MATH 209: IMPORTANT FORMULAS

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Abstract. These notes give a brief summary of the major techniques of the class, and an example for each. The descriptions are kept short so that this can be a useful, quick reference.

1. Executive Summary

We record below the types of equations we can solve. In the next sections we give more details, including conditions on the functions and, if possible, explicit solutions. This section is meant to be a quick list.

- **Linear, constant coefficient difference equations:** $a_{n+1} = c_1 a_n + c_2 a_{n-1} + c_3 a_{n-2} + \cdots + c_k a_{n-k+1}$; for example, $a_{n+1} = 3a_n + 4a_{n-1} - 2a_{n-2}$. See §2.

- **Integrating factors:** $dy/dt + p(t)y(t) = g(t)$. See §3.1.

- **Separable equations:** $M(x) + N(y)dy/dx = 0$. See §3.2.

- **Exact equations:** $M(x, y) + N(x, y)dy/dx = 0$ with $\partial M/\partial y = \partial N/\partial x$. See §3.3.

- **Second order linear constant coefficient homogenous equations:** $d^2y/dt^2 + ady/dt + by = 0$. See §4.1.

- **Method of Undermined Coefficients:** $ay'' + by' + cy = g(t)$ with $g(t) = e^{\alpha t}P_n(t)$ (with $P_n(t)$ a polynomial of degree n in t) or $g(t) = e^{\alpha t}\cos(\beta t)$ or $g(t) = e^{\alpha t}\sin(\beta t)$. See §4.2.

- **Variation of Parameters:** Let p, q, g continuous functions and consider $y''(t) + p(t)y'(t) + q(t)y = g(t)$ with known solutions to the homogenous equation. See §4.3.

- **Series expansions:** $p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0$ with $p(x_0) \neq 0$ and guessing $y(x) = \sum_{n=0}^{\infty} a_n(X - x_0)^n$. See §5.

- **Linear systems:** $\overrightarrow{x}'(t) = A\overrightarrow{x}(t) + \overrightarrow{g}(t)$ with $\overrightarrow{x}(0) = \overrightarrow{x}_0$. See §6.

Date: April 28, 2009.

2. Difference Equations

2.1. Linear, constant coefficient difference equations.

Statement: Let k be a fixed integer and $c_1, \dots, c_k$ given real numbers. Then the general solution of the difference equation

$$a_{n+1} \;=\; c_1 a_n + c_2 a_{n-1} + c_3 a_{n-2} + \cdots + c_k a_{n-k+1}$$

is

$$a_n \;=\; \gamma_1 r_1^n + \cdots + \gamma_k r_k^n$$

if the characteristic polynomial

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k \;=\; 0$$

has k distinct roots. Here the $\gamma_1, \dots, \gamma_k$ are any k real numbers; if initial conditions are given, these conditions determine these γ_i's.

Example: Consider the equation $a_{n+1} = 5a_n - 6a_{n-1}$. In this case $k = 2$ and we find the characteristic polynomial is $r^2 - 5r + 6 = (r-2)(r-3)$, which clearly has roots $r_1 = 2$ and $r_2 = 3$. Thus the general solution is $a_n = \gamma_1 2^n + \gamma_2 3^n$. If we are given $a_0 = 1$ and $a_1 = 2$, this leads to the system of equations $1 = \gamma_1 + \gamma_2$ and $2 = \gamma_1 \cdot 2 + \gamma_2 \cdot 3$, which has the solution $\gamma_1 = 1$ and $\gamma_2 = 0$.

Applications: Population growth (such as the Fibonacci equation), why double-plus-one is a bad strategy in roulette.

3. First Order Differential Equations

3.1. Integrating factors.

Statement: For a differential equation of the form $y'(t) + p(t)y(t) = g(t)$, the general solution is

$$y(t) \;=\; \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right],$$

where

$$\mu(t) \;=\; \exp\left(\int p(t)dt\right)$$

and C is a free constant (if an initial condition is given, then C can be determined uniquely).

Example: Consider the equation $y'(t) - 2ty(t) = \exp\left(t^2 + t\right)$. Then

$$\mu(t) \;=\; \exp\left(\int -2tdt\right) \;=\; \exp(-t^2),$$

and

$$\begin{aligned} y(t) &= \frac{1}{\exp(-t^2)}\left[\int \exp(-t^2)\exp(t^2+t)dt + C\right] \\ &= \frac{1}{\exp(-t^2)}\left(\exp(t)+C\right). \end{aligned}$$

If we have the initial condition $y(0)=2$ then we find $2=1+C$, or $C=1$.

