MATH 209: MODELING: FROM DIFFERENCE EQUATIONS TO RANDOM MATRIX THEORY

STEVEN MILLER

ABSTRACT. We briefly discuss some methods of modeling, emphasizing the connections between difference equations and random matrix theory.

1. FIRST MODEL

We consider the following simplified model for the number of pairs of whales alive at a given moment in time. We make the following simplifying assumptions:

(1) Time moves in discrete steps of 1 year.
(2) The number of whale pairs that are 0, 1, 2 and 3 years old in year n are denoted by a_n, b_n, c_n and d_n respectively; all whales die when they turn 4.
(3) If a whale pair is 1 year old it gives birth to two new pairs of whales, if a whale pair is 2 years old it gives birth to one new pair of whales, and no other pair of whales give birth.

Letting

$$v_n \ = \ \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix}, \qquad (1)$$

we see that

$$v_{n+1} \ = \ Av_n, \qquad (2)$$

where

$$A \ = \ \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \qquad (3)$$

Thus

$$v_{n+1} \ = \ A^{n+1}v_0, \qquad (4)$$

where v_0 is the initial populations at time 0. As discussed before, it is one thing to write down a solution and another to have be able to numerically work with it. This matrix is fortunately easily diagonalizable. The eigenvalues are $\frac{1+\sqrt{5}}{2}$, -1, $\frac{1-\sqrt{5}}{2}$, 0, and $S^{-1}AS = \Lambda$ where

$$S \ = \ \begin{pmatrix} 2+\sqrt{5} & -1 & 2-\sqrt{5} & 0 \\ \frac{3+\sqrt{5}}{2} & 1 & \frac{3-\sqrt{5}}{2} & 0 \\ \frac{1+\sqrt{5}}{2} & -1 & \frac{1-\sqrt{5}}{2} & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \qquad (5)$$

Thus, given the initial distribution of ages, we can readily calculate the number of whale pairs of each age at any time.

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Note: The matrix A is a special type of matrix, known as a Leslie matrix; see, for example, `http://en.wikipedia.org/wiki/Leslie_matrix`.

2. Second Model

We now discuss a generalization of the above model, with connections to Markov chains and Random Matrix Theory. There are numerous problems with the model above, as discussed in class. The biggest is that the matrix elements are constant. There are two ways to change this. One is to replace A with $A(n)$; however, this is not the best approach. The reason is that both this and the original model are completely deterministic, leaving nothing to chance. For this reason, a better method is to allow the matrix elements of $A(n)$ to be randomly chosen according to some reasonable procedure. We consider the simplest one below.

Let us assume that the only randomness we'll incorporate is the number of pairs a whale pair gives birth to in a given year. Currently we have

$$a_{n+1} \ = \ 2b_n + c_n. \tag{6}$$

It is of course absurd to have, every single year, the exact same birth rates. A more realistic model would be to have

$$a_{n+1} \ = \ x_n b_n + y_n c_n, \tag{7}$$

where x_n (respectively y_n) is the value of the random variable X_n (respectively Y_n). We'll assume X_n and Y_n have means 2 and 1 respectively. Depending on what we choose for X_n and Y_n, the model can become more and more complicated. We consider one of the simplest cases possible below, and leave the generalizations as an exercise.

We assume $Y_n = 1$ for all n (ie, there is no variance here); however, we assume $X_n = 2$ exactly 80% of the time, $X_n = 1$ exactly 10% of the time and $X_n = 3$ exactly 10% of the time. Thus X_n does equal 2 on average. This leads to three matrices, which we shall denote by A_1, A_2 and A_3:

$$A_1 \ = \ \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 \ = \ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_3 \ = \ \begin{pmatrix} 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{8}$$

Our assumptions imply that

$$v_{n+1} \ = \ B_n v_n, \tag{9}$$

where 80% of the time $B_n = A_1$, 10% of the time $B_n = A_2$ and 10% of the time $B_n = A_3$. Thus

$$v_{n+1} \ = \ B_n B_{n-1} B_{n-2} \cdots B_0 v_0, \tag{10}$$

where each B_i equals A_1, A_2 or A_3 with probabilities 80%, 10% and 10%. We no longer have a deterministic system; it is possible that $v_{2009} = A_3^{2010} v_0$, though this is unlikely (the probability is just 10^{-2010}!).

This leads to the following natural question: what can one say about the generic products $B_n\ B_{n-1}\ B_{n-2} \cdots\ B_0$? Not surprisingly, questions like these arise in Random Matrix Theory. Though in class we only discussed the distribution of eigenvalues of a given matrix, another very important question in the subject is the distribution of eigenvalues (or coefficients) of products of random matrices. We have just studied a simple case where at each time there are only three possibilities for the next matrix; in general one would have a continuous spectrum of possible matrices, with entries drawn from given probability distributions.