Solving Polynomial Equations

Part III

Ferrari and the Biquadratic

Ferrari's solution of the quartic (biquadratic) equation involved the introduction of a new variable and then specializing this variable to put the equation into a form that could easily be solved. Finding the right specialization involved solving a cubic equation (called the ***resolvent*** of the original quartic). Here are the details, again using modern techniques.

Consider the general quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

and rewrite it as

$$x^4 + ax^3 = -bx^2 - cx - d.$$

Now add $\frac{1}{4} a^2x^2$ to both sides to make the LHS a perfect square:

$$(x^2 + \tfrac{1}{2} ax)^2 = (\tfrac{1}{4} a^2 - b)x^2 - cx - d.$$

Ferrari and the Biquadratic

We introduce a new variable by adding $y(x^2 + \frac{1}{2} ax) + \frac{1}{4} y^2$ to both sides of the equation (this keeps the LHS a perfect square):

$(x^2 + \frac{1}{2} ax + \frac{1}{2} y)^2 = (\frac{1}{4} a^2 - b + y)x^2 + (-c + \frac{1}{2} ay)x + (-d + \frac{1}{4} y^2)$.

If we can chose a y so that the RHS is a perfect square, the resulting quartic equation would be very easy to solve. Now, a quadratic

$$Ax^2 + Bx + C$$

is a perfect square (has two equal roots) if and only if $B^2 - 4AC = 0$.

So we consider the equation (resolvent):

$$(-c + \tfrac{1}{2} ay)^2 = 4(\tfrac{1}{4} a^2 - b + y)(-d + \tfrac{1}{4} y^2) \text{ or}$$

$$y^3 - by^2 + (ac - 4d)y + 4bd - a^2d - c^2 = 0.$$

With y being any solution of this cubic, we obtain

$$(x^2 + \tfrac{1}{2} ax + \tfrac{1}{2} y)^2 = (ex + f)^2,$$

where (in general)

$$e = \sqrt{\frac{a^2}{4} - b + y} \text{ and } f = \frac{-c + \frac{ay}{2}}{2\sqrt{\frac{a^2}{4} - b + y}}.$$

Ferrari and the Biquadratic

The solutions of the quartic can now be obtained by solving the two quadratic equations:

$$x^2 + \tfrac{1}{2} ax + \tfrac{1}{2} y = ex + f \quad \text{and} \quad x^2 + \tfrac{1}{2} ax + \tfrac{1}{2} y = -ex - f.$$

We illustrate this procedure with a simple example

$$x^4 + 3 = 4x.$$

Here $a = b = 0$, $c = -4$ and $d = 3$.

The resolvent is

$$y^3 - 12y - 16 = 0.$$

We could use the Cardano formula to obtain a root, but inspection yields $y = 4$ in this case. This gives $e = 2$ and $f = 1$. Therefore, the quadratics we solve are:

$$x^2 + 2 = 2x + 1 \quad \text{and} \quad x^2 + 2 = -2x - 1$$

or

$$x^2 - 2x + 1 = 0 \quad \text{and} \quad x^2 + 2x + 3 = 0.$$

Complex Roots

From the first we obtain x = 1 (repeated) and the second gives $x = -1 \pm i\sqrt{2}$.

Cardano would have rejected the complex roots of the second equation here, but he did, in the *Ars Magna*, at least consider them in regards to a quadratic problem. He stated, after verifying that the complex roots satisfied the original quadratic:

So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless.

Full recognition of complex roots of equations came about 15 years later in the *Algebra* of Rafael Bombelli (1526 - 1572).

The Fifth Degree Equation

During the next two centuries several alternate solutions for the cubic and quartic equations were found. In all of these solutions an auxiliary equation (the resolvent) was used. In some cases, the resolvent equation had a degree larger than the equation that was under consideration, but because of its special form a solution was obtainable.

The success with the cubic and quartic equations naturally led to a search for a similar solution of the general quintic (5^{th} degree) equation. But the methods that had worked with the lower degree equations generally produced resolvent equations which could not be solved. New ideas were needed for any progress to be made, and these surfaced in the 18^{th} Century.

Lagrange

The two greatest mathematicians of the 18th Century were Euler and Lagrange, and which of the two was the greater is a matter of debate that often reflects the differing mathematical sensitivities of the debaters. - Eves, *An Introduction to the History of Mathematics*

Joseph Louis, Comte Lagrange (1736 – 1813) was born in Turin. He was Italian by birth, German by adoption, and Parisian by choice.

He began his teaching as professor of mathematics in the artillery school at Turin (1755) when he was only nineteen. He was not one of the infant prodigies in mathematics, and showed no interest in mathematics until he was seventeen; but within a few years he was recognized as the greatest living scholar in his science.