Applications: To be added.

3.2. **Separable equations.**

Statement: For a differential equation of the form $M(x)+N(y)dy/dx=0$ the general solution is

$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds \ = \ 0,$$

where we are using the shorthand notation $y_0=y(x_0)$ and $y=y(x)$. We could also write the solution as

$$\int M(x)dx + \int N(y)dy \ = \ C,$$

and then determine C from the initial conditions. *NOTE:* if we can write the differential equation as $y'=F(v)$ for $v=y/x$, then we can convert this to a separable equation: $y=vx$ so $v+xv'=F(v)$ or $-\frac{1}{x}+\frac{v'}{F(v)-v}=0$.

Example: Consider the equation $3x^2+\cos(y)y'=0$. Then $M(x)=3x^2$, $N(y)=\cos(y)$, so the solution is

$$\int 3x^2dx + \int \cos(y)dy \ = \ C,$$

or

$$x^3+\sin(y(x)) \ = \ C.$$

If we are told $y(1)=\pi$, then $C=\pi^3$.

Applications: Solow growth model in economics, population growth.

3.3. **Exact equations.**

Statement: Consider $M(x,y)+N(x,y)dy/dx=0$ with $\partial M/\partial y=\partial N/\partial x$. Then there is a function $\psi(x,y)$ such that the solution to the differential equation is given by $\psi(x,y)=C$, with C determined by the initial conditions. One way to find ψ is as follows. Our problem implies that $\partial\psi/\partial x=M$ and $\partial\psi/\partial y=N$. Thus

$$\begin{aligned} \psi(x,y) &= \int M(x,y)dx+g(y) \\ \psi(x,y) &= \int N(x,y)dy+h(x), \end{aligned}$$

and then determine $g(y)$ and $h(x)$ so that the two expressions are equal. NOTE: sometimes it is possible to multiply a differential equation by an integrating factor and convert it to an exact equation; unfortunately in practice it is hard to find an integrating factor which works.

Example: Consider

$$(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)dy/dx \ = \ 0.$$

Thus $M(x, y) = 3x^2 - 2xy + 2$, $N(x, y) = 6y^2 - x^2 + 3$ and $\partial M/\partial y = \partial N/\partial x = -2x$. Thus the differential equation is exact, and we have

$$\begin{aligned} \psi(x,y) &= \int \left(3x^2 - 2xy + 2\right) dx + g(y) \ = \ x^3 - x^2y + 2x + g(y) \\ \psi(x,y) &= \int \left(6y^2 - x^2 + 3\right) dy + h(x) \ = \ 2y^3 - x^2y + 3y + h(x). \end{aligned}$$

Therefore we need

$$x^3 - x^2y + 2x + g(y) \ = \ 2y^3 - x^2y + 3y + h(x),$$

which is possible if we let $h(x) = x^3 + 2x$ and $g(y) = 2y^3 + 3y$. In other words,

$$\psi(x,y) \ = \ x^3 - x^2y + 2x + 2y^3 - 3y$$

solves the original equation.

Applications:

4. Second Order Differential Equations

4.1. **Linear, constant coefficient homogenous equations.**

Statement: Consider $d^2y/dt^2 + ady/dt + by = 0$. Guessing solutions of the form e^{rt} leads to studying the characteristic polynomial $r^2 + ar + b$; let r_1 and r_2 be the two roots. If the roots are distinct, all solutions are of the form $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$, where c_1 and c_2 are determined by two initial conditions (often $y(0)$ and $y'(0)$, though we could have y at two times). If $r_1 = r_2 = r$, the general solution is instead of the form $y(t) = c_1e^{rt} + c_2te^{rt}$.

