Simpson's Rule

Midpoint Approx: $f\left(\frac{a+b}{2}\right) \cdot (b-a) \equiv M$

Taylor: $f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + O((x-a)^3)$

True Area: $\int_a^b f(x)dx = f(a)(b-a) + f'(a)\frac{(b-a)^2}{2} + f''(a)\frac{(b-a)^3}{6} + O((b-a)^4)$

↳ now $f\left(\frac{a+b}{2}\right) = f(a) + f'(a)\frac{b-a}{2} + f''(a)\frac{(b-a)^2}{8} + O((b-a)^3)$

so True Area − midpoint approx is

$$f''(a)\frac{(b-a)^3}{6} - f''(a)\frac{(b-a)^3}{8} + O((b-a)^4)$$

$$= f''(a)\frac{(b-a)^3}{24} + O((b-a)^4)$$

Trapezoid Approx: $\left(f(a) + f(b)\right) \cdot \frac{b-a}{2} \equiv T$

↳ algebra: True Area − trapezoid approx is

$$-\frac{1}{12}(b-a)^3 f''(a) + O((b-a)^4)$$

Weighted Average: $\frac{2M+T}{3}$

↳ has error $O((b-a)^4)$

Yields Simpsons rule:

$$\int_a^b f(x)\,dx = \left[f(a) + 4f\left(\tfrac{a+b}{2}\right) + f(b)\right]\frac{b-a}{6} + O((b-a)^4)$$

↳ note: error actually $O((b-a)^5)$ b/c of symmetry of evaluations

CHAPTER 8: NUMERICAL METHODS

Sec 8.1: THE EULER OR TANGENT LINE METHOD

↳ Review tangent line 

Application: Newton's Method

Consider first order eq: $y' = f(t, y)$ with $y(t_0) = y_0$

↳ Assume f, f_y cont in rectangle $\ni (t_0, y_0)$

By Thm 2.4.2, $\exists!$ soln in nbhood t_0

$$\boxed{\begin{array}{l} \underline{\text{Euler}}/\underline{\text{Tangent line}} \quad y_{n+1} = y_n + f(t_n, y_n)\cdot(t_{n+1} - t_n) \\ \quad\quad\quad\quad\quad\quad \text{or } y_{n+1} = y_n + f(t_n, y_n)\, h \text{ if stepsize constant} \end{array}}$$

Idea: $f(t, y)$ is slope, tells us how y is changing

<u>ERRORS</u>: ϕ soln, $E_n = \phi(t_n) - y_n$: <u>global truncation error</u>

↳ at each step make errors, input data at each step only approx

↳ if assume y_{n-1} exactly right, error in going one step forward to y_n is <u>local truncation error</u>.

↳ also have <u>round-off errors</u> from computers

↳ Lorentz equations and birth of chaos

↳ story of restarting ...

↳ We'll assume no round-off and analyze just local truncation error

Assume $y = \phi(t)$ soln @ cont second deriv (true if f, f_t, f_y cont)

↳ $\phi'(t) = f(t, \phi(t))$

$\phi''(t) = f_t(t, \phi(t)) + f_y(t, \phi(t))\phi'(t)$ Chain rule, $\phi'(t) = f(t, \phi(t))$

Taylor series: $\phi(t_n + h) = \phi(t_n) + \phi'(t_n)\, h + \frac{1}{2}\phi''(\bar{t}_n)\, h^2$

$ \overset{\|}{\phi(t_{n+1})}$ and $y_{n+1} = y_n + f(t_n, \overset{y_n}{\cancel{\phi(t_n)}})\, h$

Subtract: $\phi(t_{n+1}) - y_{n+1} = \underbrace{(\phi(t_n) - y_n)}_{\text{assuming zero as}} + h\underbrace{[f(t_n, \phi(t_n)) - f(t_n, y_n)]}_{\text{taking } y_n = \phi(t_n)} + \frac{1}{2}\phi''(\bar{t}_n)h^2$

local error $|\phi(t_{n+1}) - y_{n+1}| = \left|\frac{1}{2}\phi''(\bar{t}_n)\, h^2\right| \le \max|\phi''| \cdot \frac{h^2}{2}$

Key fact: error is like h^2!