Lagrange

In 1766, Frederick the Great wrote that “the greatest king in Europe” wanted “the greatest mathematician of Europe” at his court. As a result of this letter, Lagrange went to Berlin, succeeding Euler as mathematical director in the Berlin Academy and remained there for more than 20 years.

With the death of Frederick in 1787 and the resulting turmoil in Prussia, Lagrange accepted an invitation by Louis XVI to take up residence in Paris, and became a member of the Académie des Sciences.

Lagrange was made president of the commission that designed the metric system. This work was valued at the time of the French Revolution, and even though all foreigners were banished from France, the Committee of Public Safety expressly excepted Lagrange from this decree.

Lagrange

Lagrange was revolted by the cruelties of the Terror that followed the French Revolution. When the great chemist Lavoisier went to the guillotine, Lagrange expressed his indignation at the stupidity of the execution: “It took the mob only a moment to remove his head; a century will not suffice to reproduce it.”

Lagrange had decided to leave Paris, but was invited to establish the mathematics department at the newly created École Normale (1795). This chance to establish a training program for teachers based on the most thorough mathematical scholarship was sufficient to change his mind. Two years later, he was asked to do the same at the new École Polytechnique.

Under Napoleon he was made a senator and a count, and was awarded other honors appropriate to his genius.

Lagrange

Lagrange's mathematical work had a deep influence on later mathematical research, for he was the earliest first-rank mathematician to recognize the thoroughly unsatisfactory state of the foundations of analysis and accordingly to attempt a rigorization of the calculus.

He also gave a method for approximating the real roots of an equation by means of continued fractions, obtained general equations of motion of dynamical systems, worked on differential and partial differential equations and contributed to the calculus of variations. He also had a penchant for number theory and wrote important papers in this field also, such as the first published proof of the theorem that every positive integer can be expressed as the sum of not more than four squares.

He also did fundamental work on the theory of equations ...

Resolvents

Lagrange carried out a detailed investigation of all the various solutions of cubic and quartic equations in order to discover techniques that would work for quintic and higher degree equations. In particular, he was interested in determining the relationship between the original equations and the resolvents that were used in their solutions.

Starting with the cubic equation $x^3 + nx + p = 0$, the substitution $x = y - (n/3y)$ leads to the 6th degree resolvent $y^6 + py^3 - (n^3/27) = 0$. With $r = y^3$, this is a quadratic equation in r which has two roots r_1 and r_2. The cube roots of r_1 and r_2 are solutions of the resolvent and can then be used to obtain a value of x. Since $r_2 = -(n/3)^3/r_1$, we get

$$x = \sqrt[3]{r_1} + \sqrt[3]{r_2}.$$

Resolvents

This was Cardano's solution, but he only used the real cube roots of r_1 and r_2. Lagrange knows that there are really three cube roots of a number:

The three cube roots of a number r are

$$\sqrt[3]{r}, \quad \omega\sqrt[3]{r} \text{ and } \omega^2\sqrt[3]{r} \quad \text{where } \omega = \frac{(-1+\sqrt{-3})}{2} \text{ and } \omega^2 = \frac{(-1-\sqrt{-3})}{2}.$$

The numbers 1, ω and ω^2 are the solutions of $x^3 - 1 = 0$, called the *cube roots of unity*.

Thus, the resolvent really has 6 roots (and not just the two real roots). "Plugging" these values in, one obtains only three solutions of the cubic since each of the cubic roots is obtained by two of the resolvent roots. These solutions of the cubic are:

Resolvents

$$\begin{aligned} \boldsymbol{x}_1 &= \sqrt[3]{\boldsymbol{r}_1} + \sqrt[3]{\boldsymbol{r}_2} \\ \boldsymbol{x}_2 &= \omega\sqrt[3]{\boldsymbol{r}_1} + \omega^2\sqrt[3]{\boldsymbol{r}_2} \text{ and} \\ \boldsymbol{x}_3 &= \omega^2\sqrt[3]{\boldsymbol{r}_1} + \omega\sqrt[3]{\boldsymbol{r}_2}. \end{aligned}$$

Cardano came close to the realization that other roots of the resolvent would give other solutions to the cubic ... for he says:

I need not say whether having found another value for [a root of the resolvent] *... we would come to two other solutions* [for x]. *If this operation delights you, you may go ahead and inquire into this for yourself.*

Resolvents

Having expressed all of the roots of the cubic equation in terms of all of the roots of the resolvent, Lagrange proceeds to now write the roots of the resolvent in terms of the roots of the cubic. He finds that the six roots of the resolvent can all be obtained from

$$y = (1/3)(x' + \omega x'' + \omega^2 x''')$$

where x', x'' and x''' are the three roots of the cubic arranged in any order (there are $3! = 6$ such arrangements ... we call them ***permutations*** today.)