Example: Consider $y'' + 5y' + 6 = 0$ with $y(0) = 0$ and $y'(0) = 1$. The characteristic polynomial is $r^2 + 5r + 6 = (r + 2)(r + 3)$. Thus the roots are $r_1 = -2$ and $r_2 = -3$, and we find the general solution is $y(t) = c_1e^{-2t} + c_2e^{-3t}$. This leads to $0 = c_1 \cdot 1 + c_2 \cdot 1$ and $1 = c_1 \cdot (-2) + c_2 \cdot (-3)$. This is two equations in two unknowns; using linear algebra or solving for c_2 in terms of c_1 leads to $c_1 = 1$ and $c_2 = -1$, or $y(t) = e^{-2t} - e^{-3t}$.

Applications: Physics and engineering problems include motion with friction proportional to velocity, masses on springs, circuit theory.

4.2. **Method of Undetermined Coefficients for linear, constant coefficient non-homogenous equations.**

Statement: Consider $ay'' + by' + cy = g(t)$. Let $y_1(t)$ and $y_2(t)$ be any fundamental set of solutions to the homogenous equation $ay'' + by' + cy = 0$ and $Y_1(t)$ any solution to the non-homogenous equation $ay'' + by' + cy = g(t)$. Then the general solution is $y(t) = c_1y_1(t) + c_2y_2(t) + Y_1(t)$ (i.e., every solution is of this form). In general it is hard to find $Y_1(t)$, but for special choices of $g(t)$ this can be done. In particular, the following guesses will work:

$g(t)$	Guess	Comment
$e^{\alpha t}$	$Ae^{\alpha t}$	A is a free parameter
$P_n(t)$	$Q_n(t)$	$P_n(t) = a_0 + a_1t + \cdots + a_nt^n$ is the given polynomial, and $Q_n(t) = b_0 + \cdots + b_nt^n$, with the b_i's free parameters.
$\sin(\beta t)$ or $\cos(\beta t)$	$A\sin(\beta t) + B\cos(\beta t)$	A, B free parameters

By linearity, if we have products of the above we basically take combinations of multiples of our guesses. For example, if $g(t) = e^{\alpha t}P_n(t)$ we guess $Ae^{\alpha t}Q_n(t)$; actually it suffices to guess $e^{\alpha t}Q_n(t)$ as the free parameter A can be incorporated into the coefficients of the polynomial. Note it is very important that we have a polynomial $P_n(t)$; if we had $1/t$ then it is not at all clear what the answer is (it involves the exponential integral function).

Example: $y'' + 3y' + 2y = t\sin(t)$. The homogenous equation yields the characteristic polynomial $r^2 + 3r + 2 = 0$, or $r = -1, -2$. Thus the general solution to the homogeneous equation is $c_1e^{-t} + c_2e^{-2t}$. As $g(t) = t\sin(t)$, we guess

$$Y_1(t) \ = \ (b_0 + b_1t)\sin t + (c_0 + c_1t)\cos t).$$

Note we are *not* guessing something of the form $(b_0 + b_1t)(A\sin t + B\cos t)$; this gives us a little more flexibility (and sadly the simpler guess doesn't work). Doing lots of algebra leads to the solution is

$$Y_1(t) \ = \ \left(\frac{6}{50} + \frac{5t}{50}\right)\sin t + \left(\frac{17}{50} - \frac{15t}{50}\right)\cos t.$$

Application: See the passages in the book about spring motion with external driving forces.

4.3. **Variation of Parameters. Statement:** Let p, q, g continuous functions and consider $y''(t) + p(t)y'(t) + q(t)y = g(t)$ with known solutions $y_1(t)$ and $y_2(t)$ to the homogenous equation. Then a particular solution to the non-homogenous equation is

$$Y(t) \ = \ -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)ds}{W(y_1,y_2)(s)} \ + \ y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)ds}{W(y_1,y_2)(s)},$$

where the Wronksian is

$$W(y_1,y_2)(s) \ = \ y_1(s)y_2'(s) - y_1'(s)y_2(s),$$

and the general solution is

$$y(t) \ = \ c_1y_1(t) + c_2y_2(t) + Y(t).$$

Example: Consider $y''(t) + 4y'(t) + 4y(t) = \cosh t$, where the hyperbolic cosine is given by $\cosh t = (e^t + e^{-t})/2$ (and $\sinh t = (e^t - e^{-t})/2$). The two solutions to the homogenous differential equation are $y_1(t) = e^{2t}$ and $y_2(t) = te^{-2t}$ (as there is a repeated root in the characteristic polynomial). The Wronskian is $W(y_1,y_2)(s) = e^{-4t} \neq 0$, so the solutions are linearly independent (and we can divide by the Wronskian for all s) and are now found by performing the specified integrals.

