FOURIER ANALYSIS FOR POISSON SUMMATION

THM 11.4.6 (POISSON SUMMATION)

Assume f is twice differentiable and f, f', f'' decay like $x^{-(1+\eta)}$ for some $\eta > 0$ (means $\exists x_0, C$ st $\forall |x| > x_0, |g(x)| \le C|x|^{-(1+\eta)}$)

Then $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$ is Fourier Transform of f

↳ aside: "many" normalizations for Fourier Transform. This is the "correct" one as makes Poisson Sum nice

↳ aside: do not need many properties of Fourier Transform.

ASIDE: For more, see Sections 11.4.1, 11.4.3, 11.5 (CLT) and exercise 11.6.4 (applications to PDEs)

↳ Key ingredient in Proof of Poisson Sum:

THM 11.3.8 (Dirichlet)

Suppose f is periodic with period 1, $|f(x)|$ is bounded and f is differentiable at x_0. Then $\lim_{N \to \infty} S_N(x_0) = f(x_0)$, where $S_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}$

↳ ASIDE: Proof of This / general convergence questions of Fourier Series. Need differentiability assumption as Dirichlet kernel D_N is not an approx to the identity (Fejer kernel was).
Remember $S_N(x_0) = (f * D_N)(x_0)$.
See Exe 11.6.1 for what can happen at discontinuities

PROOF OF POISSON SUMMATION

Natural to study $F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ and $F(0)$

Clearly F is periodic with period 1, To use Dirichlet's Thm need to show F bounded and differentiable for all x.

<u>ASIDE</u>: If only knew F cont, could we use Fejér's Thm and get something?

↳ caveat: Exe 11.4.7 shows why we need some conds on f, f', f'' to ensure F is cont and diff

↳ ex: $f(x) = \begin{cases} n^6 \left| \frac{1}{n^4} - |n-x| \right| & |n-x| \le \frac{1}{n^4} \\ 0 & \text{otherwise} \end{cases}$

↳ for this f, $F(0)$ (or $F(n)$) does not exist

<u>LEMMA 11.4.8</u>: g decays like $x^{-(1+\eta)}$ then $G(x) = \sum_{n=-\infty}^{\infty} g(x+n)$ converges for all x and is continuous

<u>Proof</u>: $|G(x)| \le \sum_{|n| \le N} |g(x+n)| + \sum_{|n| > N} |g(x+n)|$

$\le \text{Const}(N) + C \sum_{|n|>N} \left(\frac{1}{x+n}\right)^{1+\eta}$

as $\eta > 0$, sum converges

(N large st $x+n \ne 0$)

Must show Continuous

PROOF OF POISSON SUM (CONT)

Lemma 11.4.10: f, f', f'' decay like $x^{-(1+\eta)}$ Then F is cont diff

Proof: from decay of f and previous lemma know F is continuous

Must show F^* is differentiable. Natural candidate for $F'(x)$ is $\sum_{n=-\infty}^{\infty} f'(x+n)$. If this is the deriv, then done as conds on f' imply this is continuous

↳ Set $F'(x) = \sum_{n=-\infty}^{\infty} f'(x+n)$

Must show $\forall \varepsilon > 0\ \exists \delta > 0$ st $\left|\frac{F(x+h)-F(x)}{h} - F'(x)\right| < \varepsilon$ for all $|h| < \delta$.

Have $\frac{F(x+h)-F(x)}{h} - F'(x) = \sum_{|n| \le N} + \sum_{|n| > N} \left(\frac{f(x+n+h)-f(x+n)}{h} - f'(x+n)\right)$

(ok b/c of decay)

Apply MVT (Thm A.2.2) to f:

$$\frac{f(x+n+h)-f(x+n)}{h} = h f'(x+n+c_n), \quad c_n \in [0,h]$$

$$\Rightarrow \frac{F(x+h)-F(x)}{h} - F'(x) = \sum_{|n| \le N} + \sum_{|n|>N} \left(f'(x+n+c_n) - f'(x+n)\right)$$

$$= \underbrace{\sum_{|n| \le N} \left[f'(x+n+c_n) - f'(x+n)\right]}_{\text{Use continuity: } \delta \text{ suff small each at most } \frac{\varepsilon}{2(2N+1)}} + \underbrace{\sum_{|n|>N} f''(x+n+d_n) \text{ with } d_n \in [0, c_n]}_{N \text{ large} \ldots \le \sum_{|n|>N} \frac{C}{n^{1+\eta}} < \frac{\varepsilon}{2}}$$

Thus absolute value at most ε for $|h| < \delta$

ASIDE: Exe 11.4.11: Weakening assumps in lemma

Proof of Poisson Sum (cont)

Know F, F' exist and are cont, F is periodic @ period 1

(can show F is bounded).

By Dirichlet's Thm:

$$F(x) = \sum_{m=-\infty}^{\infty} \hat{F}(m) e^{2\pi i m x} \quad \text{with} \quad \hat{F}(m) = \int_0^1 F(x) e^{-2\pi i m x}\,dx$$

$$\text{Clearly } F(0) = \sum_{m=-\infty}^{\infty} \hat{F}(m) = \sum_{m=-\infty}^{\infty} f(m)$$

With abuse of notation, reduced to showing $\hat{F}(m) = \hat{f}(m)$,
where $\hat{f}$ is the Fourier Transform of f: $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\,dx$

↳ Have $\hat{F}(m) = \int_0^1 \left(\sum_{n=-\infty}^{\infty} f(x+n) \right) e^{-2\pi i m x}\,dx$

Here $\int \Sigma = \Sigma \int$ by Fubini's Thm (Thm A.2.8)

↳ note $\int \Sigma \neq \Sigma \int$ always (Exe 11.4.12)

For us, need to know $\int \Sigma |\,|$ is finite

↳ trivial: $\int_0^1 \left(\sum_{n=-\infty}^{\infty} |f(x+n)| \right) dx \leq \int_{-\infty}^{\infty} |f(x)|\,dx < \infty$

as f decays like $x^{-(1+\eta)}$ (and is cont)

↳ Have $\sum_{n=-\infty}^{\infty} \int_0^1 f(x+n) e^{-2\pi i m x}\,dx$

$$= \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n) e^{-2\pi i m (x+n)} \underbrace{e^{2\pi i m n}}_{1}\,dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-2\pi i m x}\,dx = \hat{f}(m) \quad \blacksquare$$

ASIDE: Prove the Fubini–Tonelli Theorem to justify the interchange