Lagrange then makes three observations concerning the roots of the resolvent equation.

1. Because there are 6 permutations of the roots of the cubic, the degree of the resolvent equation is $6 = 3!$.

Resolvents

2. Even though there are 6 values of y, there are only 2 values of y^3. There are two sets of three permutations each, and each set corresponds to one of the values of y^3. Since there are only two values of y^3, y^3 must satisfy a quadratic equation.

3. The coefficients of the resolvent equation are rational functions of the roots of the cubic equation.

In the case we are looking at, $y^6 + py^3 - (n^3/27)$, the 0 coefficients of y^5, y^4, y^2 and y are all equal to $x_1 + x_2 + x_3$, while $p = -x_1x_2x_3$ and

$$\frac{-n^3}{27} = \frac{-(x_1x_2+x_1x_3+x_2x_3)^3}{27}.$$

Lagrange looks at other solutions of the cubic and finds the same relationships with the resolvents.

Quartic Resolvents

Lagrange next looks at Ferrari's solution of the quartic. The roots of the resolvent in this case look like,

$$y = \tfrac{1}{2}\,(x'x'' + x'''x'''')$$

where the x's are the four roots of the quartic in any of the 4!= 24 possible arrangements. This time, however, there are only three distinct values of the y's. Thus, the resolvent has degree 3 and its coefficients are again rational functions of the roots of the quartic.

Lagrange is now ready to tackle the general problem. He was going to look for a resolvent of degree k (< n, the degree of the equation to be solved) whose roots would be certain functions of the roots of the original equation, functions that take on only k values when the roots of the original equation are permuted in n! ways.

Lagrange's Attempt

In his search for such functions he did prove that the degree of the resolvent he sought would be a divisor of n! (but he could not show that it was less than n!).

Furthermore, he proved that if all the permutations of the roots which leave some function f fixed also leave another function g fixed, then g can be expressed as a rational function of f together with the coefficients of the original equation. Furthermore, if g is not fixed by the permutations which fix f, but is changed to one of a set of r different functions, then g will be a root of an equation of degree r whose coefficients are rational functions of f and the coefficients of the original equation.

Lagrange's Attempt

Lagrange hoped to use these results to solve the general polynomial equation of degree n. He would start with a symmetric function of the roots (a function fixed by all the permutations of the roots, such as $x_1 + x_2 + ... + x_n$) and then find a function g that takes r values under these permutations. This function would satisfy an equation of degree r, with coefficients rational in the original coefficients – the f here is one of the original coefficients. If this equation could be solved then he would try to find a new function h which would take on s values under the permutations which leave g fixed. He would continue in this manner until the solutions of the original were found. Unfortunately, Lagrange was unable to find a general method of determining these intermediate functions that could be solved by known methods. He had to quit.

Ruffini

An Italian physician, Paola Ruffini (1765 – 1822) proposed that the general quintic equation could not be solved using radicals. His proof was privately published and revised in 1803, 1805 and 1813. It is claimed that no contemporary mathematicians could understand this purported proof. It does have a gap. However, Ruffini does develop all the essential ideas of group theory that are used in this proof, and his approach is essentially correct.

Abel

Niels Henrik Abel (1802 – 1829) was born in Findö, Norway, the son of a country minister. Plagued by poverty and suffering from a pulmonary condition throughout his short life, this brilliant young mathematician died at 26 from an attack of tuberculosis at Froland in Norway.

His native abilities in mathematics were discovered by his mathematics teacher in high school who encouraged Abel to read advanced mathematics texts. After graduating from the university, he obtained a small stipend which enabled him to tour Europe and advance his knowledge. He wrote several papers during this year of visiting France, Germany and Italy. These were highly regarded by the European mathematicians. Five of them appeared in the first issue of the new *Journal für die reine und angewandte Mathematik* also known as *Crelle's Journal*.

Abel

The editor of this journal, August Crelle, soon became one of Abel's best friends and promoter. Abel returned to Norway in 1927 when his money was exhausted only to find that the only opening at the university had just been filled by his former teacher. Unable to obtain employment, he attempted to support himself by tutoring and doing substitute teaching. Two days after his death, a delayed letter arrived from Crelle informing him that a position had been secured for him in Berlin.

Abel wrote papers in various areas of mathematics dealing with the convergence of infinite series, the theory of doubly periodic functions, elliptic functions and the theory of equations. His name is associated with fundamental results in all these areas today.

