MATH 406: ANALYSIS AND NUMBER THEORY

INTRODUCTION

↳ Explain my background
basics of class
↳ several levels: broad overview + details
↳ options to read more
↳ options to do research / explore

↳ emphasize techniques and universalities

QUESTION

↳ What do you think of when hear "analysis" or "number theory"

Move to a brief description of topics:

$\{n^k\alpha\}$ and Poissonian behavior (FA, P, NT)
Random matrices and L-fns (P, L, NT)

Spacings b/w events
↳ Fourier Analysis (FA)
Probability (P)
Linear Algebra (L)
Number Theory (NT)

Benford's Law (P, FA)
Circle Method (FA, NT)
PNT (FA, NT)
MSTD (P)

COUNTING EVENTS
↳ Fourier Analysis (FA)
Probability (P)
Number Theory (NT)

Other: $3X+1$, algebraic structure of numbers (Champernowne)

RESULT	$n^k\alpha$ and Poisson Sum	RMT	Circle Method	Benford	$\zeta(s)$ and L-fns
BASIC FOURIER SERIES	trig functions to help analyze problem		trig functions are the generating fns for problem, get coeffs from Complex Analysis	Fourier series and transform to study probability distr	useful expansions, Complex analysis to get coefficients
↳ Poisson Summation				↳ Many processes are Benford	↳ Functional Equations
↳ Fejér/Dirichlet's Theorem	↳ approximate step fns with trig polynomials		↳ Modulo 1 CLT products converge to Benford behavior		
Algebraic Structure of numbers	Approximating irrationals with rationals to understand dynamics		Waring type problems/ understanding sizes of exponential sums	Rates of convergence (quantified equidistribution)	Special values of L-fns encode arithmetically interesting information
↳ Kronecker/Weyl	↳ equidistribution			↳ Certain data sets are Benford	
Combinatorics		moments (matchings and Catalan numbers), Toeplitz matrices, ...	Counting solutions to Diophantine Eqs		Unique factorization and the Euler Product, coeff of L-fns encode arithmetic
Philosophy of Square-root Cancellation		Scale for studying the eigenvalues	Estimating size of exponential sums		"Random Primes": The CLT is equiv to the Riemann hypothesis
Probability	distributions, moments	distributions moments	heuristic models	distributions moments	heuristic models
PROBLEMS					
↳ Counting Events	Equidistribution	Semicircle Law/ McKay's Law, ...	Waring, Goldbach, Germain, ...	Digit frequencies	Prime number thm, rank of elliptic curve, class number...
↳ Spacings b/w Events	Poissonian Behavior	GOE spacings	Germain, Twin Primes...	(order statistics and Benford's Law)	Zeros ⇒ Class number, races, ...

$\{n^k\alpha\}$ AND POISSONIAN BEHAVIOR

PROBLEM: Fix k and α, look at $n^k\alpha \bmod 1$ for $n \le N$

What can you say about this sequence?

↳ Dense?

Equidistributed?

Spacings?

} DO EXAMPLES WITH MATHEMATICA CODE AFTER STUDENTS CONJ

Applications: • $k=2$ arises in special Hamiltonians (see refs in RSZ)

• At some scale, see same behavior in

↳ primes

↳ waiting at banks

Needed Material: FA, P, NT

FA ① Fejér's Thm: f cont then $\forall \epsilon\ \exists N_0$ st $\forall N \ge N_0$ $|f(x) - T_N(x)| < \epsilon$

where $T_N(x)$ is the "averaged" Fourier Series: hardest input

② $e^{2\pi i x} = e^{2\pi i (x \bmod 1)}$: This is why so useful

P: ① Point masses and induced probability distr

② Order statistics for uniform rand vars

NT: ① Dirichlet's Pidgeon-hole Principle

② Approx irrationals by rationals

CHAPTER 12: $\{n^k\alpha\}$ AND POISSONIAN BEHAVIOR

Questions: Dense, Equidistr, Spacings

Notation:
- Char fn: $\chi_{(a,b)}(x)$
- Wrapped unit interval and norm $\|x-y\| = \min_{n\in\mathbb{Z}} |x-y-n|$
- probability density: $\int_{\mathbb{R}} f(x)\,dx = 1$, $f(x) \geq 0$
  - ↳ X rv with density f means $\text{Prob}(X \in [a,b]) = \int_a^b f(x)\,dx$
  - ↳ key densities: uniform, standard exponential
  - ↳ mean: $\mathbb{E}[X] = \mu = \int x f(x)\,dx$ (if exists)
    - ↳ will do variance later (Chebyshev, CLT)

① DENSENESS

Pre-reqs:
- Dirichlet's Pidgeon-hole principle (Appendix A.4)
  - ↳ ex: A.4.2: $S \subset \{1,\ldots,2n\}$ with $|S| = n+1$
    Then $\exists a,b \in S$ st $a|b$
    - ↳ proof: two elements in S have same odd part

- Exponent/Order of Approx: $\xi \in \mathbb{R}$ has approx order $\tau(\xi)$ if $\tau(\xi)$ smallest number st $\forall e > \tau(\xi)$ there are only finitely many solns to $|\xi - p/q| < 1/q^e$ (Section 5.5)
  - ↳ "many" properties in arithm dynamics governed by approx exponent

① Denseness (cont)

Thm (Kronecker): $\alpha \notin \mathbb{Q} \implies n\alpha \bmod 1$ is dense

Proof: Dirichlet's Pidgeon-hole principle

wlog enough to show $\forall \varepsilon\ \exists N$ st $0 < N\alpha \le \varepsilon$ (WHY)

$\alpha \notin \mathbb{Q} \to \exists\ \infty$ many p, q st $|\alpha - p/q| < 1/q^2$

↳ Proof: Q large, consider $\alpha \bmod 1, \ldots, (Q+1)\alpha \bmod 1$

two in same box: $|q_1\alpha - q_2\alpha - p| \le \frac{1}{Q}$

so $\left|\alpha - \frac{p}{q}\right| < \frac{1}{Qq} < \frac{1}{q^2}$ ← why $<$ not $\le$?

as $Q \to \infty$, must have ∞ many q

Thus if $1/q < \varepsilon$: $|\alpha - p/q| < \frac{1}{q^2} \to |q\alpha - p| < \frac{1}{q} < \varepsilon \ldots$

Remark: Proof aided by "linearity": can keep walking in blocks of N

Situation harder for $n^k\alpha \bmod 1$ when $k \ge 2$

As Equidistr $\implies$ Denseness, won't spend too much time here in general.

Idea for $k=2$, assuming α has approx exponent $4+\eta$ for some $\eta > 0$

↳ now given $x \in [0,1]$ and $\varepsilon > 0$, show $\exists N$ st $\|N^2\alpha - x\| < \varepsilon$

Let $\frac{1}{q} < \frac{\varepsilon}{100}$ and choose p, q st $\alpha - \frac{p}{q} = \frac{\delta}{q^4}$ $\quad 0 \le \delta \le 1$, (wlog $\delta > 0$)

Thus $(qm)^2\alpha - pqm^2 = (\delta m^2)\frac{1}{q^2}$ and can ~~assume~~ choose m st $1 \le \delta m^2 \le 4$

As $\frac{x}{\delta m^2} \in [0,1]$, choose n st $\frac{n}{q}$ is within $\frac{1}{q}$ of $\sqrt{\frac{x}{\delta m^2}}$ (and smaller)

Gives $(qmn)^2\alpha - \underbrace{pqm^2n^2}_{\text{int}} = \underbrace{x}_{\text{desired}} - \underbrace{\frac{(\delta m^2)(2n\theta + \theta^2)}{q^2}}_{< 12/q < \varepsilon}$ $\quad \theta \le 1,\ n \le q$

Aside: Algebraic Structure of Numbers

How well can an irrational be approx by rationals?

↳ Can generalize: theory of heights, ell curves, ...