Application:

5. Series Solution

Statement: Consider $p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0$ with $p(x_0) \neq 0$. One can guess $y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ and attempt to determine a series expansion for the solution. Doing so involves finding recurrence relations for the a_n's (we'll have sums of infinite series expansions equalling zero, and this can only happen if the coefficient of x^m vanishes for all m). Typically there will be two free parameters, and one checks to see if the Wronskian is non-zero to see if the solutions are linearly independent (ie, if they generate a fundamental set of solutions). To do so does *not* require us to know all the coefficients a_n of each solution, but only the constant and linear terms. One must investigate the convergence properties of the expansion, which is not surprisingly related to properties of $p(x), q(x)$ and $r(x)$. For a review of Taylor series, you can see my notes at
`http://www.williams.edu/go/math/sjmiller/public_html/`
`103/MVT_TaylorSeries.pdf`.

Example: $(1-x)y''(x) + y(x) = 0$ about $x = 0$. Note the coefficient of $y''(x)$ is non-zero at $x = 0$. Guessing $y(x) = \sum_{n=0}^{\infty} a_nx^n$ leads to

$$y'(x) \ = \ \sum_{n=1}^{\infty} na_nx^{n-1}$$

$$y''(x) \ = \ \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}.$$

Unfortunately the $1-x$ factor complicates things a bit, as we don't want to combine $1-x$ and x^n. The solution to this is to expand things out, rewriting the differential equation as

$y''(x) - xy''(x) + y(x) = 0$. We now have

$$xy''(x) \;=\; \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}.$$

This leads to

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \;=\; 0.$$

We want all sums to be of x^m, so we must shift the indices of summation. The last term is the easiest – we simply let $m = n$. For the other two, it is easiest to do the shift of summation slowly. For the first term, let $m = n - 2$ (we choose this as x^{n-2} will become x^m). Thus $n = m + 2$ and as n ran from 2 to ∞, m runs from 0 to ∞. Thus this sum becomes $\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$. A similar analysis shows the second term becomes $\sum_{m=1}^{\infty}(m+1)ma_{m+1}x^m$. We thus find

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty}(m+1)ma_{m+1}x^m + \sum_{m=0}^{\infty} a_m x^m \;=\; 0.$$

We combine the terms. Note that two sums start at $m = 0$ while one starts at $m = 1$. We thus group the two $m = 0$ terms together and then combine the three sums from $m = 1$ to ∞ and find

$$(2 \cdot 1a_2 + a_0) + \sum_{m=1}^{\infty} \left[(m+2)(m+1)a_{m+2} - (m+1)ma_{m+1} + a_m\right] x^m \;=\; 0.$$

Thus $a_2 = -a_0/2$ and

$$a_{m+2} \;=\; \frac{(m+1)ma_{m+1} - a_m}{(m+2)(m+1)}.$$

Note that a_0 and a_1 are free, and once specified then all the remaining a_i are uniquely determined (as a_{m+2} is determined by its two predecessors, once we know the first two terms of the sequence we know all the terms). We can compute the first few values of a_n by hand: a_0, a_1, $a_2 = -a_0/2$, $a_3 = -a_0/6 - a_1/6$, ... This leads to

$$y(x) \;=\; a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots\right) \;+\; a_1\left(x - \frac{1}{6}x^3 + \cdots\right).$$

We have thus found two solutions,

$$\begin{aligned} y_1(x) &= 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots \\ y_2(x) &= x - \frac{1}{6}x^3 + \cdots. \end{aligned}$$

Are the two solutions linearly independent (ie, do they generate all solutions)? We must calculate the Wronskian at $x = 0$, which is

$$W(y_1, y_2)(0) \;=\; \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} \;=\; \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \;=\; 1 \;\neq\; 0.$$

Note we are able to show $y_1(x)$ and $y_2(x)$ are linearly independent *without* having computed all terms in the expansion! Finally, it is worth noting that we can show the two series expansions converge for all $|x| < 1$. To see this, note

$$\begin{aligned} |a_{m+2}| &\le \frac{(m+1)m|a_{m+1}| + |a_m|}{(m+2)(m+1)} \\ &\le \frac{m^2+m+1}{m^2+2m+1} \max\left(|a_{m+1}|, |a_m|\right) \ < \ \max\left(|a_{m+1}|, |a_m|\right). \end{aligned}$$

