\int_a^b a_{ij}(t)dt \right)$$

↳ $\frac{d}{dt}\left(C A(t) \right) = C \frac{dA}{dt}$ C const matrix

$$\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt}$$

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B$$

$$e^A e^B \neq e^{A+B} \text{ unless } [A,B] = AB - BA = 0$$

↳ Baker - Campbell - Hausdorff formula

CLICKER QUESTIONS

↳ Eigenvalues / Eigenvectors: usually $A\vec{v}$ different direction than $\vec{v}$

$$A\vec{v} = \lambda\vec{v} \Rightarrow (A - \lambda I)\vec{v} = 0$$

$$\Rightarrow A - \lambda I \text{ not invertible so } \operatorname{Det}(A - \lambda I) = 0$$

Things very nice if all evalues distinct

↳ implies matrix is diagonalizable

↳ $\exists$ invertible T st $T^{-1}AT = \Lambda$ = diagonal matrix

Clicker Questions

Are evalues of real matrices real?

Are evectors of real matrices real?

Does every matrix have an evalue?

Is every matrix diagonalizable?

↳ benefits: FAST Computations:

↳ $A = T\Lambda T^{-1}$ so $A^n = T\Lambda^n T^{-1}$!

↳ any real symm matrix is diagonalizable with real evalues

Needed Aside: How do we diagonalize a matrix?

Let A be $n \times n$ constant matrix with evalues $\lambda_1, \dots, \lambda_n$ (not nec distinct!) but with n linearly independent eigenvectors $\vec{v}_1, \dots, \vec{v}_n$.

Let $S = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}$, S^{-1} exists as lin indep columns

Note $S\vec{e}_i = \vec{v}_i$ and $S^{-1}\vec{v}_i = \vec{e}_i$

Let $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

Claim: $A = S\Lambda S^{-1}$ or $S^{-1}AS = \Lambda$

If $S^{-1}AS\vec{e}_i = \Lambda\vec{e}_i$ $\forall i$, done as $\vec{e}_i$ a basis

$$\begin{aligned}(S^{-1}AS)\vec{e}_i &= (S^{-1}A)(S\vec{e}_i) \\ &= S^{-1}(A\vec{v}_i) \\ &= S^{-1}(\lambda_i\vec{v}_i) \\ &= \lambda_i S^{-1}\vec{v}_i \\ &= \lambda_i\vec{e}_i \\ &= \Lambda\vec{e}_i \quad \blacksquare\end{aligned}$$

So if we can find n linearly indep evectors, done

If know λ is an evalue, to find evector must solve

$(A - \lambda I)\vec{v} = \vec{0}$: Gaussian Elimination / Gauss-Jordan...

7.5. Homogeneous & Linear Systems @ Constant Coeffs

Consider $\vec{X}'(t) = A\vec{X}(t)$ with A a constant matrix.

Case 1: A is diagonal

↳ Can "uncouple": $\begin{pmatrix} X_1'(t) \\ \vdots \\ X_n'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}$

See $X_i'(t) = \lambda_i X_i(t)$ or $X_i(t) = c_i e^{\lambda_i t}$

Let $\vec{e}_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, the vector with zeros everywhere but ith spot, where we have a 1.

Then general soln is

$$\vec{X}(t) = c_1 e^{\lambda_1 t} \vec{e}_1 + \cdots + c_n e^{\lambda_n t} \vec{e}_n,$$

and note $\vec{e}_1, \ldots, \vec{e}_n$ are evectors of A with evalues $\lambda_1, \ldots, \lambda_n$

General A: Guess $\vec{X}(t) = \vec{\xi} e^{rt}$ for some vector $\vec{\xi}$

↳ $\vec{X}' = A\vec{X} \Rightarrow r e^{rt} \vec{\xi} = e^{rt} A \vec{\xi}$

or $(A - rI)\vec{\xi} e^{rt} = 0 \Rightarrow r$ evalue, $\vec{\xi}$ evector

Case 2: A is diagonalizable

Say $\exists\, T$ st $T^{-1}AT = \Delta$ or $A = T\Delta T^{-1}$

Then $\vec{X}' = T\Delta T^{-1}\vec{X}$

or $T^{-1}\vec{X}' = \Delta T^{-1}\vec{X}$

↳ let $\vec{Y} = T^{-1}\vec{X}$ (no loss as T invertible)

Then $\vec{Y}' = \Delta \vec{Y}$, reduce to case 1!

