MATH 209: FINDING SECOND SOLUTIONS

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Abstract. In studying second order linear constant coefficient homogenous differential or difference equations, we saw that our 'divine inspiration' guess only generates one solution if the characteristic polynomial has a repeated root. We discuss below how one is led to guessing the other solution.

1. Statement of problem

For second order linear constant coefficient homogenous difference equations, say

$$x_{n+2} - ax_{n+1} - bx_n \ = \ 0, \tag{1.1}$$

we saw that guessing $x_n = r^n$ often leads to a complete solution. Specifically, trying this leads to the characteristic polynomial

$$r^2 - ar - b \ = \ 0, \tag{1.2}$$

and if the roots are distinct, say r_1 and r_2, then the general solution to the difference equation is

$$x_n \ = \ c_1 r_1^n + c_2 r_2^n, \tag{1.3}$$

where c_1, c_2 are chosen to satisfy the two initial conditions. Similarly if we have the differential equation

$$y''(t) - ay'(t) - by(t) \ = \ 0 \tag{1.4}$$

we guess $y(t) = e^{rt}$. This leads to the same characteristic polynomial for r, and if the roots are distinct the general solution is then

$$y(t) \ = \ c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{1.5}$$

Unfortunately, this method breaks down if the two roots are equal. In that case, we only find one solution, either $x_n = r^n$ or $y(t) = e^{rt}$. We need to find another solution. While direct inspection shows $x_n = nr^n$ and $y(t) = te^{rt}$ work, the goal of this note is to explain how one is led to making such a guess. This is not to say that it not important to know how to solve these problems when there are repeated roots; this is a very important part of the subject. Rather, we want to emphasize how one can search for additional solutions to a problem when we know some solutions. This is a very general technique, and can be used fruitfully in many situations.

2. Combinations of Solutions

Before explaining the logic behind the guess which finds the other solutions, it is worthwhile to consider some specific differential equations and various linear combinations of solutions. The standard one to study is

$$y''(t) + y(t) \ = \ 0. \tag{2.1}$$

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If we guess $y(t) = e^{rt}$, we find $r^2 + 1 = 0$, so $r = \pm i$ so the solutions are e^{it} and e^{-it}. Thus the general solution is $c_1 e^{it} + c_2 e^{-it}$.

A little thought and inspection, however, turns up two other solutions. We are looking for a function whose second derivative is the negative of the original function; thus we see $\cos(t)$ and $\sin(t)$ also solve the original equation, and hence the general solution should be $b_1 \cos(t) + b_2 \sin(t)$.

What is going on here? There should only be two free parameters, not four. Thus, somehow give a choice of b_1 and b_2 there should be a choice of c_1 and c_2 so that $b_1 \cos(t) + b_2 \sin(t) = e^{it} + c_2 e^{-it}$ (and similarly the other way around).

We can see this by recalling Euler's formula: $e^{i\theta} = \cos(\theta) + i \sin(\theta)$, so $e^{-i\theta} = \cos(\theta) - i \sin(\theta)$. Simple algebra yields

$$\begin{aligned} \frac{1}{2}e^{it} + \frac{1}{2}e^{-it} &= \cos(t) \\ \frac{1}{2i}e^{it} - \frac{1}{2i}e^{-it} &= \sin(t). \end{aligned} \tag{2.2}$$

Thus if we are given b_1 and b_2, we can re-express $b_1 \cos(t) + b_2 \sin(t)$ in terms of e^{it} and e^{-it}; specifically, we have

$$\begin{aligned} b_1 \cos(t) + b_2 \sin(t) &= b_1 \left(\frac{1}{2}e^{it} + \frac{1}{2}e^{-it}\right) + b_2 \left(\frac{1}{2i}e^{it} - \frac{1}{2i}e^{-it}\right) \\ &= \left(\frac{b_1}{2} + \frac{b_2}{2i}\right) e^{it} + \left(\frac{b_1}{2} - \frac{b_2}{2i}\right) e^{-it} \\ &= \frac{b_1 - ib_2}{2} e^{it} + \frac{b_1 + ib_2}{2i} e^{-it}. \end{aligned} \tag{2.3}$$

In other words, given b_1 and b_2 we take $c_1 = (b_1 - ib_2)/2$ and $c_2 = (b_1 + ib_2)/2i$.