Measure cost by size of denom

<u>Thm</u> (Hurwitz, Thm 7.9.4): $\forall \alpha \notin \mathbb{Q}$ $\exists \infty$ many p, q (rel prime) st $|\alpha - p/q| \leq \frac{1}{\sqrt{5}} \frac{1}{q^2}$. Taking $\alpha = \frac{1+\sqrt{5}}{2}$ we see this is sharp (ie, golden mean "most" irrational)

Comments: • Continued fraction expansions (Chapter 7)

• $\gamma = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \dots]$

• Note Dirichlet easily gives $\frac{1}{q^2}$: save $\frac{1}{\sqrt{5}}$

<u>Thm</u> (Liouville): $\alpha \in \mathbb{R}$ algebraic of deg d. Then $\tau(\alpha) \leq d$ where $\tau(\alpha)$ is the approx exponent (Thm 5.6.1)

Proof: $f(x) = \sum_{k=0}^{d} a_k (x - \alpha)^k$ and $a_0 = 0$ since $f(\alpha) = 0$

Chose p/q "close" to α, say within 1 unit

$$\left|f\left(\frac{p}{q}\right)\right| = \left|\frac{\text{int}}{q^d}\right| \leq \left|\frac{p}{q} - \alpha\right| \cdot \sum_{k=1}^{d} |a_k| \cdot 1^{k-1} \leq A \left|\frac{p}{q} - \alpha\right|$$

If $\tau(\alpha) > d$ then ∞ many p, q st $\left|\frac{p}{q} - \alpha\right| < 1/q^{d+\varepsilon}$

Get $\left|f\left(\frac{p}{q}\right)\right| = \left|\frac{\text{int}}{q^d}\right| \leq \frac{A}{q^{d+\varepsilon}}$ q large $\Rightarrow$ int $= 0$

Thus infinitely many rational roots!

<u>Corr</u>: Liouville Numbers Transcendental! (Thm 5.6.4)

↳ $\sum 10^{-m!}$ (or my generalization)

<u>Comment</u>: Roth's Thm (Thm 5.7.1, Chapter 6) that $\tau(\alpha) = 2$ if $\alpha \notin \mathbb{Q}$ is algebraic of deg $d \geq 2$.

Other: $\tau(e)$, $\tau(\pi)$, Thm 5.5.9 ("small measure" with $\tau(\alpha) \geq 2+\varepsilon$)

ASIDE (CONT)

Algebraic/Transcendental numbers: Chapter 5

↳ countable and uncountable sets

↳ Cantor's Thm ("most" transcendental, Thm 5.3.24)

↳ Continuum Hypothesis

↳ Gödel: ZF consistent $\Rightarrow$ ZF + Continuum consistent

Cohen (my math grandfather): ZF $\Rightarrow$ ZF + not Continuum!

(ie, Continuum Hyp indep standard axioms set theory)

Paradoxes: Russell (Section 5.1)

Irrationality of e easy (Thm 5.4.5) (Due to Euler) (1737)

Transcendence of e harder but "elementary" (Hermite 1873, Thm 5.4.6)

Good exercise to show $\pi^2 \notin \mathbb{Q}$ (Exercise 5.4.17)

↳ consequence: infinitely many primes! (See Exe 3.1.7, 3.3.28)

idea: $\zeta(2) = \pi^2/6 = \prod_p (1 - 1/p^2)^{-1}$

Can we get information on $\pi(x)$ from this?

<u>Some suggested exercises:</u>

5.5.2, 5.5.5, 5.6.7, 5.6.8

Harder: 5.5.6, 5.6.5, 5.6.9

Very hard: 5.5.13

② EQUIDISTRIBUTION OF $n^k\alpha$ mod 1

Will do for $k=1$, see book for general k

If $x_n = n\alpha \bmod 1 = n\alpha - [n\alpha]$, have $\mathbb{e}(x_n) = \mathbb{e}(n\alpha)$

where $\mathbb{e}(z) = e^{2\pi i z}$: See utility FA

as can "drop" the modulo 1.

THM (WEYL, THM $\begin{Bmatrix}12.3.5\\12.3.2\end{Bmatrix}$): $\alpha \notin \mathbb{Q} \to n^k\alpha \bmod 1$ is equidistr

Proof ($k=1$): Must show $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{a,b}(x_n) = b-a$

where $\chi_{a,b}$ is char fn of (a,b) and $x_n = n\alpha \bmod 1$, ~~[illegible]~~

↳ Caveat: book looks at $\{x_n\}_{n=-\infty}^{\infty}$: typo

doesn't change anything

Geometric series: $\frac{1}{N}\sum_{n=1}^{N} \mathbb{e}_m(\cancel{x_n}) = \begin{cases} 1 & m=0 \\ \frac{1}{N}\left(\frac{\mathbb{e}_m(\alpha) - \mathbb{e}_m((N+1)\alpha)}{1-\mathbb{e}_m(\alpha)}\right) & \text{otherwise} \end{cases}$

with $\mathbb{e}_m(z) = \mathbb{e}(mz) = e^{2\pi i m z}$

As $\alpha \notin \mathbb{Q}$, for m fixed have $|1-\mathbb{e}_m(\alpha)| > 0$ (this is where use $\alpha \notin \mathbb{Q}$)

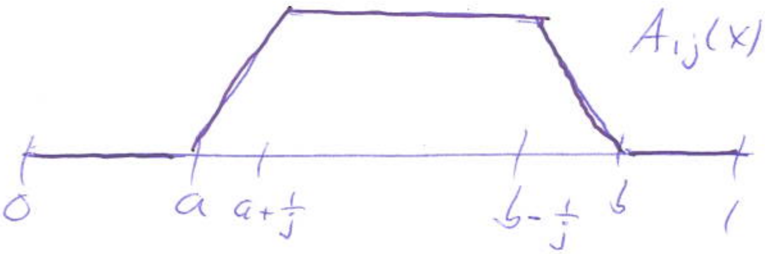
$\Rightarrow$ $P(x)$ any finite trig poly over a symmetric range, so

$P(x) = \sum_{m=-M}^{M} a_m \mathbb{e}_m(x)$, Then $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P(x_n) = a_0 = \int_0^1 P(x)\,dx$

↳ key fact: M fixed, let $C = \max_{\substack{|m|\le M\\ m\neq 0}} \frac{1}{|1-\mathbb{e}_m(\alpha)|}$

② EQUIDISTR (CONT)

Approx step fn by cont fn
Approx cont fn by trig poly } common analysis technique



$0 \le A_{1j}(x) \le \chi_{a,b}(x) \le A_{2j}(X)$

Use Fejer: For each j, given ε $\exists$ symm trig poly P_{1j}, P_{2j} st $|P_{ij}(X) - A_{ij}(x)| \le \varepsilon$.

$$\Longrightarrow \frac{1}{N}\sum_{n=1}^{N} A_{ij}(x_n) \text{ and } \frac{1}{N}\sum_{n=1}^{N} P_{ij}(x_n)$$

differ by at most ε, and know the P_{ij}-sums tend to $\int_0^1 P_{ij}(x)dx$ as $N \to \infty$.

These integrals are just $b-a \pm 1/j$ ∎

③ POISSONIAN BEHAVIOR

Probability review

- Dirac Delta Functional: "point mass": $\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$

- Seq $\{x_n\}$ induces a discrete measure

$$\mu_N(x)\,dx = \frac{1}{N}\sum_{n=1}^{N}\delta(x-x_n)\,dx$$

or $$\int_{-\infty}^{\infty} f(x)\mu_N(x)\,dx = \frac{1}{N}\sum_{n=1}^{N} f(x_n)$$

↳ will see again in RMT

BIG APPLICATION: MONTE CARLO INTEGRATION

↳ Called one of most influential papers in 20th Cent

allows approx multi-dim integration

↳ See Exercise 12.5.2

THM 12.7.3: X_n iidrv Unif(0,1) Then adj spacings (normalized) Converges to standard exponential.

Proof: "unfold" to have unit mean spacing (KEY CONCEPT)

↳ why twin primes so hard (Nicely + Pentium Bug)

Let $\{Y_n\}_{n=1}^{N}$ be $\{X_n\}_{n=1}^{N}$ in increasing order, $Z_n = N Y_n$

Study $Y_{n+1} - Y_n$ or $Z_{n+1} - Z_n$

↳ For y's: as ave spacing size $1/N$, natural to look at diff of size t/N and send $N \to \infty$

③ POISSONIAN BEHAVIOR (CONT)

By symm of wrapped around interval, all look same

(1) Calc prob all $N-1$ other X_n's are at least $\frac{t}{N}$ units to the right of X_1: $P_N(t) = \left(1 - \frac{t}{N}\right)^{N-1}$

(2) Calc prob all $N-1$ other X_n's are at least $\frac{t+\Delta t}{N}$ units to the right of X_1: $P_N(t+\Delta t) = \left(1 - \frac{t+\Delta t}{N}\right)^{N-1}$

Thus prob ~~next~~ spacing is b/w t/N and $\frac{t+\Delta t}{N}$ is

$$\left(1 - \frac{t}{N}\right)^{N-1} - \left(1 - \frac{t+\Delta t}{N}\right)^{N-1}$$

As $N \to \infty$ this goes to $e^{-t} - e^{-(t+\Delta t)}$

$$= e^{-t}\left(1 - e^{-\Delta t}\right)$$

$$= e^{-t}\Delta t + O((\Delta t)^2)$$

Thus converges to standard exponential.

