FOURIER ANALYSIS FOR $\{n^k\alpha\}$

Fourier analysis big subject. Will prove just enough to get to Fejér's Thm. Other main results we might need is Poisson Summation/CLT

11.1. INNER PRODUCT SPACES

$e^{ix} = \cos x + i\sin x$, infinite series; $e_n(x) = e^{2\pi i n x}$

When does $f(x)$ "$=$" $\sum_{n=-\infty}^{\infty} a_n e_n(x)$?

↳ What do we mean by "$=$"? Pointwise? L^p?

↳ What are the a_n?

Doing ∞-dim lin alg: Are the $e_n(x)$ a basis (for what?)

$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \vec{w}$ or $\vec{v}^* \vec{w}$ becomes

$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\,dx$

↳ $\langle f, f \rangle$ is ≥ 0 and $= 0$ iff $f \equiv 0$

$\langle f, g \rangle = \overline{\langle g, f \rangle}$

$\langle af + bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle$

$\langle e_m(x), e_n(x) \rangle = \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & \text{other} \end{cases}$ (orthogonality relations)

Question: Why this defn of $\langle f, g \rangle$?

$f \Leftrightarrow \left(f(0), f(\tfrac{1}{n}), \ldots, f(\tfrac{n-1}{n}) \right)$ and take dot product

11.2. Fourier Series

11.2.1. Intro

n^{th} Fourier coeff: $\hat{f}(n) = \langle f(x), e_n(x) \rangle = \int_0^1 f(x) e^{-2\pi i n x} dx$

- $\langle f(x) - \hat{f}(n) e_n(x), e_n(x) \rangle = 0$

- N^{th} partial Fourier Series: $S_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e_n(x)$

↳ lots of nice properties (Do Exe 11.2.2)
- $\langle f - S_N, e_n \rangle = 0 \quad \forall |n| \leq N$
- $|\hat{f}(n)| \leq \int_0^1 |f(x)| dx = \|f\|_1$

Question: When does $S_\infty(x)$ equal $f(x)$?

↳ if $|f|$ is bounded, f period 1, f diff at x_0
then $\lim_{N \to \infty} S_N(x_0) = f(x_0)$ (Dirichlet! Thm 11.3.8)

↳ unfortunately need diff: for us just want cont
(b/c want to use  approx to characteristic fns)

↳ General caveat in Series Expansions!
(1) existence and uniqueness questions
↳ ex! are Taylor Series Unique?
↳ Exe A.2.7: $f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$
has all Taylor coeff zero
↳ thus radius of conv is zero!

11.2.2. APPROX TO THE IDENTITY

Seq. $\{A_n(x)\}_{n=1}^{\infty}$ is an approx to the identity if

- $A_n(x) \geq 0$
- $\int_0^1 A_n(x)\,dx = 1$ (or could do on $(-\infty, \infty)$)
- $\forall \delta \in (0, 1/2)$: $\lim_{N \to \infty} \int_\delta^{1-\delta} A_N(x)\,dx = 0$

  ↳ often do $\lim_{N \to \infty} \int_{|x| > \delta} A_N(x)\,dx = 0$

Give examples: Λ, normal distr, ...

Recall defn of Dirac Delta Fnal: $\int f(x)\,\delta(x-a)\,dx = f(a)$

11.2.3. DIRICHLET AND FEJÉR KERNELS

$$D_N(x) = \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x} \quad \text{Dirichlet}$$

$$F_N(x) = \frac{1}{N} \sum_{n=0}^{N} D_n(x) = \frac{\sin^2(N\pi x)}{N \sin^2 \pi x} \quad \text{Fejér}$$

KERNEL

Convolution: $(f * g)(x) = \int_0^1 f(y)\, g(x-y)\,dy = \int_0^1 f(x-y)\, g(y)\,dy$

$$(T_N f)(x) = (f * F_N)(x)$$

THM: Fejér kernels are an approx to the identity

Proof:
- $F_N(x) \geq 0$ clear
- $F_N(x) = e_0(x) + \frac{N-1}{N}\left(e_{-1}(x) + e_1(x)\right) + \cdots$ so $\int$ to 1.
- $\delta < x < 1-\delta$ then $|F_N(x)| \leq \frac{1}{N \sin^2 \pi \delta}$

REMARK: $S_N f(x) = f * D_N(x)$

↳ analysis much harder as D_N is not an approx to identity (see Exe 11.2.12)

11.3.1. CONVERGENCE OF FEJÉR SERIES

THM (FEJÉR): f cont, periodic on $[0,1]$. $\forall \varepsilon > 0\ \exists N_0$ st $\forall N \geq N_0$ $|f(x) - T_N f(x)| \leq \varepsilon$. Hence $\lim_{N\to\infty} T_N(x) = f(x)$

Ideas of Proof: Multiply by 1, 3ε proof.

$$T_N(x) - f(x) = \int_0^1 f(x-y) F_N(y)\,dy - f(x)\cdot 1$$

$$= \int_0^1 [f(x-y) - f(x)] F_N(y)\,dy \quad \text{as } \int F_N(y)\,dy = 1$$

$$= \underbrace{\int_0^{\delta} + \int_{\delta}^{1-\delta} (f(x-y) - f(x)) F_N(y)\,dy}_{} + \underbrace{\int_{\delta}^{1-\delta} (f(x-y) - f(x)) F_N(y)\,dy}_{}$$

Since f cont, f unif cont (Thm A.3.7). Choose δ so small that diff b/w f values at most $\varepsilon/3$ for $0 \leq y < \delta$. Note $\int F_N(y)\,dy \leq 1$

Thm A.3.13 have f bounded. Thus this is $\leq B \cdot \int_{|y| \geq \delta}^{1-\delta} F_N(y)\,dy$. Choose N so large at most $\frac{\varepsilon}{3}$

Thus $|T_N(x) - f(x)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon$

CORR: WEIERSTRASS APPROX THM (THM 11.3.4): Any cont periodic fn can be unif approx by trig poly

↳ ORIGINAL VERSION was by poly, not trig poly, and done before Fejér's Thm

ASIDE: ① Dirichlet's Convergence Thm

② General f: what can be said?

↳ Kolmogorov

↳ Carleson-Fefferman

-FAY-