Abel

While he was a student at the university, he thought that he had a solution of the quintic in terms of radicals. As his Norwegian professors could not understand his proof, he sent it to Denmark. Before it was published, he was asked to provide some numerical examples, and while searching for these he realized that his method was incorrect. He continued to work on the problem for the next few years and eventually proved that no such solution is possible. He privately published this result as a pamphlet in 1824. In order to save printing costs, he had to give the paper in a very summary form, which in a few places affects the lucidity of his reasoning. A longer version appeared later in *Crelle's Journal*.

Abel

After showing that the general quintic (and any higher degree) equation could not be solved by radicals, Abel set himself the problems of

1) Finding all equations of any degree which could be solved by radicals, and

2) Finding a decision procedure which would determine whether a given equation is algebraically solvable or not.

He did not live long enough to carry out this program, but he did make some headway. In particular, in the case where all the roots of an equation can be expressed as a rational function of one of them, say x, then if for any two roots $\alpha(x)$ and $\beta(x)$, we have $\alpha\beta(x) = \beta\alpha(x)$, then the equation is algebraically solvable.

Galois

Évariste Galois (1811 – 1832) had an even shorter and more tragic life than Abel. Galois was born in Bourg-la-Reine, a town not far from Paris in which his father was elected mayor in 1815. He had mixed success in preparatory school (mostly due to trouble in the humanities), but began to shine in mathematics to the exclusion of all other subjects. He published a short paper before he was 18, and submitted a memoir on the solvability of equations of prime degree to the French Academy at the same time. However, he twice failed the entrance exam to the École Polytechnique (the first time because he wasn't prepared and the second attempt was a few days after his father had committed suicide). Galois enrolled at the École Normale to prepare himself for a teaching career.

Galois

After the director locked the students into the building so that they could not participate in the political activities leading to the July revolution of 1830, Galois attacked the director in a letter for favoring "legitimacy" over "liberty". He was expelled.

Galois joined a heavily republican division of the National Guard, a division that was soon dissolved because of its perceived threat to the throne of King Louis-Phillipe. He was arrested twice for his political activities and sent to prison for six months for the second offense (wearing the uniform of the dissolved National Guard division).

Although heavily involved in political activities, he continued to study mathematics. He mastered the mathematical textbooks of his time with the ease of reading novels and went on to the important papers of Legendre, Jacobi and Abel, and then to creating his own.

Galois

Two of his memoirs, sent to the French Academy were mislaid and lost, and a revised version of his memoir on the solvability of equations was rejected because the referee could not understand the proofs.

While he was in prison his anger at the Academy for their failure to appreciate his work grew to such a degree that he lashed out at France's “official scientists” in a vicious diatribe intended to be a preface to the private publication of his work.

At this time he also met Stéphanie-Felicie du Motel, daughter of a physician who lived nearby. After he left prison she rebuffed his advances and Galois was drawn into a duel of honor with a fellow republician Pécheux D'Herbinville. On May 30, 1832, five months before his 21st birthday he was shot in the stomach and died the following day of peritonitis.

Galois

The evening before the duel, expecting to die on the morrow, he wrote a letter to his friend Auguste Chevalier in which he set forth briefly his discovery of the connection of the theory of groups with the solution of equations by radicals. At the end of the letter he wrote:

I have often in my life ventured to advance propositions of which I was uncertain; but all that I have written here has been in my head nearly a year, and it is too much to my interest not to deceive myself that I have been suspected of announcing theorems of which I had not the complete demonstration.

Ask Jacobi or Gauss publicly to give their opinion, not as to the truth, but as to the importance of the theorems.

Subsequently there will be, I hope, some people who will find it to their profit to decipher all this mess.

Galois

This letter was published, as requested, later that year in the *Revue encyclopédique*, but his other manuscripts lay unread until they were finally published in 1846 by Liouville in his *Journal des mathématiques*. Within the next few years, several mathematicians included Galois' material in university lectures or published commentaries on the work. It was not until 1866, however, that Galois theory was included in a text, the third edition of the *Cours d'algebre* of Paul Serret (1827-1898). Four years later, Camille Jordan (1838-1922) published his monumental *Traité des substitutions et des équations algébriques* (Treatise on Substitutions and Algebraic Equations), which contains a somewhat revised version of Galois theory.

Galois

Galois' life has been the subject of many biographies (one fictional). Several of these have been criticized for being overly imaginative and selective in what they report about Galois – trying to make a case rather than reporting the facts. One of the worst offenders is E.T. Bell in his very popular *Men of Mathematics*. Bell seems to want to make the case for a gentile misunderstood genius who was battered around by a cruel world. While Bell mentions his sources, he seems to chose only certain items and distorts others in order to support his point of view. A closer reading of the historical documents do not paint Galois in as rosy a light as Bell would have it. While his genius is beyond question, “brash”, “impatient”, and “hothead” would be a bit more descriptive of his personality.