Thus for $m \ge 3$, $|a_m| \le \max\left(|a_0|, |a_1|\right)$ and by the comparison test $\sum_{m=0}^{\infty} a_m x^m$ will converge for $|x| < 1$. It is not surprising that our analysis for convergence breaks down at $x = 1$ (to be fair, the above analysis just doesn't provide any information about what happens at $x = 1$; to have the series converge for $|x| \ge 1$ we would need the $|a_m|$ to decay rapidly, and a more involved analysis shows that this is not the case). Note that the coefficient of $y''(x)$ is $1 - x$, and thus when $x = 1$ this coefficient is zero. This means that at $x = 1$ we do not have an ordinary point, and there is a marked change in the nature of the differential equation.

Example: Consider $xy''(x) + y'(x) + xy(x) = 0$ about $x = 1$. Note $x = 1$ is an ordinary point, and we guess a solution $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$. Unfortunately this does not combine well when we substitute, as we get terms such as $(x-1)^m$ and $x(x-1)^n$. The easiest way to proceed is to ***add zero***, one of the most important techniques in mathematics. We note $x = (x-1) + 1$, and thus our differential equation is the same as

$$(x-1)y''(x) + y''(x) + y'(x) + (x-1)y(x) + y(x) \ = \ 0,$$

expanded about $x = 1$. Now the $(x-1)$ factors will combine nicely with the series expansion.

Application:

6. Systems of Linear Differential Equations

Statement: Consider $\overrightarrow{x}'(t) = A\overrightarrow{x}(t) + \overrightarrow{g}(t)$ with $\overrightarrow{x}(0) = \overrightarrow{x}_0$. If $\overrightarrow{g}(t) = \overrightarrow{0}$ then the solution is just $\overrightarrow{x}(t) = \exp(At)\overrightarrow{x}_0$, where $\exp(At) = I + At + A^2t^2/2! + \cdots = \sum_{n=0}^{\infty} A^n t^n/n!$. For computational purposes, it is often convenient to diagonalize the matrix. While it is not the case that every matrix is diagonalizable, most will be. This is always the case for $n \times n$ matrices with n distinct eigenvalues (or n linearly independent eigenvectors); recall that a non-zero vector $\overrightarrow{v}$ is an eigenvector of A with eigenvalue λ if $A\overrightarrow{v} = \lambda\overrightarrow{v}$ (the eigenvalues are found by solving $\det(A - \lambda I) = 0$, and then one solves $(A - \lambda)\overrightarrow{v} = \overrightarrow{0}$). In many cases we can explicitly write down a diagonalizing matrix. Let $\overrightarrow{v}_1, \dots, \overrightarrow{v}_n$ be n linearly independent eigenvectors for the $n \times n$ matrix A (such vectors always exist if A has n distinct eigenvalues). Let S be the matrix where the i^{th} column is $\overrightarrow{v}_i$, and let Λ be the diagonal matrix with λ_i (the eigenvalue of $\overrightarrow{v}_i$) in the i^{th} place on the main diagonal. Then $A = S\Lambda S^{-1}$ or $\Lambda = S^{-1}AS$. This is of great use as

$$\exp(At) \ = \ \exp(S\Lambda S^{-1}t) \ = \ S\exp(\Lambda t)S^{-1} \ = \ S\widetilde{\Lambda}_t S^{-1},$$

where $\widetilde{\Lambda}_t$ is the $n \times n$ diagonal matrix with entries $e^{\lambda_i t}$.

Thus for $\overrightarrow{x}'(t) = A\overrightarrow{x}(t)$ the solution is $\overrightarrow{x}(t) = S\widetilde{\Lambda}_t S^{-1}\overrightarrow{(\ x)}_0$ if A is diagonalizable with the matrix S of eigenvectors and $\widetilde{\Lambda}_t$ is the diagonal matrix with entries $e^{\lambda_i t}$.