(See also Remark 12.7.4)

Remark: These are order statistics, quite useful

↳ Miller-Nigrini and Benford

↳ Median (Exe 12.7.8) (and Exe 12.7.9)

↑ great project for prob/stat minded person

Call behavior of iidrv Uniform(0,1) order statistics Poissonian

③ POISSONIAN BEHAVIOR (CONT)

Section 12.8: What do we know about $\{n^k \alpha\}$?

↳ "Know" much, can "prove" only a little

• $k=1$: $\alpha \notin \mathbb{Q} \Rightarrow$ at most 3 spacings!

↳ Good Challenge Problem: Exe 12.6.3!

CONJ: $k \geq 2$, for almost all $\alpha \notin \mathbb{Q}$ in sense of measure, Then $n^k \alpha \bmod 1$ is Poissonian as $N \to \infty$.

↳ many different notions of what a "generic" element is

See Appendix A.5 for a brief introduction to measure Theory (or Lebesgue Theory).

↳ contrast with Thm 5.5.9

↳ See book: know Poissonian or not-Poissonian along some subseq depending on how well α approx by rationals (see proof of Thm 12.8.5)

GOING FURTHER

• Several research projects given in Section 12.9

• Project 12.9.4 (might still be open, should be doable) may require basic knowledge of continued fractions (Chap 7)

FOURIER ANALYSIS FOR $\{n^k \alpha\}$

Fourier analysis big subject. Will prove just enough to get to Fejér's Thm. Other main results we might need is Poisson Summation/CLT

11.1. INNER PRODUCT SPACES

$e^{ix} = \cos x + i \sin x$, infinite series; $e_n(x) = e^{2\pi i n x}$

When does $f(x)$ "=" $\sum_{n=-\infty}^{\infty} a_n e_n(x)$?

↳ What do we mean by "="? Pointwise? L^p?

↳ What are the a_n?

Doing ∞-dim lin alg: Are the $e_n(x)$ a basis (for what?)

$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \vec{w}$ or $\vec{v}^* \vec{w}$ becomes

$\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)}\, dx$

↳ $\langle f, f \rangle$ is ≥ 0 and $= 0$ iff $f \equiv 0$

$\langle f, g \rangle = \overline{\langle g, f \rangle}$

$\langle af + bg, h \rangle = a \langle f, h \rangle + b \langle g, h \rangle$

$\langle e_m(x), e_n(x) \rangle = \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & \text{other} \end{cases}$ (orthogonality relations)

Question: Why this defn of $\langle f, g \rangle$?

$f \Leftrightarrow \left(f(0), f(\frac{1}{n}), \dots, f(\frac{n-1}{n}) \right)$ and take dot product

11.2. Fourier Series

11.2.1. Intro

n^{th} Fourier coeff: $\hat{f}(n) = \langle f(x), e_n(x) \rangle = \int_0^1 f(x) e^{-2\pi i n x} dx$

- $\langle f(x) - \hat{f}(n) e_n(x), e_n(x) \rangle = 0$

- N^{th} partial Fourier Series: $S_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e_n(x)$

↳ lots of nice properties (Do Exe 11.2.2)

- $\langle f - S_N, e_n \rangle = 0 \quad \forall |n| \leq N$
- $|\hat{f}(n)| \leq \int_0^1 |f(x)| dx = \|f\|_1$

① Question: When does $S_\infty(x)$ equal $f(x)$?

↳ if $|f|$ is bounded, f period 1, f diff at x_0

Then $\lim_{N \to \infty} S_N(x_0) = f(x_0)$ (Dirichlet: Thm ~~2.3.8~~ 11.3.8)

↳ unfortunately need diff: for us just want cont (b/c want to use approx to characteristic fns)

↳ General caveat in Series Expansions!

(1) existence and uniqueness questions

↳ ex: are Taylor Series Unique?

↳ Exe A.2.7: $f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$

has all Taylor coeff zero

↳ thus radius of conv is zero!

11.2.2. APPROX TO THE IDENTITY

Seq. $\{A_n(x)\}_{n=1}^{\infty}$ is an approx to the identity if

- $A_N(x) \geq 0$
- $\int_0^1 A_N(x)\,dx = 1$ (or could do on $(-\infty, \infty)$)
- $\forall \delta \in (0, 1/2)$: $\lim_{N\to\infty} \int_\delta^{1-\delta} A_N(x)\,dx = 0$

↳ often do $\lim_{N\to\infty} \int_{|x|>\delta} A_N(x)\,dx = 0$

Give examples: Λ, normal distr, ...

Recall defn of Dirac Delta Fnal: $\int f(x)\,\delta(x-a)\,dx = f(a)$

11.2.3. DIRICHLET AND FEJÉR KERNELS

$$D_N(x) = \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x} \quad \text{Dirichlet}$$

$$F_N(x) = \frac{1}{N}\sum_{n=0}^{N} D_n(x) = \frac{\sin^2(N\pi x)}{N \sin^2 \pi x} \quad \text{Fejér}$$

KERNEL

Convolution: $(f*g)(x) = \int_0^1 f(y)\,g(x-y)\,dy = \int_0^1 f(x-y)\,g(y)\,dy$

$(T_N f)(x) = (f * F_N)(x)$

THM: Fejér kernels are an approx to the identity

Proof:
- $F_N(x) \geq 0$ clear
- $F_N(x) = e_0(x) + \frac{N-1}{N}\left(e_{-1}(x) + e_1(x)\right) + \cdots$ so $\int$ to 1.
- $\delta < x < 1-\delta$ then $|F_N(x)| \leq \frac{1}{N \sin^2 \pi\delta}$

REMARK: $S_N f(x) = f * D_N(x)$

↳ analysis much harder as D_N is not an approx to identity (see Exe 11.2.12)

11.3.1. CONVERGENCE OF FEJÉR SERIES

THM (FEJÉR): f cont, periodic on $[0,1]$. $\forall \varepsilon > 0\ \exists N_0$ st $\forall N \geq N_0$ $|f(x) - T_N f(x)| \leq \varepsilon$. Hence $\lim_{N \to \infty} T_N(x) = f(x)$

Ideas of Proof: Multiply by 1, 3 ε proof.

$$T_N(x) - f(x) = \int_0^1 f(x-y) F_N(y)\, dy - f(x) \cdot 1$$

$$= \int_0^1 [f(x-y) - f(x)] F_N(y)\, dy \quad \text{as } \int F_N(y)\, dy = 1$$

$$= \underbrace{\int_0^{\delta} + \int_{\delta}^{1-\delta} (f(x-y) - f(x)) F_N(y)\, dy}_{} + \underbrace{\int_{\delta}^{1-\delta} (f(x-y) - f(x)) F_N(y)\, dy}_{}$$

Since f cont, f unif cont (Thm A.3.7). Choose δ so small that diff b/w f values at most $\varepsilon/3$ for $0 \leq y < \delta$. Note $\int F_N(y)\, dy \leq 1$

Thm A.3.13 have f bounded. Thus this is $\leq B \cdot \int_{|y| \geq \delta}^{1-\delta} F_N(y)\, dy$. Choose N s.t. large at most $\frac{\varepsilon}{3}$

Thus $|T_N(x) - f(x)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon$

CORR: WEIERSTRASS APPROX THM (THM 11.3.4): Any cont periodic fn can be unif approx by trig poly

↳ ORIGINAL VERSION was by poly, not trig poly and done before Fejér's Thm

ASIDE: ① Dirichlet's Convergence Thm

② General f: what can be said?

↳ Kolmogorov

↳ Carleson - Fefferman

RANDOM MATRIX THEORY

PROBLEMS: Study eigenvalues of ensembles of matrices

↳ Density of eigenvalues

↳ Spacings b/w eigenvalues

APPLICATIONS: Physics: $H\Psi_n = E_n \Psi_n$

↳ sadly, H usually ∞-dim @ unknown entries!

Graph Theory: Eigenvalues Adjacency matrices

↳ building efficient, cheap networks

Number Theory: Zeros of L-fns

↳ density of primes, prime races, class numbers, ...

NEEDED MATERIAL: FA, P, LA, NT

FA: ① CLT (Thm 11.5.1), though could use Chebyshev (Exe 8.1.55)

P: ① Chebyshev's Thm (Exe 8.1.55):

② Moments of a distr: $\mu_k = \int x^k p(x)dx$

↳ when does knowing moments mean know distr?