Completely solved for diagonalizable matrices!

↳ includes real symm, complex Hermitian, unitary, normal, ...

↳ not all matrices diagonalizable

↳ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$: leads to Jordan Canonical Form (See Section 7.8)

↳ See Conway's See and Say Sequence for fun application of Jordanizable

7.5 Homog Lin Systems (Cont) and 7.7 Fundamental Matrices

Will not do much of Chapter 7 sections 7 and 8 as not assuming LinAlg

Consider $\vec{x}'(t) = A\vec{x}(t)$, A constant matrix, $\vec{x}(0) = \vec{x}^0$

Claim $\exists!$ solution, namely $\vec{x}(t) = \exp(At)\,\vec{x}^0$.

↳ Proof: $\exp(At) = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$ note A, t commute; note A, A commute

↳ Question: does the sum converge?

↳ always go back to 1-dim for intuition:

$\exp(at) = 1 + at + \frac{a^2t^2}{2!} + \cdots$ converges: $(at)^m \ll m!$ (grows a lot slower)

let $M = \max_{1 \le i,j \le n} |a_{ij}|$

let M_k be the maximum of the absolute value of entries of A^k

Prove $M_k \le \frac{(nM)^k}{n}$ where A is an $n \times n$ matrix

↳ implies series for $\exp(At)$ converges

↳ Note if $|t|$ small then $A \approx I$ and invertible

↳ Aside: $\exp(At)$ always invertible: inverse $\exp(-At)$

$$\boxed{\text{While } \exp(At)\vec{x}_0^0 \text{ is the unique sol'n, in general hard to compute } \exp(At)!}$$

↳ $e^A e^B = \left(\sum \frac{A^k}{k!}\right)\left(\sum \frac{B^l}{l!}\right)$: if $AB \ne BA$ can't rearrange

↳ BIG difference from 1-dimension!

↳ However: $e^{TAT^{-1}} = I + TAT^{-1} + \frac{TA\cancel{T^{-1}T}AT^{-1}}{2!} + \frac{TA\cancel{T^{-1}T}A\cancel{T^{-1}T}AT^{-1}}{3!} + \cdots$

$$= T\left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\right)T^{-1} = Te^AT^{-1}$$

So if ~~A~~ $TAT^{-1} = \Lambda$ (I know, backwards from before, my bad):

$$e^A = T^{-1}e^{TAT^{-1}}T = T^{-1}e^{\Lambda}T = T^{-1}\begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}T \quad \boxed{\text{doable Calculation}}$$

7.9: Nonhomogeneous Linear Systems

$\vec{x}'(t) = A\vec{x}(t) + \vec{g}(t)$, A constant matrix

Assume A is diagonalizable: $T^{-1}AT = \Lambda$ or $A = T\Lambda T^{-1}$

Change variables: $\vec{x}(t) = T\vec{y}(t)$ or $\vec{y}(t) = T^{-1}\vec{x}(t)$

$$T\vec{y}'(t) = AT\vec{y}(t) + \vec{g}(t)$$

or $\vec{y}'(t) = T^{-1}AT\vec{y}(t) + T^{-1}\vec{g}(t)$

or $\vec{y}'(t) = \Lambda\vec{y}(t) + \vec{h}(t)$, $\vec{h}(t) = T^{-1}\vec{g}(t)$

ie,
$$\begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} + \begin{pmatrix} h_1(t) \\ \vdots \\ h_n(t) \end{pmatrix}$$

So after algebra we've reduced to

$$y_i'(t) = \lambda_i y_i(t) + h_i(t) \qquad \mu_i(t) = \exp\left(\int -\lambda_i\, dt\right) = e^{-\lambda_i t}$$

$$\text{so } y_i(t) = e^{\lambda_i t}\left(\int_{t_0}^{t} e^{-\lambda_i s} h_i(s)\,ds + c_i\right)$$

and then take $T\vec{y}(t)$ to get $\vec{x}(t)$!