The situation is only slightly more involved if $\overrightarrow{g}(t)$ is not zero (ie, the non-homogenous case), provided that A is diagonalizable. If $A = S\Lambda S^{-1}$ (with S as above), then we can solve uncouple this system and reduce to n first order differential equations which can be solved by integrating factors. In particular, let $\overrightarrow{x}(t) = S\overrightarrow{y}(t)$ or $\overrightarrow{y}(t) = S^{-1}\overrightarrow{x}(t)$ (we can do this change of variables as S is invertible). As $\overrightarrow{x}'(t) = S\overrightarrow{y}'(t)$, we find

$$\begin{aligned} \overrightarrow{x}'(t) &= A\overrightarrow{x}(t) + \overrightarrow{g}(t) \\ S\overrightarrow{y}'(t) &= AS\overrightarrow{y}(t) + \overrightarrow{g}(t) \\ \overrightarrow{y}'(t) &= S^{-1}AS\overrightarrow{y}(t) + S^{-1}\overrightarrow{g}(t) \\ \overrightarrow{y}'(t) &= \Lambda\overrightarrow{y}(t) + \overrightarrow{h}(t), \end{aligned}$$

where $\overrightarrow{h}(t) = S^{-1}\overrightarrow{g}(t)$. This leads to n uncoupled first order linear differential equations

$$y_i'(t) = \lambda_i y_i(t) + h_i(t),$$

which can be solved by integrating factors: if $\mu_i(t) = \exp(-\lambda_i t)$ then

$$\begin{aligned} y_i(t) &= \frac{1}{\mu_i(t)}\left[\int \mu_i(t)h_i(t)dt + C\right] \\ &= e^{\lambda_i t}\left[\int e^{-\lambda_i t}h_i(t)dt + C\right]. \end{aligned}$$

It is worth noting that if we have $\overrightarrow{x}'(t) = A\overrightarrow{x}(t)$ with A an $n \times n$ matrix with linearly independent eigenvectors $\overrightarrow{v}_i$ with eigenvalues λ_i (which will always be the case if A has n distinct eigenvalues or if A is a real symmetric matrix, which means $A = A^T$), then the solution can be written as

$$\overrightarrow{x}(t) = c_1 e^{\lambda_1 t}\overrightarrow{v}_1 + \cdots + c_n e^{\lambda_n t}\overrightarrow{v}_n,$$

where the c_i are chosen so that $\overrightarrow{x}(0) = \overrightarrow{x}_0$.

Example: Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

and consider the system of differential equations $\overrightarrow{x}'(t) = A\overrightarrow{x}(t)$ with $\overrightarrow{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The solution is $\overrightarrow{x}(t) = \exp(At)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. To compute $\exp(At)$, note the eigenvalues are found by solving $\det(A - \lambda I) = 0$ or $(1-\lambda)^2 - 4 = 0$, which after some algebra yields $\lambda = 3$ or -1. Solving the linear algebra equation gives the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\lambda_1 = 3$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $\lambda_2 = -1$. This yields

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix},$$

so

$$\exp(At) = S\exp(\Lambda t)S^{-1} = S\begin{pmatrix} e^3 & 0 \\ 0 & e^{-1} \end{pmatrix}S^{-1} = \begin{pmatrix} \frac{e^{-t}}{2} + \frac{e^{3t}}{2} & -\frac{e^{-t}}{2} + \frac{e^{3t}}{2} \\ -\frac{e^{-t}}{2} + \frac{e^{3t}}{2} & \frac{e^{-t}}{2} + \frac{e^{3t}}{2} \end{pmatrix}.$$

Thus the solution is

$$\overrightarrow{x}(t) \;=\; \begin{pmatrix} \frac{e^{-t}}{2}+\frac{e^{3t}}{2} & -\frac{e^{-t}}{2}+\frac{e^{3t}}{2} \\ -\frac{e^{-t}}{2}+\frac{e^{3t}}{2} & \frac{e^{-t}}{2}+\frac{e^{3t}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} \frac{e^{-t}}{2}+\frac{e^{3t}}{2} \\ -\frac{e^{-t}}{2}+\frac{e^{3t}}{2} \end{pmatrix}.$$

We could also solve this system of differential equations by finding c_1, c_2 such that

$$\overrightarrow{x}(t) \;=\; c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

satisfies the initial condition, as this clearly solves the differential equation. Algebra yields $c_1 = c_2 = 1/2$, and we recover the solution as before.

Application: