CHAPTER 7: SYSTEMS OF FIRST ORDER LINEAR EQS

Defn of system of first order differential eqs

$$x_1'(t) = F_1(t, x_1, x_2, \ldots, x_n)$$

$$\vdots$$

$$x_i'(t) = F_i(t, x_1, x_2, \ldots, x_n)$$

$$\vdots$$

$$x_n'(t) = F_n(t, x_1, x_2, \ldots, x_n)$$

Linear if $F_i(t, x_1, \ldots, x_n) = a_{i1}(t)\,x_1(t) + \cdots + a_{in}(t)\,x_n(t) + g_i(t)$

↳ often write (use arrows as can't do bold well on blackboard)

$$\vec{x}' = A\vec{x} + \vec{g}(t), \text{ with } \vec{x} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{ij}(t) \end{pmatrix}$$

Note: book often uses $p_{ij}(t)$ for $a_{ij}(t)$

↳ why care?

↳ ① Theory just ordinary first order (integrating factors) plus linear algebra (mostly eigenvalues/vectors), VERY solvable

↳ ② VERY applicable: can rewrite many eqs as a system.

↳ example (pg 357): $u''(t) + \frac{1}{8}u'(t) + u(t) = 0$

↳ let $x_1(t) = u(t)$, $x_2(t) = u'(t)$ (so $x_1'(t) = u'(t) = x_2(t)$)

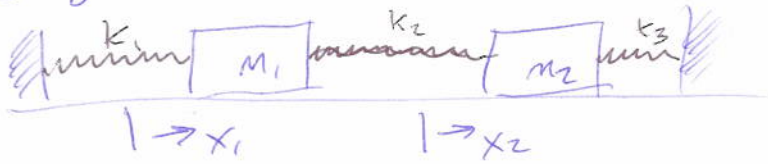
Note $u''(t) = x_2'(t)$. Thus we find

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = -x_1(t) - \frac{1}{8}x_2(t)$$

So
$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1/8 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Section 7.1: Introduction

Example: pgs 405 – 406



Physics: $m_1 x_1'' = -(k_1+k_2)x_1 + k_2 x_2$

$m_2 x_2'' = k_2 x_1 - (k_2+k_3)x_2$

Set $y_1 = x_1$ $\quad y_3 = x_1'$ (so $y_1' = y_3$) (need y_3', y_4')

$y_2 = x_2$ $\quad y_4 = x_2'$ (so $y_2' = y_4$)

Find $y_3' = -\frac{k_1+k_2}{m_1} y_1 + \frac{k_2}{m_1} y_2$

$y_4' = \frac{k_2}{m_2} y_1 - \frac{k_2+k_3}{m_2} y_2$

or

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

THM 7.1.1: F_i and $\partial F_i / \partial x_j$ cont in region R defined by $t \in [\alpha, \beta]$ and $x_j \in [\alpha_j, \beta_j]$, $(t_0, x_1^0, \dots, x_n^0) \in R$. Then for some $h > 0$ there is a soln $x_i(t) = \varphi_i(t)$ to the initial value problem and the soln is unique.

THM 7.1.2: Assume in the system of linear equations each $p_{ij}(t)$ and each $g_j(t)$ cont on open interval (α, β). Then there is a unique soln to the initial value problem which exists throughout (α, β).

7.2. REVIEW OF MATRICES and 7.3 AlgEqs, LinIndep, Evalues...

↳ talk about cA, $A+B$, $A \cdot B$, $A\vec{v}$, $A(c_1\vec{v}_1 + c_2\vec{v}_2)$: linear

↳ talk about $\det(A)$: volume parallelpiped, independence

↳ talk about lin independent / dependent

↳ talk about Vector Space of Solns

↳ Won't get into invertible, Gaussian Elimination, Gauss Jordan

↳ Matrix operations:

$$A = \left(a_{ij}(t) \right) \text{ Then } \frac{dA}{dt} = \left(\frac{da_{ij}}{dt} \right) \text{ and } \int_a^b A(t)\,dt = \left(\int_a^b a_{ij}(t)\,dt \right)$$

↳ $\dfrac{d}{dt}\left(C A(t)\right) = C \dfrac{dA}{dt}$ C const matrix

$$\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt}$$

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B$$

CLICKER QUESTIONS

$$e^A e^B \neq e^{A+B} \text{ unless } [A,B] = AB - BA = 0$$

↳ Baker - Campbell - Hausdorff formula

↳ Eigenvalues / Eigenvectors: usually $A\vec{v}$ different direction than $\vec{v}$

$$A\vec{v} = \lambda\vec{v} \Rightarrow (A - \lambda I)\vec{v} = 0$$

$$\Rightarrow A - \lambda I \text{ not invertible so } \mathrm{Det}(A - \lambda I) = 0$$

Things very nice if all evalues distinct

↳ implies matrix is diagonalizable

↳ $\exists$ invertible T st $T^{-1}AT = \Lambda$ = diagonal matrix

Clicker Questions
Are evalues of real matrices real?
Are evectors of real matrices real?
Does every matrix have an evalue?
Is every matrix diagonalizable?

↳ benefits: FAST Computations:

↳ $A = T\Lambda T^{-1}$ so $A^n = T\Lambda^n T^{-1}$!

↳ any real symm matrix is diagonalizable w/ real evalues

Needed Aside: How do we diagonalize a matrix?

Let A be $n \times n$ constant matrix with evalues $\lambda_1, \ldots, \lambda_n$ (not necc distinct!) but with n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$.

Let $S = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}$, S^{-1} exists as lin indep columns

Note $S\vec{e}_i = \vec{v}_i$ and $S^{-1}\vec{v}_i = \vec{e}_i$

Let $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

Claim: $A = S \Lambda S^{-1}$ or $S^{-1}AS = \Lambda$

If $S^{-1}AS\vec{e}_i = \Lambda \vec{e}_i$ $\forall i$, done as $\vec{e}_i$ a basis

$$\begin{aligned}(S^{-1}AS)\vec{e}_i &= (S^{-1}A)(S\vec{e}_i) \\ &= S^{-1}(A\vec{v}_i) \\ &= S^{-1}(\lambda_i \vec{v}_i) \\ &= \lambda_i S^{-1}\vec{v}_i \\ &= \lambda_i \vec{e}_i \\ &= \Lambda \vec{e}_i \quad \blacksquare\end{aligned}$$

So if we can find ~~the~~ n linearly indep evectors, done

If know λ is an evalue, to find evector must solve

$(A - \lambda I)\vec{v} = \vec{0}$: Gaussian Elimination / Gauss-Jordan...

7.5. Homogeneous & Linear Systems @ Constant Coeffs

Consider $\vec{x}'(t) = A\vec{x}(t)$ with A a constant matrix.

<u>Case 1: A is diagonal</u>

↳ Can "uncuple": $\begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$

See $x_i'(t) = \lambda_i x_i(t)$ or $x_i(t) = c_i e^{\lambda_i t}$

Let $\vec{e}_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, the vector with zeros everywhere but i^{th} spot, where we have a 1.

Then general soln is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\,\vec{e}_1 + \cdots + c_n e^{\lambda_n t}\,\vec{e}_n,$$

and note $\vec{e}_1, \ldots, \vec{e}_n$ are evectors of A with evalues $\lambda_1, \ldots, \lambda_n$

<u>General A</u>: Guess $\vec{x}(t) = \vec{\xi} e^{rt}$ for some vector $\vec{\xi}$

↳ $\vec{x}' = A\vec{x} \Rightarrow r e^{rt} \vec{\xi} = e^{rt} A \vec{\xi}$

or $(A - rI)\vec{\xi} e^{rt} = 0 \Rightarrow r$ evalue, $\vec{\xi}$ evector

<u>Case 2: A is diagonalizable</u>

Say $\exists\, T$ st $T^{-1}AT = \Delta$ or $A = T\Delta T^{-1}$

Then $\vec{x}' = T\Delta T^{-1}\vec{x}$

or $T^{-1}\vec{x}' = \cancel{A}\,\Delta T^{-1}\vec{x}$

↳ let $\vec{y} = T^{-1}\vec{x}$ (no loss as T invertible)

Then $\vec{y}' = \Delta\vec{y}$, reduce to case 1!

Completely solved for diagonalizable matrices!

↳ includes real symm, complex Hermitian, unitary, normal, ...

↳ not all matrices diagonalizable

↳ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$: leads to Jordan Canonical Form (See Section 7.8)

↳ See Conway's See and Say Sequence for fun application of Jordanizable

7.5 Homog Lin Systems (cont) AND 7.7 Fundamental Matrices

Will not do much of Chapter 7 sections 7 and 8 as not assuming LinAlg

Consider $\vec{x}'(t) = A\vec{x}(t)$, A constant matrix, $\vec{x}(0) = \vec{x}^0$

Claim $\exists!$ solution, namely $\vec{x}(t) = \exp(At)\,\vec{x}^0$.

↳ Proof: $\exp(At) = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$ note A, t commute; note A, A commute

↳ Question: does the sum converge?

↳ always go back to 1-dim for intuition:

$\exp(at) = 1 + at + \frac{a^2t^2}{2!} + \cdots$ converges: $(at)^m << m!$ (grows a lot slower)

let $M = \max_{1 \le i,j \le n} |a_{ij}|$

let M_k be the maximum of the absolute value of entries of A^k

Prove $M_k \le \frac{(nM)^k}{n}$ where A is an $n \times n$ matrix

↳ implies series for $\exp(At)$ converges

↳ Note if $|t|$ small then $A \approx I$ and invertible

↳ Aside: $\exp(At)$ always invertible: inverse $\exp(-At)$

$\boxed{\text{While } \exp(At)\vec{x}_0^0 \text{ is the unique soln, in general hard to compute } \exp(At)!}$

↳ $e^A e^B = \left(\sum \frac{A^k}{k!}\right)\left(\sum \frac{B^l}{l!}\right)$: if $AB \ne BA$ can't rearrange

↳ BIG difference from 1-dimension!

↳ However: $e^{TAT^{-1}} = I + TAT^{-1} + \frac{TA\cancel{T^{-1}T}AT^{-1}}{2!} + \frac{TA\cancel{T^{-1}T}A\cancel{T^{-1}T}AT^{-1}}{3!} + \cdots$

$$= T\left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\right)T^{-1} = Te^AT^{-1}$$

So if ~~A~~ $TAT^{-1} = \Lambda$ (I know, backwards from before, my bad):

$e^A = T^{-1}e^{TAT^{-1}}T = T^{-1}e^{\Lambda}T = T^{-1}\begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}T$: $\boxed{\text{doable Calculation}}$

7.9: Nonhomogeneous Linear Systems

$\vec{x}'(t) = A\vec{x}(t) + \vec{g}(t)$, A constant matrix

Assume A is diagonalizable: $T^{-1}AT = \Delta$ or $A = T\Delta T^{-1}$

Change variables: $\vec{x}(t) = T\vec{y}(t)$ or $\vec{y}(t) = T^{-1}\vec{x}(t)$

$T\vec{y}'(t) = AT\vec{y}(t) + \vec{g}(t)$

or $\vec{y}'(t) = T^{-1}AT\vec{y}(t) + T^{-1}\vec{g}(t)$

or $\vec{y}'(t) = \Delta\vec{y}(t) + \vec{h}(t)$, $\vec{h}(t) = T^{-1}\vec{g}(t)$

ie, $$\begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} + \begin{pmatrix} h_1(t) \\ \vdots \\ h_n(t) \end{pmatrix}$$

So after algebra we've reduced to

$y_i'(t) = \lambda_i y_i(t) + h_i(t)$ $\quad \mu_i(t) = \exp(\int -\lambda_i\,dt) = e^{-\lambda_i t}$

so $y_i(t) = e^{\lambda_i t}\left(\int_{t_0}^{t} e^{-\lambda_i s} h_i(s)\,ds + c_i\right)$

and then take $T\vec{y}(t)$ to get $\vec{x}(t)$!

Ex: Solve $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$ (pg 433)

↳ Boyce-Book Thm: evalues $-1, -3$

evector with -1 is $\binom{1}{1}$, with -3 is $\binom{1}{-1}$ as real symm

$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $\quad T^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\vec{y}' = \begin{pmatrix} -3 & \\ & -1 \end{pmatrix}\vec{y} + \frac{1}{\sqrt{2}}\begin{pmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{pmatrix}$

Solns $y_1 = \frac{\sqrt{2}}{2}e^{-t} - \frac{3}{\sqrt{2}}\left(\frac{t}{3} - \frac{1}{9}\right) + c_1 e^{-3t}$

$y_2 = \sqrt{2}\,t e^{-t} + \frac{3}{\sqrt{2}}(t-1) + c_2 e^{-t}$

KEY FACT
For small enough time intervals, any matrix is approximately constant, use for approximations
See Chapter 9

Can generalize: method undet coeff, non-constant A and variation of params...

= CHAPTER 7-6 =