LA: ① Eigenvalues (especially of real symm matrices)

② Diagonalization / Triangularization

③ $Tr(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$

④ $Tr(A^k) = \sum_{i=1}^{N} \lambda_i(A)^k$

NT: ① Combinatorics: Catalan numbers / Moments of Gaussian

② L-fns and primes: $\zeta(s) = \sum \frac{1}{n^s} = \prod (1-p^{-s})^{-1}$ $\zeta(1), \zeta(2)$

RANDOM MATRIX THEORY

History: 3 body problem → Uranium

Stat Mech: 

Approx H by random $N \times N$, average

↳ hope ave is good approx for "generic"

KEY INGREDIENTS:

All problems involve 3 key inputs:

① Normalization/Determining scale

↳ often CLT

② Trace Formula

↳ ex $Tr(A^k) = \sum \lambda_i(A)^k$

connect evalues (which we want to know) with matrix elements (which we do know)

③ Averaging formulas

↳ often combinatorics (hard)

↳ Trace formula useless if can't do anything with matrix coeffs

↳ ave formulas better in RMT than NT

15.1.3. Random Matrix Ensembles

p prob distr with mean 0, variance 1, finite moments

↳ $\int x p(x) dx = 0 \quad \int x^2 p(x) dx = 1 \quad \int |x|^k p(x) dx < \infty$

Consider real symm $N \times N$ matrices $A = (a_{ij})$

↳ have $\frac{N^2-N}{2} + N = \frac{N^2+N}{2}$ indep entries

$\text{Prob}(A)\, dA = \prod_{1 \le i \le j \le N} p(a_{ij})\, da_{ij}$

↳ do 2×2 example

↳ Show $\int \text{Prob}(A)\, dA = 1$ (still true if had $p_{ij}(a_{ij})$)

15.2. EIGENVALUE PRELIMINARIES

15.2.1 EIGENVALUE TRACE FORMULA

$\text{Trace}(A) = a_{11} + \cdots + a_{NN}$

$\text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} a_{ji} = \sum_{i,j=1} \sum a_{ij}^2$

$\text{Tr}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}$

Key Fact: "Most important" Thm in Lin Alg

THM 15.2.4: EIGENVALUE TRACE LEMMA: $\text{Tr}(A^k) = \sum_{i=1}^{N} \lambda_i(A)^k$

↳ Proof: $k=1$ from char poly: $\det(A - \lambda I) = 0$

$k \ge 2$: Triangularize $A \Rightarrow U^{-1} A U = T$, eigenvalues A = those of T

↳ if A real symm (our case): diagonalize

Idea behind moments: $N \times N$ matrix, know $\sum_{i=1}^{N} \lambda_i(A)^k \quad 1 \le k \le N$

Then know the $\lambda_i(A)$

15.2.2. NORMALIZATIONS

Study primes on "right" scale
↳ Nicely + Pentium bug (twin primes hard)

Claim: $\lambda_i(A)$ is of size $\sqrt{N}$

"Proof": $\mathrm{Tr}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2$

$$\mathbb{E}[\mathrm{Tr}(A^2)] = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}[a_{ij}^2]$$ as expectation's linear

Thus see it is of size $N^2 \cdot 1$, N eigenvalues...

"better": $\mathrm{Tr}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2$

↳ CLT or Chebyshev says sum is like N^2

so $\sum \lambda_i(A)^2 \sim N^2$

$\Rightarrow \langle \lambda_i(A)^2 \rangle_{ave} \sim N$ so $\langle \lambda_i(A) \rangle_{ave} \sim \sqrt{N}$

↳ can't pass $\sqrt{\ }$ into ave...

<u>Chebyshev's Thm</u>: Prob distr p with mean μ, var $\sigma^2 < \infty$. Then for random variable X with density p: $\mathrm{Prob}(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$

Note: "natural" to do $k\sigma$ as "units" of X, μ are σ

Proof: $\mathrm{Prob}(|X-\mu| \geq k\sigma) = \int_{|x-\mu| \geq k\sigma} p(x)\,dx$

$$= \int_{\left|\frac{x-\mu}{k\sigma}\right| \geq 1} \left(\frac{x-\mu}{k\sigma}\right)^2 p(x)\,dx$$

extend integral $\leq \frac{1}{k^2\sigma^2} \int (x-\mu)^2 p(x)\,dx = \frac{1}{k^2}$

ASIDE

(Note: Can also use CLT, Thm 8.4.1. Chebyshev is weaker, but applies to more densities: like divide + conquer vs Newton's method)

15.2.3. EIGENVALUE DISTRIBUTION

A is $N \times N$, eigenvalues $\lambda_i(A)$

Define $\mu_{A,N}(x)\,dx = \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right) dx$

↳ $\sqrt{N}$ is from "natural" scale, to have limit as $N \to \infty$

2 is to show off (semi-circle vs semi-ellipse)

Normalized evalues in $[a,b]$ is $\int_a^b \mu_{A,N}(x)\,dx$

Denote k^{th} moment by $\mathbb{E}[x^k]_A$ or $M_{N,k}(A)$

IMPORTANT

$$M_{N,k}(A) = \frac{\mathrm{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}$$

Proof: $M_{N,k}(A) = \int x^k \mu_{A,N}(x)\,dx$

do algebra...

15.3. SEMI-CIRCLE LAW

Idea: $A \to \mu_{A,N}(x)dx \to \{M_{N,k}(A)\}_{k=1}^{\infty}$

Study $\mathbb{E}[M_{N,k}(A)]$ (expectation over A)

on average these equal moments of semi-circle

want to claim that as $N \to \infty$ "most" $\mu_{A,N}$ are "close" to semi-circle

Caveats: • Exe 15.3.2: non-uniqueness of moments

• Remark 15.3.3: $\{a_{n,2N}\}_{n=1}^{2N}$ st half $+1$, half -1

then average 0, none close

need to control variances

Idea of Proof of Wigner's Semicircle Law when $k=2$

$$\text{Let } M_{N,2} = \int_A M_{N,k}(A)\,\text{Prob}(A)\,dA$$

$$= \frac{1}{2^2 N^{\frac{2}{2}+1}} \int_A \text{Tr}(A^2)\,\text{Prob}(A)\,dA$$

$$= \frac{1}{4N^2} \sum_{i=1}^{N}\sum_{j=1}^{N} \int_{a_{11}=-\infty}^{\infty} \cdots \int_{a_{NN}=-\infty}^{\infty} a_{ij}^2\, p(a_{11})\cdots p(a_{NN})\,da_{11}\cdots da_{NN}$$

as finite sums, $\sum\sum\int\int = \int\int\sum\sum$

integrals factor $\int a_{ij}^2 p(a_{ij})\,da_{ij} \int\cdots\int p(a_{k\ell})\,da_{k\ell} = 1$

assuming variance $=1$, mean $=0$

$$= \frac{N^2}{4N^2} = \frac{1}{4}$$

$k \geq 3$: Combinatorics

note have integrals of poly in matrix elements

CHAPTER 15: ODDS AND ENDS

Show plots of semi-circle law

↳ especially @ Cauchy distr, violating finite moments

↳ very good distr to test universalities

Mention GOE conj for spacings:

$$V_{A,N}(s)\,ds = \frac{1}{N-1}\sum_{i=2}^{N} \delta\left(s - \frac{\lambda_i(A) - \lambda_{i-1}(A)}{2\sqrt{N}}\right) ds'$$

Show pictures of spacings

Talk about d-regular graphs

ASIDE

THM 15.5.9: McKay's Law

d-regular graphs. As $N \to \infty$ $\mu_{A(G),N}(x)$ converges to

Kesten's Measure: $f(x) = \begin{cases} \frac{d}{2\pi(d^2 - x^2)}\sqrt{4(d-1) - x^2} & |x| \leq 2\sqrt{d-1} \\ 0 & \text{other} \end{cases}$

Note: as $d \to \infty$, if change scale becomes semi-circle

Conj spacings are GOE even though density of states is not semi-circle

↳ distr of spacings more "fundamental"

Number Theory: Mention zeros of L-fns

↳ GUE, Mont-Odlyzko Law

CHAPTER 16: RMT: EIGENVALUE DENSITIES

Semi-circle Law (Wigner)

p mean 0, variance 1, finite higher moment. Then as $N \to \infty$

The typical $\mu_{A,N}(x)$ converges to the semi-circle density

$$P(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

↳ Can prove @ weaker assumps

↳ Will sketch the proof using Method of Moments
(other techniques more powerful but "harder")

Note: Moments of semi-circle are $C(k) = \begin{cases} 2\frac{(2m-1)!!}{(2m+2)!!} & k=2m \\ 0 & k=2m+1 \end{cases}$

↳ Proof is change of variable: $\sqrt{1-x^2}\,dx \to \cos^2\theta\, d\theta$
finish with Mathematical Induction (Appendix A.1)

↳ $k!!$ means $k(k-2)(k-4)(k-6)\cdots$ to 2 or 1

aside: investigate combinatorial aspects
arises in Toeplitz matrices
↳ part of matching $2m$ objects in m pairs

Moment Preliminaries

$$M_{N,k}(A) = \int x^k \mu_{A,N}(x)\,dx$$
$$= \frac{1}{N}\sum\left(\frac{\lambda_i(A)}{2\sqrt{N}}\right)^k$$
$$= \frac{1}{2^k N^{\frac{k}{2}+1}}\,\mathrm{Tr}(A^k)$$

Thus need $\int \mathrm{Tr}(A^k)\,\mathrm{Prob}(A)\,dA$: as $N \to \infty$ terms $o(N^{\frac{k}{2}+1})$ negligible

Wigner's Semi-Circle Law

$$M_{N,k} = \mathbb{E}[M_{N,k}(A)]_A = \int M_{N,k}(A)\,\mathrm{Prob}(A)\,dA$$

$$= \frac{1}{2^k N^{\frac{k}{2}+1}} \int \cdots \int a_{i_1 i_2} \cdots a_{i_k i_1} \prod_{1 \le i \le j \le N} p(a_{ij})\, da_{ij}$$

note $a_{ij} = a_{ji}$ as A is real symm

Observations

Write $a_{i_1 i_2} \cdots a_{i_k i_1}$ as $a_{x_1 y_1}^{r_1} \cdots a_{x_\ell y_\ell}^{r_\ell}$
with $\{x_i, y_i\} \neq \{x_j, y_j\}$ if $i \neq j$

Then $\int \cdots \int a_{i_1 i_2} \cdot a_{i_k i_1} \mathrm{Prob}(A)\,dA$

$$= \int \cdots \int a_{x_1 y_1}^{r_1} \cdots a_{x_\ell y_\ell}^{r_\ell} \,\mathrm{Prob}(A)\,dA$$

$$= P_{r_1} \cdots P_{r_\ell} \quad \text{where } P_r = \int x^r p(x)\,dx$$

Exercise: do first three moments / read in book

Lemma: If an a-tuple has an $r_j = 1$ then it vanishes

Proof: $P_1 = 0$

Thus each $r_j \geq 2$, and everything is at least paired.
Will see later no contribution if a triple or more.

Do third moment: triple gives $N p_3$, divide by $N^{5/2}$ and
thus negligible in the limit.

Wigner's Semicircle Law (cont)

Lemma 16.1.12: $M_{N;k} = 0$ for k odd

Proof: Trivial if density p is even: $p(x) = p(-x)$
↳ odd moments of p vanish
k odd means in $P_{r_1} \cdots P_{r_\ell}$ at least one r is odd

Counting argument for general P
↳ dividing by $N^{\frac{k}{2}+1} = N^{m+3/2}$ if $k = 2m+1$
proof completed by showing $O(N^{m+1})$ matchings
where no ~~a_{ij}~~ a-term by itself (no P_1).
Thus in our $a_{x_1y_1}^{r_1} \cdots a_{x_\ell y_\ell}^{r_\ell}$ each $r \geq 2$ and one is odd (and thus ≥ 3)
have $k = 2m+1$ indices initially free
↳ how many result in matchings in pairs?...
As higher moments bounded, each term at most B_k

Aside on Combinatorics

Don't really need

Suffices to note the number of matchings depends on $k = 2m+1$ and NOT on N.

Harder: # of solns to $r_1^2 + \cdots + r_\ell^2 = 2m+1$ Leads to Circle Method

Combinatorics: $r_1 + \cdots + r_\ell = 2m+1$
each $r_i \geq 2$, one is ≥ 3 (as odd)

↳ Soln: Cookie Problem (Pages 11-13)
10 cookies, 5 people $\rightarrow \binom{10+5-1}{5-1}$

all 2's and a 3 for r_j gives $\ell = m$
Could do $r_1 = 2m+1$ and $\ell = 1$
Fixed ℓ count/solve $r_1 + \cdots + r_\ell = 2m+1$
↳ so $\tilde{r}_1 + \cdots + \tilde{r}_\ell = 2(m-\ell)$
number solns is $\binom{2(m-\ell)+\ell-1}{\ell-1} = \binom{2m-\ell-1}{\ell-1}$

Sum over $\ell = 1$ to m
↳ all that matters is answer depends ONLY on $k = 2m+1$ and NOT on N.

Wigner's Semicircle Law (cont)

↳ some number (depending on $k = 2m+1$) matchings, say $\mathcal{M}_k$

Initially have $N^k = N^{2m+1}$ free variables, $i_1, \ldots, i_k$

Every time an $a_{i_j i_{j+1}}$ is paired with an $a_{i_n i_{n+1}}$ lose at least one degree of freedom and gain at most a factor of 2

↳ if adjacent: $a_{i_j i_{j+1}} — a_{i_{j+1} i_{j+2}}$

Then $i_{j+1} = i_j$

↳ if not adjacent

a_{i_{n-1}, i_n}

$a_{i_j, i_{j+1}}$ — $a_{i_n, i_{n+1}}$ $\begin{cases} i_n = i_j \text{ or } i_{j+1} \\ i_{n+1} = i_{j+1} \text{ or } i_j \end{cases}$

$a_{i_{n+1}, i_{n+2}}$

Thus lose the freedom to choose i_n, i_{n+1} but can "flip" and so have a factor of 2.

This number of "free" variables at most $\frac{k}{2} = m + \frac{1}{2}$ (is at most m as an integer and have a triple) but can multiply by $2^{k/2}$ (or 2^m) from "flips"

Thus $$|M_{N,k}| \le \cancel{B_k} \cdot \underbrace{\mathcal{M}_k}_{\#\text{ matchings}} \cdot \underbrace{B_k}_{\text{max contr}} \cdot 2^{k/2} \cdot \underbrace{N^{m+1/2}}_{\text{choices for free vars}} \Big/ 2^k N^{\frac{k}{2}+1}$$

$$\le \frac{\mathcal{M}_k B_k}{2^{k/2}} \cdot \frac{1}{N} \xrightarrow[N\to\infty]{} 0$$

WIGNER'S SEMICIRCLE LAW

Lemma 16.1.14: k even, as $N \to \infty$ no contribution unless all $r = 2$

Main term thus $k = 2m$ and all $r_j = 2$

$$M_{N,2m} = \frac{1}{2^m N^{1+m}} \sum_{1 \le i_1 \le \cdots \le i_{2m} \le N}^{*} 1$$

where $*$ means restrict to $(i_1, \ldots, i_{2m})$ matched in pairs

ASIDE

As this is a main term, estimates are not enough

Must do combinatorics

The soln involves the Catalan numbers $C_k = \frac{1}{k+1}\binom{2k}{k}$

↳ see [Leh] for details

Non-Semi-circle Behavior

① Band matrices (especially diagonal!)

② Toeplitz and Palindromic Toeplitz (student papers)

↳ 4th moment first "real" moment for ~~sym~~ even density

↳ Stories of results

③ Truncated Cauchy

ASIDE: Lots of Research Projects

ASIDE

Just scratched the surface on rich, fascinating subject

Combinatorics essential: lots of times "believe" two expressions are equal but need the "aha" inspiration to count.

↳ ex: telescoping series

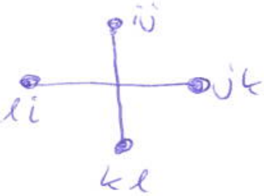
ex: matching coeff: $\sum_{k=0}^{n} \binom{n}{k}^2$ from $(x+y)^{2n}$

Real Symmetric (Palindromic) Toeplitz Matrices

↳ can view probabilistically or number theoretically

↳ I prefer the latter and see Diophantine Eqs

Comes down to counting sol'ns

4th Moment 

$\Rightarrow \frac{1}{N^3} \sum_{\substack{i,j,k,l=1 \\ i=j+l-k}}^{N} \cdots \sum 1$

↳ number of tuples is $\frac{2}{3}N^3 + \frac{1}{3}N$

↳ need value, though easy to see at most $\left(1-\frac{1}{27}\right)N^3$

↳ each index in $\{1,\ldots,N\}$: trouble if $j,l \geq \frac{2}{3}N$, $k < \frac{1}{3}N$

Chapter 17: Spacings! GOE

↳ more probability, elementary diff eq: $y' = ay$

linear alg (orthog rotations)