The point of all this is that we can use either $\cos(t)$ and $\sin(t)$ as our fundamental set of solutions, or e^{it} and e^{-it}, whichever is more convenient for us. We shall see an appropriate combination of solutions of related difference and differential equations 'suggest' our guess.

3. Heuristic for our guess

We now describe the heuristic which suggests the guess for second order linear constant coefficient difference equations when the characteristic polynomial has a repeated root; a similar analysis works for differential equations.

For a general difference equation $x_{n+2} - ax_{n+1} - bx_n$, the characteristic polynomial $r^2 - ar - b$ has two distinct roots and there are no difficulties in writing down all solutions; all solutions are linear combinations of r_1^n and r_2^n.

A little algebra shows we can also write all solutions as linear combinations of $r_2^n + r_1^n$ and $r_2^n - r_1^n$. To see this, we try and solve

$$b_1(r_2^n + r_1^n) + b_2(r_2^n - r_1^n) \;=\; c_1 r_1^n + c_2 r_2^n. \tag{3.1}$$

If we are given b_1 and b_2, then clearly we just take $c_1 = b_1 - b_2$ and $c_2 = b_1 + b_2$. Conversely, if we are given c_1 and c_2 we see we may take $b_1 = (c_1 + c_2)/2$ and $b_2 = (c_2 - c_1)/2$. Thus $r_2^n + r_1^n$ and $r_2^n - r_1^n$ are also a fundamental set of solutions.

If we multiply a solution by a constant, we still have a solution. Thus, we are led to the following pair of solutions (which still generate all solutions):

$$\frac{r_2^n + r_1^n}{r_2 + r_1}, \quad \frac{r_2^n - r_1^n}{r_2 - r_1}, \tag{3.2}$$

as $r_2 + r_1$ and $r_2 - r_1$ are independent of n and thus constant.

We now explore what happens as we deform the difference equation so that the two roots collapse into a common value, say r. Let us write $r_1 = r$ and $r_2 = r + h$, so $h \to 0$. Note $(r_2^n + r_1^n)/(r_2 + r_1)$ becomes $2r^n/2r$ or r^{n-1}; as we can multiply a solution by a constant, we see that this is the same as our original guess of r^n.

What happens to the second solution, $(r_2^n - r_1^n)/(r_2 - r_1)$? This becomes

$$\lim_{h \to 0} \frac{(r+h)^n - r^n}{h}; \tag{3.3}$$

this is the definition of the derivative of the function $g(r) = r^n$ with respect to r. As $g'(r) = nr^{n-1}$, we see that this solution becomes $nr^{n-1} = nr^n/r$; again as r is a constant we see that this is the same as guessing nr^n. A similar analysis for the differential equation gives the second guess is te^{rt}.

We would like to use some kind of continuity argument to claim that the limit of the second guess is also a solution. In class, we substituted these guesses into the original equations to show that they worked; if $(r_2^n - r_1^n)/(r_2 - r_1)$ is a solution of the difference equation $x_{n+2} - ax_{n+1} - bx_n$ whenever there are distinct roots, as we vary a and b so that the roots collapse must the limit of $(r_2^n - r_1^n)/(r_2 - r_1)$ also be a solution? One must sadly be careful about claims such as these. Linear algebra is replete with examples where something holds almost all the time but can suddenly fail. Examples range from varying an $N \times N$ matrix A and going from there is a unique solution to $Ax = b$ to suddenly there are none or infinitely many (as A becomes singular, i.e., non-invertible) to whether or not the matrix A is diagonalizable. Thus we should check our second guess and see that it does solve the original difference or differential equation.

We are thus left with the question of why we considered these combinations, specifically $(r_2^n + r_1^n)/(r_2 + r_1)$ and $(r_2^n - r_1^n)/(r_2 - r_1)$. The best answer I can give is that we can see one of the solutions collapsing to zero, *but* the division by $r_2 - r_1$ means we get $0/0$, and thus there is the hope that something useful will remain in the limit. Another reason is that we saw combinations like this were useful when looking at e^{it}, e^{-it} and $\cos(t), \sin(t)$.