Math 209: Soln Key to Difference Equation Exercises

Below are solutions to the difference equation exercises; This is also a good template for what your HW should look like (of course you don't need to use so many colors!). It's good to state the problem, write clearly, and if appropriate box the answer.

Exercise 1.1: Let $\alpha_0, \dots, \alpha_{k-1}$ be fixed integers and consider The recurrence relation $x_{n+k} = \alpha_{k-1} x_{n+k-1} + \dots + \alpha_0 x_n$.

(1) Show once k values of x_m are specified then all values of x_m are determined.

(2) Let $f(r) = r^{k-1} - \alpha_{k-1} r^{k-1} - \dots - \alpha_0$. If $f(\rho) = 0$ Show $x_n = c\rho^n$ solves the recurrence relation.

① The relation implies:

$(n=0)$ $x_k = \alpha_{k-1} x_{k-1} + \alpha_{k-2} x_{k-2} + \dots + \alpha_0 x_0$

Thus if we know $x_0, \dots, x_{k-1}$ (the first k values), Then we know x_k, and in fact This is The only choice for x_k. Now consider The recurrence relation with $n=1$:

$(n=1)$ $x_{k+1} = \alpha_{k-1} x_k + \alpha_{k-2} x_{k-1} + \dots + \alpha_0 x_1$

We know The k values on The right hand side (we were given $x_1, \dots, x_{k-1}$ and we just found x_k). Thus we now know x_{k+1}, and in fact there is only 1 possible choice for x_{k+1}. Arguing along These lines shows That The whole series is uniquely determined from The first k values.

More formally, we proceed by induction (let me know if you want a refresher on induction).

1 Continued

We've done the inductive step, showing that if we know $X_0, \ldots, X_{k-1}$ then we know X_k.

Assume now we know $X_0, \ldots, X_m$ for some m; we must show X_{m+1} is uniquely determined.

By the recurrence relation,

$$X_{m+1} = \alpha_{k-1} X_m + \alpha_{k-2} X_{m-1} + \cdots + \alpha_0 X_{m-k-1}$$

By the inductive assumption, all the X's on the right hand side are uniquely determined, thus X_{m+1} is uniquely determined.

(2) Assume $f(\rho) = 0$ so $\rho^k - \alpha_{k-1}\rho^{k-1} - \cdots - \alpha_0 = 0$

We must show $X_n = c\rho^n$ solves the recurrence relation:

Thus $X_{n+k} = c\rho^{n+k}$, $X_{n+k-1} = c\rho^{n+k-1}, \ldots$

Does $c\rho^{n+k} \stackrel{?}{=} \alpha_{k-1} c\rho^{n+k-1} + \alpha_{k-2} c\rho^{n+k-2} + \cdots + \alpha_0 c\rho^n$

↳ if we divide by $c\rho^n$ we get

$$\rho^k \stackrel{?}{=} \alpha_{k-1}\rho^{k-1} + \alpha_{k-2}\rho^{k-2} + \cdots + \alpha_0$$

or $\rho^k - \alpha_{k-1}\rho^{k-1} - \cdots - \alpha_0 \stackrel{?}{=} 0$

But we are given that the left hand side is 0 (since this is just $f(\rho)$). Thus, if $f(\rho) = 0$ then $X_n = c\rho^n$ is a solution.