↳ lots of open problems

↳ Key idea behind GOE: nature doesn't care which axes choose

$P(A) = \prod_{i \leq j} p(a_{ij})$ and $P(QAQ^T) = P(A)$ $\forall$ orthog Q

↳ rotate by ε and then coeff of ε term must vanish

THE CIRCLE METHOD

PROBLEMS:

- Waring: Is there an s st all pos ints (or all suff large pos ints) are a sum of at most s k-th powers?
  ↳ $x_1^k + \cdots + x_s^k = n$
- Goldbach: $P_1 + P_2 + P_3 = N$
  $P_1 + P_2 = N$
- Germain/Twin: P st $P, \frac{P-1}{2}$ prime or $P, P+2$ prime

How to solve? Especially Goldbach

↳ primes are multiplicative, these sumset questions

NEEDED INPUT: Number Theory

See section 13.2.6

PRIME NUMBER THM (PNT): $\sum_{p \le x} \log p = x + O(x \exp(-c\sqrt{\log x}))$

for some $c>1$. By Partial Sum, $\sum_{p \le x} 1 = Li(x) + O(x \exp(-\frac{c\sqrt{\log x}}{2}))$

↳ See Section 2.2.2

↳ $Li(x) \approx \frac{x}{\log x}$

SIEGEL-WALFISZ: $\sum_{\substack{p \le x \\ p \equiv a(q)}} \log p = \frac{x}{\varphi(q)} + O\left(\frac{x}{\log^C x}\right)$ $\forall q \le \log^B x$

$\forall C, B > 0$ and $(a,q) = 1$

↳ wish could do $q \le x^\delta$

↳ Main term >> error term if C suff large

↳ $\varphi(q)$ Euler's φ-function: See Section 2.1

Need basic relation from Fourier Analysis:

$$\int_0^1 e(nx)\, e(-mx)\, dx = \begin{cases} 1 & n = m \\ 0 & \text{otherwise} \end{cases}$$

ASIDE: PNT and SIEGEL–WALFISZ

Counting Primes

① Euclid: $P_1 \cdots P_n + 1$

↳ investigate on OEIS where q_ℓ is ℓ^{th} prime generated by this

↳ can generalize to some but not all arithm prog
$p \equiv 4n+1$ no $p \equiv 4n+3$ yes (see Exe 2.3.5)

② See "Proofs from THE BOOK"

↳ Two favorites: $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = \prod_{p \text{ prime}} (1 - 1/p^s)^{-1}$ $Re(s) > 1$

$\lim_{s \to 1^+} \zeta(s) = +\infty \Rightarrow \infty$ many primes

$\zeta(2) = \pi^2/6$ and $\pi^2 \notin \mathbb{Q} \Rightarrow \infty$ many primes (See also § 11.3.4)

③ Chebyshev's Thm (Thm 2.3.9) $\exists A, B$ with $0 < A < 1 < B < \infty$ st

$\forall x \geq 30 \quad \frac{Ax}{\log x} \leq \pi(x) \leq \frac{Bx}{\log x}$

↳ many proofs, one in book uses nice approx technique and telescoping sums

↳ Corr: Bertrand's Postulate: $\forall n \geq 1 \; \exists p \in [n, 2n]$!

Questions on primes in short gaps

Good Exercise: Exe 2.3.15 (Prime Deserts)

↳ also Exe 2.3.16, 2.3.17

④ Dirichlet's Thm (Section 3.3): $(a,m) = 1$. Then

$\pi_{m,a}(x) = \frac{1}{\varphi(m)} \frac{x}{\log x} + O_a\left(\frac{1}{\varphi(m)} \frac{x}{\log x}\right)$

↳ Siegel–Walfisz is controlling error

THE CIRCLE METHOD

ORIGINS / BASIC PROBLEMS

"Cookie Problem": N cookies, P people: $\binom{N+P-1}{P-1}$

↳ equiv to $x_1 + \cdots + x_P = N$

(note: easy add conds such as $x_i \geq a_i$)

↳ solving by <u>combinatorics</u>; doesn't generalize to higher powers such as $x_1^k + \cdots + x_s^k = N$

Generating Functions

↳ look at $x_1 + \cdots + x_s = N$ again

$$f(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$$

$$f(x)^s = \left(\sum_{m_1} x^{m_1}\right) \cdots \left(\sum_{m_s} x^{m_s}\right) = \sum_{n=0}^{\infty} r_{1,s}(n) x^n$$

↳ Thus $r_{1,s}(n)$ is answer, but how to recover?

<u>BUT</u> $f(x)^s = \left(\frac{1}{1-x}\right)^s = \frac{1}{(s-1)!} \frac{d^{s-1}}{dx^{s-1}} \frac{1}{1-x}$

$$\Rightarrow f(x)^s = \frac{1}{(s-1)!} \frac{d^{s-1}}{dx^{s-1}} \left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} x^n$$

$$\Rightarrow r_{1,s}(n) = \binom{n+s-1}{s-1}$$

<u>WARING</u>: $f_{k,N}(x) = \sum_{n=0}^{N} x^{n^k}$, $f_{k,N}^s(x) = \cdots$

<u>Goldbach</u>: $f_N(x) = \sum_{p \leq N} e(px)$, $f_N^s(x) = \cdots$ $\quad e(u) = e^{2\pi i u}$

↳ technically easier to do <u>finite</u> sums and <u>exponentials</u>

↳ get say $r_{N,s}(m) = r_s(m)$ if $m \leq N$

> <u>ASIDE</u>: Differentiating identities POWERFUL technique, especially for calculating moments in prob / generating identities. See handout

THE CIRCLE METHOD

Write $n = p_1 + \cdots + p_s$

$$F_N(x) = \sum_{p \leq N} \log p \cdot e(px)$$

↳ $\log p$ is to simplify some sums
technically easier to count primes @ log weight

ASIDE: Section 2.3.4 and 3.2.2 for
why we count primes with $\log p$ weight

$$F_N^s(x) = \sum_{m=0}^{\infty} R_{N,s}(m)\, e(mx)$$

where $R_{N,s}(m)$ is # ways write m as a sum
of s primes at most N. Note $R_{N,s}(m) = R_s(m)$
for $m \leq N$

Basic Fourier analysis

$$R_{N,s}(n) = \langle F_N^s, e(-nx) \rangle = \int_0^1 F_N^s(x)\, e(-nx)\, dx$$

↳ Note: we have "solved" our problem!
Unfortunately, not very illuminating

Goal: Show $R_{N,s}(N) > 1$ → at least one way

↳ amazing that can often only prove one item exists
by showing ∞ many do
↳ ex: general a, q rel prime: to show $\exists p \equiv a(q)$
with p prime requires proving Dirichlet's
Thm ($\pi_{a,q}(x) \sim \frac{1}{\varphi(q)} \pi(x)$)

THE CIRCLE METHOD: SIZE OF ~~ALL~~ $F_N(X)$

$$F_N(x) = \sum_{p \le N} \log p \cdot e(px)$$

<u>Trivial</u>: $|F_N(x)| \le \sum_{p \le N} \log p = N + o(N)$

<u>Average</u>: On average $|F_N(x)|^2$ is like $N \log N + o(N \log N)$

↳ <u>Proof</u>: VERY IMPORTANT TECHNIQUE
works for L^2 norm, not others...

$$\int_0^1 |F_N(x)|^2 dx = \int_0^1 F_N(x) F_N(-x)\, dx \dots$$

Gives $\sum_{p \le N} \log^2 p = N \log N + o(N \log N)$

↳ could see by $p \le \frac{N}{\log^2 N}$ and $\frac{N}{\log^2 N} \le p \le N$

<u>ASIDE</u>: Philosophy of Square root cancellation

<u>Special Values</u>: $F_N(0) = F_N(1) = N + o(N)$

$F_N(1/2) = -N + o(N)$

↳ <u>much</u> larger than ave value

<u>Idea</u>: • F_N "large" at x near a/q with q "small"
and F_N "small" otherwise

• Split $\int_0^1 F_N(x)\, e(-nx)\, dx$ into two integrals,
one where F_N "large", other "small"

↳ technicalities galore!