Ex: Solve $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$ (pg 433)

↳ Brook-Book then: evalues $-1, -3$

evector with -1 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, with -3 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as real symm

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad T^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\vec{y}' = \begin{pmatrix} -3 & \\ & -1 \end{pmatrix}\vec{y} + \frac{1}{\sqrt{2}}\begin{pmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{pmatrix}$$

Solns $y_1 = \frac{\sqrt{2}}{2}e^{-t} - \frac{3}{\sqrt{2}}\left(\frac{t}{3} - \frac{1}{9}\right) + c_1 e^{-3t}$

$$y_2 = \sqrt{2}\, t e^{-t} + \frac{3}{\sqrt{2}}(t-1) + c_2 e^{-t}$$

KEY FACT

For small enough time intervals, any matrix is approximately constant, use for approximations. See Chapter 9

Can generalize: method undet coeff, non-constant A and variation of params...

= CHAPTER 7-6 =

CHAPTER 6: THE LAPLACE TRANSFORM

SECTION 6.1: DEFN OF THE LAPLACE TRANSFORM

Danger with improper integrals: $\lim_{\substack{A\to\infty \\ B\to\infty}} \int_{-A}^{B} f(x)dx$ could depend on how go to infinity

↳ ex: $\int_{-A}^{2A} f(t)dt$ vs $\int_{-A}^{A} f(t)dt$ for $f(t) = \frac{t}{t^2+1}$

Piecewise cont fn: finite # points $\alpha = t_0 < t_1 < t_2 < \cdots < t_n = \beta$ st f is cont on (t_{i-1}, t_i) and ~~limit exists~~ left and right hand limits exist for each t_i

THM 6.1.1: f piecewise cont for $t \geq a$, $|f(t)| \leq g(t)$ $\forall t \geq M$ then
(integral test)
- $\int_M^\infty g(t)dt$ conv $\Rightarrow \int_a^\infty f(t)\,dt$ conv
- If instead $f(t) \geq g(t) \geq 0$ $\forall t \geq M$ then
- $\int_M^\infty g(t)dt$ diverges $\rightarrow \int_a^\infty f(t)\,dt$ diverges

Integral Transforms: $F(s) = \int_\alpha^\beta K(s,t) f(t)\,dt$, K the kernel.

Laplace Transform: $\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

↳ Why Useful?
1. convert DiffEq for f to algebraic problem for $\mathcal{L}[f(t)] = F(s)$
2. solve for $F(s)$
3. invert and find f from F (can be hard!)

THM 6.1.2: f piecewise cont on $[0,A]$ for ~~some~~ all A, $|f(t)| \leq Ke^{at}$ $\forall t \geq M$ for constants a, $K \geq 0$, $M \geq 0$
Then $\mathcal{L}[f(t)]$ exists for $s > a$

Proof: Use $\int_0^\infty e^{-st} f(t)\,dt = \int_0^M e^{-st} f(t)dt + \int_M^\infty e^{-st} f(t)dt$

Note: "Big" table of Laplace Transforms in Section 6.2

↳ ex: $\mathcal{L}[1] = 1/s \quad s > 0$

$\mathcal{L}[e^{at}] = 1/(s-a) \quad s > a$

$\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2} \quad s > 0$

Linear Operator: $\mathcal{L}[c_1 f_1(t) + \cdots + c_k f_k(t)] = c_1\mathcal{L}[f_1(t)] + \cdots + c_k\mathcal{L}[f_k(t)]$

SECTION 6.2: SOL'N OF INITIAL VALUE PROBLEMS

THM 6.2.1: f cont, f' piecewise cont on any $[0, A]$, $|f(t)| \le K e^{at}$ $t \ge M$.
Then $\mathcal{L}[f'(t)]$ exists $\forall t \ge a$ and $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$

Proof: assume for simplicity f, f' cont
integrate by parts: $\int_0^A e^{-st} f'(t)\,dt$ and let $A \to \infty$

CORR: $f, f', \dots, f^{(n-1)}$ cont, $f^{(n)}$ piecewise cont on any $[0, A]$
and $|f^{(i)}(t)| \le K e^{at}$. Then $\mathcal{L}[f^{(n)}(t)]$ exists and equals

$$s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Ex: $y'' - y' - 2y = 0$ with $y(0) = 1$, $y'(0) = 0$
↳ know sol'n comes from e^{rt}, get $\frac{2}{3} e^{-t} + \frac{1}{3} e^{2t}$

↳ Now use Laplace Transform

$\mathcal{L}[y''] - \mathcal{L}[y'] - \mathcal{L}[2y] = 0$, set $Y(s) = \mathcal{L}[y(t)]$

$\Rightarrow \left(s^2 Y(s) - s\,y(0) - y'(0)\right) - \left(s Y(s) - y(0)\right) - 2Y(s) = 0$

$(s^2 - s - 2) Y(s) + (1 - s)\,y(0) - y'(0) = 0$ but $y(0) = 1$, $y'(0) = 0$

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}$$

↳ partial fractions! $\frac{a}{s-2} + \frac{b}{s+1} = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}$

know $\mathcal{L}[e^{at}] = \frac{1}{s-a}$

Thus $y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$

Question: $\mathcal{L}[y_1] = \mathcal{L}[y_2]$ iff $y_1 = y_2$?

↳ For us want $\mathcal{L}^{-1}$ unique: can appeal to sol'ns to diff eq unique....

Converts solving diff eq to algebra and inverting Laplace Transform!

CHAPTER 8: NUMERICAL METHODS

Sec 8.1: THE EULER OR TANGENT LINE METHOD

↳ Review tangent line 

Application: Newton's Method

Consider first order eq: $y' = f(t, y)$ with $y(t_0) = y_0$

↳ Assume f, f_y cont in rectangle $\ni (t_0, y_0)$

By Thm 2.4.2, $\exists!$ soln in nbhood to

$$\boxed{\begin{aligned} &\underline{\text{Euler/Tangent line}} \quad y_{n+1} = y_n + f(t_n, y_n)\cdot(t_{n+1} - t_n) \\ &\qquad\qquad \text{or } y_{n+1} = y_n + f(t_n, y_n)\, h \text{ if stepsize constant} \end{aligned}}$$

Idea: $f(t, y)$ is slope, tells us how y is changing

<u>ERRORS</u>: φ soln, $E_n = \varphi(t_n) - y_n$: <u>global truncation error</u>

Q ↳ at each step make errors, input data at each step only approx

↳ if assume y_{n-1} exactly right, error in going one step forward to y_n is <u>local truncation error</u>.

↳ also have <u>round-off errors</u> from computers

↳ Lorentz equations and birth of chaos

↳ story of restarting ...

↳ We'll assume no round-off and analyze just local truncation error

Assume $y = \varphi(t)$ soln @ cont second deriv (true if f, f_t, f_y cont)

↳ $\varphi'(t) = f(t, \varphi(t))$

$\varphi''(t) = f_t(t, \varphi(t)) + f_y(t, \varphi(t))\varphi'(t)$ Chain rule, $\varphi'(t) = f(t, \varphi(t))$

Taylor series: $\underset{\varphi(t_{n+1})}{\overset{\|}{\varphi(t_n + h)}} = \varphi(t_n) + \varphi'(t_n) h + \frac{1}{2}\varphi''(\bar{t}_n) h^2$

and $y_{n+1} = y_n + f(t_n, \overset{y_n}{\cancel{\varphi(t_n)}})\, h$

Subtract: $\varphi(t_{n+1}) - y_{n+1} = \underbrace{(\varphi(t_n) - y_n)}_{\text{assuming zero as}} + h\underbrace{[f(t_n, \varphi(t_n)) - f(t_n, y_n)]}_{\text{taking } y_n = \varphi(t_n)} + \frac{1}{2}\varphi''(\bar{t}_n) h^2$

local error $|\varphi(t_{n+1}) - y_{n+1}| = \left|\frac{1}{2}\varphi''(\bar{t}_n) h^2\right| \le \max|\varphi''| \cdot \frac{h^2}{2}$

Key fact: error is like h^2!