THE CIRCLE METHOD: SIZE OF $F_N(X)$ (CONT)

<u>THM</u>: Fix B, set $Q = \log^B N$, take a, q rel prime with $q \le Q$ then $\forall C$ have
$$F_N\left(\frac{a}{q}\right) = \frac{N}{\varphi(q)} \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{ar}{q}\right) + O\left(\frac{N}{\log^{C-B} N}\right)$$

Proof: $F_N\left(\frac{a}{q}\right) = \sum_{p \le N} \log p \cdot e\left(p\frac{a}{q}\right)$

only depends on $p \bmod q$

$$= \sum_{r=1}^{q} e\left(\frac{ar}{q}\right) \sum_{\substack{p \equiv r(q) \\ p \le N}} \log p$$

trivial estimation + Siegel–Walfisz

<u>MAJOR ARCS</u>: $\mathcal{M} = \bigcup_{q=1}^{Q} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{M}_{a,q}$

where $\mathcal{M}_{a,q} = \left\{ x \in [0,1] : \left|x - \frac{a}{q}\right| < \frac{Q}{N} \right\}$

and $\mathcal{M}_{1,1} = \left[0, \frac{Q}{N}\right) \cup \left(1 - \frac{Q}{N}, 1\right]$

<u>MINOR ARCS</u>: $m = [0,1] - \mathcal{M}$

Trivially have $|\mathcal{M}| \le \sum_{q=1}^{Q} q \cdot \frac{2Q}{N} \le \frac{2Q^3}{N}$

as $Q = \log^B N$, see $\lim_{N\to\infty} |\mathcal{M}| = 0$, $\lim_{N\to\infty} |m| = 1$

↳ Discuss terminology for major/minor arc

13.3.5. The Major Arcs and the Singular Series

Major Arc Contribution (Heuristic)

lets do $s=3$ and $m=N$ (ie, $p_1+p_2+p_3=N$)

$$\int_0^1 F_N(x)^3 \, e(-Nx)\,dx = \sum_{\substack{p_1,p_2,p_3 \le N \\ p_1+p_2+p_3=N}} \cdots \sum \log p_1 \log p_2 \log p_3$$

Major arcs

$$\int_{\mathcal{M}} F_N(x)^3 e(-Nx)\,dx = \sum_{q=1}^{Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{a,q}} F_N(x)^3 e(-Nx)\,dx$$

↳ estimate $F_N(x)$ by $F_N(\frac{a}{q})$

ignore all lower order terms, note $|\mathcal{M}_{a,q}| = 2Q/N$

yields $$\frac{2Q}{N} \sum_{q=1}^{Q} \sum_{(a,q)=1} \left(\frac{N}{\varphi(q)} \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{ar}{q}\right) \right)^3 e\left(-\frac{Na}{q}\right)$$

$$= \mathfrak{S}(N) \cdot \left(N^2 \cdot 2Q\right) \quad \text{with } Q = \log^B N$$

where $$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{1}{\varphi(q)^3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{ar}{q}\right) \right)^3 e\left(-N\frac{a}{q}\right)$$

↳ More careful analysis (see Chap 14) gives $\mathfrak{S}(N) \cdot \frac{N^2}{2}$

Comments: ① Close: correct power of N, missed by a few logarithms which could matter

② not surprising, as doing a zeroth order Taylor Expansion.

↳ surprisingly, do much better (ie, easier) NOT to Taylor Expand $F_N(x)$ but to find a fn $U_N(x)$ st $U_N(a/q) = F_N(a/q)$ and U_N easily integrated

③ Main term is from this (we hope!), but tough to look at $\mathfrak{S}(N)$ and "see" its behavior: ie: $\mathfrak{S}(1776) = 0$!

THE SINGULAR SERIES

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{1}{\varphi(q)^3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{ar}{q}\right) \right)^3 e\left(-N\frac{a}{q}\right)$$

Using methods of Chapter 14:

$$\mathfrak{S}(N) = \prod_p \left(1 - \frac{c_p(N)}{\varphi(p)^3}\right) \quad \text{with} \quad c_p(N) = \begin{cases} p-1 & \text{if } p \mid N \\ 0 & \text{otherwise} \end{cases}$$

↳ very important

Says $\mathfrak{S}(N)$ is a multiplicative function (see Defn 2.1.2)

view as $\mathfrak{S}(N) = \prod_p \delta_p(N)$, each $\delta_p(N)$ a

"local density", measuring obstructions

↳ if $2 \mid N$ then $\delta_2(N) = 0$ and main term $\mathfrak{S}(N)\frac{N^2}{2} = 0$!

↳ What does this mean?

Circle Method knows Goldbach ($p_1 + p_2 = 2m$) hard

Say $p_1 + p_2 + p_3 = 2N = 2m+2$

Then wlog $p_3 = 2$ and writing $2m$ as sum of two primes!

ASIDE ON OBSTRUCTIONS: Hasse Principle, Section 4.4

Exe 13.3.16: Important

N odd $\to \exists\, c_1, c_2 > 0$ (and indep of N) st $c_1 < \mathfrak{S}(N) < c_2$

Minor Arcs Contribution

Know $|F_N(x)|$, on average, is about $\sqrt{N \log N}$ and at most N

Can be as large as order N near a/q with $q \ll N$.

Vinogradov: $\max_{x \in m} |F_N(x)| \ll \frac{N}{\log^D N}$ $D = D(B,C) > 1$

$$\left| \int_m F_N^3(x) e(-Nx)\, dx \right| \le \int_m |F_N(x)|^2 \cdot |F_N(x)|\, dx$$

$$\le \max_{x \in m} |F_N(x)| \cdot \int_m |F_N(x)|^2\, dx$$

$$\ll \frac{N}{\log^D N} \cdot \int_0^1 |F_N(x)|^2\, dx$$

$$= \frac{N}{\log^D N} \cdot N \log N = \frac{N^2}{\log^{D-1} N}$$

Thus a "small" cancellation is enough to win when $s = 3$

Aside: Very common/important technique
easy to evaluate $\int_0^1 |F_N(x)|^{2k} dx$, and
need slight savings in an $|F_N(x)|$ for $x \in m$

WHY IS GOLDBACH HARD?

Cauchy-Schwarz (ver 6): $\left|\int_0^1 fg\right| \le \left(\int_0^1 f^2\right)^{\frac{1}{2}} \left(\int_0^1 g^2\right)^{\frac{1}{2}}$

↳ Proof: Lemma A.6.9

When $s=2$ ($p_1+p_2=N$) Major arcs give $\sim \mathfrak{S}_2(N)\cdot N$

Estimate minor arcs: insert abs values (lose <u>all</u> oscillation)

↳ $\int_m |F_N(x)|^2 dx \sim N \log N$ too big

↳ Pull out $|F_N(x)|$: $\max_{x\in m}|F_N(x)| \cdot \int_m^* |F_N(x)| dx$

↳ real bad: even if $|F_N(x)|$ is always its average value, $\sqrt{N\log N}$ we get at least $N \log N$ again

↳ what if "most" of the time $|F_N(x)|$ small?

Using Cauchy-Schwarz:

$$\int_m |F_N(x)| dx \le \left(\int_m |F_N(x)|^2 dx\right)^{1/2} \left(\int_m 1^2 dx\right)^{1/2}$$

$$\le (N \log N)^{1/2} \cdot 1$$

(note: applying Cauchy-Schwarz good on finite intervals)

↳ still need more than average value cancellation!

<u>ASIDE</u>: Littlewood Problem (includes results of Paul Cohen)

<u>Remark</u>: Is Goldbach hopeless?

↳ This is too crude of a method: moving absolute value inside the integration loses <u>all</u> oscillation.

<u>ASIDE</u>: Research projects / detailed calculations in Chapter 14 for Germain Primes

BENFORD'S LAW OF DIGIT BIAS

PROBLEM: Given a sequence of data or some random variables, determine the distribution of the first (or leading) digits.

↳ data $\{a_n\}_{n=1}^{\infty}$ calculate (if it exists!)

$$\lim_{N\to\infty} \frac{\#\{n \le N : \text{first digit of } a_n \text{ is } d\}}{N}$$

Applications: ① leads to some good, interesting math questions

↳ ex: quantified equidistribution in $\{n\alpha \bmod 1\}$

② data integrity / fraud detection (IRS story)

NEEDED INPUT: FOURIER ANALYSIS + NUMBER THEORY

Amount needed depends on application.

Number Theory: Equidistribution of $n\alpha$

Fourier Analysis: • Fejér's Thm and a generalization:

↳ Lebesgue's Thm: $f \in L^1 \to T_N f$ converges in L^1 to f

• Poisson Summation: $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$

Then f nice $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$

↳ Why useful? Can convert a slowly converging sum to a rapidly converging one, and often small error in replacing with $\hat{f}(0)$

BENFORD'S LAW

Newcomb (1880s), Benford (1930s)

Many processes: Prob(1st digit d) $= \log_{10}\left(\frac{d+1}{d}\right)$

↳ check prob distr by adding and see sums to 1

More generally: $X>0$ write $X = M_{10}(X) \cdot 10^k$ $k \in \mathbb{Z}$

Then $\text{Prob}(M_{10}(X) \le s) = \log_{10} s$

Not all seqs are Benford: some don't even have a limit!