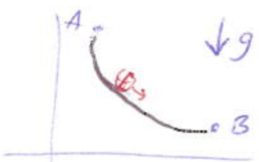
CALCULUS OF VARIATIONS (GELFAND-FOMIN)

Goal: find function to max/min certain expressions
↳ can do modern phys from this point of view
↳ $F=ma$ consequence

CHAPTER 1: ELEMENTS OF THE THEORY

Study expressions of form $\int_a^b F(x,y,y')\,dx$ with $y=y(x)$
↳ Find y satisfying boundary conds that max/min above

Exs: ① Find shortest plane curve joining A and B (ie, a line!)
↳ $\int_a^b \sqrt{1+y'(x)^2}\,dx$ with $y(a)=A$ $y(b)=B$

② 

path for particle to slide under gravity from A to B in least time: brachistochrone
↳ Lion's Paw

③ Of all curves of a given length l, which has largest area?

Let $J[y] = \int_a^b F(x,y,y')\,dx$ with $y(a)=A$, $y(b)=B$
↳ J maps fns to real numbers, called a functional
↳ J can act on many spaces: natural is $\mathcal{C}^1$ (cont diff fns)
↳ These ($\mathbb{R}^n, \mathcal{C}^1, \mathcal{C}^2, \ldots$) Vector Spaces (lin alg)
↳ want to study normed vector-spaces (sense of dist b/w elements)
↳ ① $\|\vec{x}\| = 0 \iff \vec{v} = \vec{0}$
② $\|\alpha\vec{x}\| = |\alpha|\,\|\vec{v}\|$ α scalar
③ $\|\vec{v}+\vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$
↳ Examples: $\mathcal{C}(a,b)$: $\|y\|_0 = \max\limits_{a\le x\le b} |y(x)|$

$\mathcal{C}^1(a,b)$: $\|y\|_1 = \max\limits_{a\le x\le b} |y(x)| + \max\limits_{a\le x\le b} |y'(x)|$

$\mathcal{C}^n(a,b)$: $\|y\|_n = \sum\limits_{i=1}^{n} \max\limits_{a\le x\le b} |y^{(i)}(x)|$

↳ note: what happens with other norms?

CHAPTER 1 (CONT): ELEMENTS OF THE THEORY

<u>DEFN</u>: Functional $J[y]$ cont at $\hat{y}$ if $\forall \epsilon > 0\ \exists \delta > 0$ st

$$|J[y] - J[\hat{y}]| < \epsilon \text{ whenever } \| y - \hat{y} \| < \delta$$

<u>Caveat</u>: $J[y] = \int_a^b \sqrt{1 + y'(x)^2}\, dx$: arc length

$\hookrightarrow$ cont in $\mathcal{C}'(a,b)$

$\hookrightarrow$ not cont in $\mathcal{C}(a,b)$ 

<u>DEFN</u>: Normed vector space $\mathcal{R}$, functional φ is a cont linear functional if

① $\varphi[h]$ is cont $\forall h \in \mathcal{R}$

② $\varphi[\alpha h] = \alpha \varphi[h]$ for $h \in \mathcal{R}$ and α scalar

③ $\varphi[h_1 + h_2] = \varphi[h_1] + \varphi[h_2]$ $\forall h_i \in \mathcal{R}$.

<u>LEMMA 1</u>: $\alpha \in \mathcal{C}(a,b)$ and $\int_a^b \alpha(x) h(x)\, dx = 0$ $\forall h \in \mathcal{C}(a,b)$ with $h(a) = h(b) = 0$.

Then $\alpha(x) = 0$ $\forall x \in [a,b]$.

<u>Proof</u>: $\alpha(x_0) > 0$ (wlog at interior point)

$\alpha(x) > 0$ x close to x_0, say within δ

Take $h(x) = \begin{cases} \cancel{(x-x_0)^2}\ \left(x - (x_0 - \frac{\delta}{2})\right)\left(x - (x_0 + \frac{\delta}{2})\right) & |x - x_0| \le \frac{\delta}{2} \\ 0 & \text{otherwise} \end{cases}$

then $\int_a^b \alpha(x) h(x)\, dx > 0$, contradiction

<u>CORR</u>: Using $h(x)^n$ shows true if assume $h \in \mathcal{C}^n(a,b)$

<u>LEMMA 2</u>: $\alpha \in \mathcal{C}'(a,b)$ and $\forall h \in \mathcal{C}'(a,b)$ have $\int_a^b \alpha(x) h'(x)\, dx = 0$ when $h(a) = h(b) = 0$.

Then $\exists c$ st $\alpha(x) = c$ $\forall x \in [a,b]$

<u>LEMMA 3</u>: $\alpha \in \mathcal{C}'(a,b)$, assume $\forall h \in \mathcal{C}^2(a,b)$ with $h(a) = h'(a) = h(b) = h'(b) = 0$

Then $\exists c_1, c_2$ st $\alpha(x) = c_0 + c_1 x_n$ for all $x \in [a,b]$.

<u>LEMMA 4</u>: $\alpha, \beta \in \mathcal{C}(a,b)$ and $\forall h \in \mathcal{C}'(a,b)$ with $h(a) = h(b) = 0$, assume

$\int_a^b [\alpha(x) h(x) + \beta(x) h'(x)]\, dx = 0$. Then β is diff and $\beta'(x) = \alpha(x)$ on $[a,b]$

CHAPTER 1 (CONT): ELEMENTS OF THE THEORY

Variation of a Function:

Define $\Delta J[h] = J[y+h] - J[y]$

↳ change when increment the "variable" y (a function) by $h = h(x)$

Suppose $\Delta J[h] = \varphi[h] + \varepsilon \|h\|$ where φ is a linear functional and $\varepsilon \to 0$ as $\|h\| \to 0$. Then say the functional $J[h]$ is <u>differentiable</u> and the <u>variation (or differential)</u>, $\varphi[h]$, is denoted $\delta J[h]$.

<u>THM</u>: The differential of a differentiable function is unique.

<u>Proof</u>: Note if $\lim_{\|h\| \to 0} \frac{\varphi[h]}{\|h\|} = 0$ then $\varphi[h] = 0$

↳ if not, Assume $\varphi[h_0] \neq 0$ some h_0, set $h_n = \frac{h_0}{n}$ and $\|h_n\| \to 0$

If not unique: $\varphi_1[h] + \varepsilon_1 \|h\| = \varphi_2[h] + \varepsilon_2 \|h\|$

$\Rightarrow$ lin functional $\varphi_1 + \varphi_2$ must be zero, clear

<u>THM</u>: A nece cond for diff functional $J[y]$ to have an extremum at $\hat{y}$ is that its variation is zero at $\hat{y}$: $\delta J[h] = 0$ $\forall$ admissible h

<u>Proof</u>: Assume have min, so $\|h\|$ small have $\Delta J[h]$ and $\delta J[h]$ same sign

Suppose $\exists h_0$ st $\delta J[h_0] \neq 0$. Then $\forall \alpha$, $\delta J[-\alpha h_0] = -\delta J[\alpha h_0]$

can either be pos or neg for arb small h

↳ violates assump $J[y]$ min at $\hat{y}$, or $\Delta J[h] = J[\hat{y}+h] - J[\hat{y}] \geq 0$ for h small

CHAPTER 1 (CONT): ELEMENTS OF THE THEORY

SIMPLEST VARIATIONAL PROBLEM: EULER'S EQ

<u>THM</u>: $J[y] = \int_a^b F(x, y, y')\,dx$, $y \in \mathcal{C}^1(a,b)$, $y(a) = A$, $y(b) = B$. Then $J[y]$ has extremum at y if y satisfies Euler Eq: $F_y - \frac{d}{dx} F_{y'} = 0$

<u>Note</u>: This is a second order diff eq

<u>Proof</u>: Say increment $y(x)$ by $h(x)$ to $y(x) + h(x)$

Clearly need $h(a) = h(b) = 0$ to still satisfy boundary conds

$$\begin{aligned}\Delta J[h] &= J[y+h] - J[y] \\ &= \int_a^b [F(x, y+h, y'+h') - F(x, y, y')]\,dx \\ &= \int_a^b [F_y(x,y,y')\,h + F_{y'}(x,y,y')\,h']\,dx + \ldots\end{aligned}$$

higher order terms in h, h'

Necc cond for extremum of $J[y]$ at y is $\delta J[h] = 0$, with

$$\delta J[h] = \int_a^b (F_y\, h + F_{y'}\, h')\,dx$$

By Lemma 4, implies $F_y - \frac{d}{dx} F_{y'} = 0$

<u>Ex</u>: Among all curves joining (x_0, y_0) and (x_1, y_1), find one which generates minimum area when revolve about x-axis (ans: catenary/catenoid)

$$J[y] = \int_{x_0}^{x_1} 2\pi y \sqrt{1 + y'(x)^2}\,dx \quad \text{(can ignore } 2\pi)$$

$\hookrightarrow$ Thus $F_y - \frac{d}{dx} F_{y'} = 0$ with $F(x, y, y') = y\sqrt{1 + y'(x)^2}$

<u>Note</u>: doesn't depend on x explicitly

$F_y - \frac{d}{dx} F_{y'} = F_y - F_{y'y}\, y' - F_{y'y'}\, y''$ (do for above)

mult by y': $F_y\, y' - F_{y'y}\, y'^2 - F_{y'y'}\, y' y'' = 0 = \frac{d}{dx}(F - y' F_{y'})$

Thus $\exists$ C st $F - y' F_{y'} = 0$

or $y\sqrt{1+y'^2} - \frac{y\, y'^2}{\sqrt{1+y'^2}} = C \Rightarrow y' = \sqrt{\frac{y^2 - C^2}{C^2}}$

Get $\frac{C\,dy}{\sqrt{y^2 - C^2}} = dx \Rightarrow x + C_1 = C \ln\left(\frac{y + \sqrt{y^2 - C^2}}{C}\right) \Rightarrow y = C\cosh\left(\frac{x + C_1}{C}\right)$

CHAPTER 4: CANONICAL FORM OF EULER EQ AND RELATED TOPICS

Generalization: $J[y_1, \dots, y_n] = \int_a^b F(x, y_1, \dots, y_n, y_1', \dots, y_n')\,dx$

↳ Euler Eq becomes $F_{y_i} - \frac{d}{dx} F_{y_i'} = 0 \quad i = 1, \dots, n$

Principle of Least Action

n particles, i^{th} has mass m_i and coords $(x_i(t), y_i(t), z_i(t))$ and velocity $(x_i'(t), y_i'(t), z_i'(t))$. Assume system has potential energy $U = U(t, x_1, y_1, z_1, \dots, x_n, y_n, z_n)$ st force on i^{th} is

$X_i = -\frac{\partial U}{\partial x_i}, \quad Y_i = -\frac{\partial U}{\partial y_i}, \quad Z_i = -\frac{\partial U}{\partial z_i} \quad (\text{Force} = -\nabla U)$

Let $T = \sum_{i=1}^{n} \frac{1}{2}(x_i'(t)^2 + y_i'(t)^2 + z_i'(t)^2)$ be ~~potential~~ kinetic energy

Let $L = T - U$ (the action). Then motion of n particles is such that $\int_{t_0}^{t_1} L\,dt$ is a minimum.

Euler eqs become $\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial x_i'} = 0 \quad (= L_{x_i} - \frac{d}{dt} L_{x_i'})$ (same y_i, z_i)

$$\Rightarrow -\frac{\partial U}{\partial x_i} - m_i x_i''(t) = 0 \Rightarrow \left.\begin{aligned} X_i &= m_i x_i''(t) \\ Y_i &= m_i y_i''(t) \\ Z_i &= m_i z_i''(t) \end{aligned}\right\} \vec{F} = m\vec{a}$$

One of most important systems of 2nd order diff eqs!

Note: alternate formulations

↳ Hamiltonian: $T + U$, generalizes better for quantum

↳ generalized coords

↳ get cons of energy, momentum, angular momentum, ...