↳ ex: 1, 2, 3, ... : first dig freq oscillates b/w 1/9 and 5/9

<u>ASIDE</u>: Analytic Density can help for questions like this

Ex: First digit of primes (Serre's comment about Bombieri): $\text{Density}_{\text{analytic}}(A) = \lim_{s \to 1} \frac{\sum_{n \in A} 1/n^s}{\sum_{n=1}^{\infty} 1/n^s}$

<u>Examples</u>: Fibonacci, Recurrence relations, <u>L-fns and RMT</u>, financial data (IRS!), hydrology data ....

Also $3X+1$ (Tell IRS Story, writing @ AlexK..., Hawaii)

↳ Erdös + Kakutani quotes

$$a_{n+1} = \begin{cases} 3a_n + 1 & a_n \text{ odd} \\ a_n/2 & a_n \text{ even} \end{cases} \quad \text{or } a_{n+1} = \frac{3a_n+1}{2^k}$$

where $2^k \| 3a_n + 1$

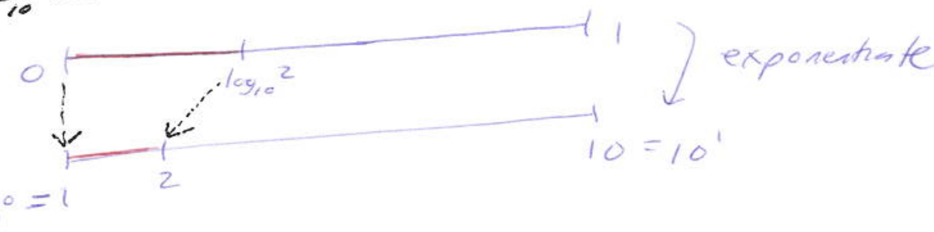
RECURRENCE RELATIONS + BENFORD'S LAW

Why is $\alpha \notin \mathbb{Q} \Rightarrow n\alpha \bmod 1$ so useful?

- 10^u and 10^v same first digits iff $u \equiv v \bmod 1$
  - Thus instead of studying seq $\{X_n\}_{n=1}^{\infty}$ study $\{Y_n\}_{n=1}^{\infty}$ with $Y_n = \log_{10} X_n \bmod 1$
    - Note first digit of X_n is d iff $M_{10}(X_n) \in [d, d+1)$ ie, iff $Y_n \in [\log_{10} d, \log_{10}(d+1))$

<u>Thm (9.2.4)</u>: If $\log_{10} X_n \bmod 1$ is equidistr then X_n is Benford

Proof: Calculation:



<u>Example</u>: $\log_{10} r \notin \mathbb{Q} \Rightarrow X_n = ar^n$ is Benford

Proof: $\log_{10} X_n = \underbrace{n \log_{10} r}_{\text{equidistr}} + \underbrace{\log_{10} a}_{\text{constant}}$

<u>Example</u>: Fibonacci numbers are Benford

- "Many" recurrence relations Benford (Thm 9.3.1)

  $a_{n+k} = c_1 a_{n+k-1} + \dots + c_k a_n$ with $c_1, \dots, c_k$ fixed constants

  (ex: $a_{n+1} = a_n + a_{n-1}$: Fibonacci)
  - solve by divine inspiration: $a_n = u_1 \lambda_1^n + \dots + u_k \lambda_k^n$
    - Binet's formula: $n-1^{st}$ Fibonacci is $\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

      note n large main term dominates: book-keeping

<u>ASIDE</u>: Theory of difference/differential eqs

ASIDE

① Values of L-fns, RMT
② 3X+1: need {quantified equidistribution, irrationality exponent} } Poisson Sum
③ Mod 1 CLT: need Fejér/Lebesgue

↳ Many noticed amalgamated data more Benford than constituents

Analysis of products of random variables involves an enormous amount of prob and Fourier Analysis

④ Order Statistics

Other asides

- Proof of quantified equidistribution: Erdös-Turan, irrationality exponent plays huge role.

- 3X+1 problem
  ↳ what is known...
  heuristics: $\mathbb{E}[\log a_{n+1}]$ "$=$" $\sum_{k=1}^{\infty} \frac{1}{2^k} \log\left(\frac{3a_n}{2^k}\right)$
  ↳ This involves differentiating an identity as must evaluate $\sum_{k=1}^{\infty} \frac{k}{2^k}$

THE RIEMANN ZETA FN $\zeta(s)$

Problems: $\zeta(s) = \sum_n 1/n^s = \prod_p (1-p^{-s})^{-1}$ $Re(s)>1$

↳ extend to all s

↳ understand values / zeros

↳ generalize to $\sum_n a_n/n^s = \prod_p L_p(s)$

↳ many choices of a_n lead to interesting connections

↳ local ↔ global interplay

↳ examples: Dirichlet characters (primes in arithm progression, class number)

Elliptic Curves...

NEEDED INPUT: (FA, CA, NT/Alg)

Fourier Analysis: ① Fourier Transform

② Poisson Summation

Complex Analysis: ① terminology (zero, pole, residue, meromorphic cont, ...)

② logarithmic derivatives

③ Cauchy Residue Formula

↳ converts integrals to algebra (Taylor Series)

$$\frac{1}{2\pi i}\oint_\gamma f(z)\,dz = \sum_{z\in A} \mathrm{Res}_z(f) \quad \partial A = \gamma$$

Number Theory: ① Basic group theory etc. for Dirichlet L-fns

② Elliptic curves for elliptic curve L-fns..

ANALYTIC / MEROMORPHIC CONTINUATION OF $\zeta(s)$

Gamma Fn: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \quad Re(s) > 0$

↳ $\Gamma(s+1) = s\Gamma(s)$ for $Re(s) > 1$ ($\int$ by parts)

↳ This is the fnl eq of Γ, allows us to extend to $s \in \mathbb{C}$

↳ corr: $\Gamma(n+1) = n!$ (note $0! = 1$)

↳ aside: $\Gamma(1/2) = \sqrt{\pi}$

↳ occurs in prob (normalization constant of Gaussian)

ASIDE: Prove properties of $\Gamma(s)$, investigate occurances

THM 3.1.19 (Riemann): Analytic Cont of $\zeta(s)$

Define completed zeta function by $\xi(s) = \frac{1}{2} s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$

Then, though originally defined for $Re(s) > 1$, extends to all $s \in \mathbb{C}$ with only a simple pole of residue 1 at $s = 1$, and $\xi(s) = \xi(1-s)$

Proof: $\int_0^\infty x^{\frac{1}{2}s-1} e^{-n^2\pi x} dx = \frac{\Gamma(s/2)}{n^s \pi^{s/2}}$ (defn Γ-fn)

Sum over n with $Re(s) > 1$ so converge and can have $\sum \int = \int \sum$

(See Exe 11.4.12 for example where cannot interchange $\begin{matrix} & \uparrow & +1 \\ +1 & -1 & \\ -1 & & \rightarrow \end{matrix}$)

Setting $\omega(x) = \sum_{n=1}^\infty e^{-n^2\pi x}$, $\theta(x) = \sum_{n=-\infty}^\infty e^{-n^2\pi x}$, $\omega(x) = \frac{\theta(x)-1}{2}$

have $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1}\omega(x)dx$

$= \int_1^\infty x^{\frac{1}{2}s-1}\omega(x)dx + \int_1^\infty x^{-\frac{1}{2}s-1}\omega(\frac{1}{x})dx$

↳ Common technique: $\int_0^\infty = \int_1^\infty + \int_0^1$ and in $\int_0^1$ send $x \to x^{-1}$

Poisson Sum gives $\omega(\frac{1}{x}) = -\frac{1}{2} - \frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$

$\Rightarrow \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-1})\omega(x)dx$

↳ as ω decays RAPIDLY, integral makes sense for all s

↳ Fnl eq for $\omega(x)$ from applying Poisson Sum to show $x^{\frac{1}{2}}\theta(x) = \theta(x^{-1}) = \sum_{n=-\infty}^\infty e^{-\pi n^2 x^{-1}}$ $(x > 0)$

3.2. Zeros of $\zeta(s)$ and Primes

Heuristic: why zeros give info on primes?

↳ Polynomials: $P(X) = A(X-r_1)\cdots(X-r_n)$
$$= A\left(X^n + a_{n-1}(r_1,\dots,r_n)X^{n-1} + \dots + a_0(r_1,\dots,r_n)\right)$$

Thus know zeros know coeffs and vice versa

<u>ASIDE</u>: Relations b/w roots and coeffs
Newton's identities + symmetric polynomials

Von Mangoldt fn: $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$

Set $\psi(x) = \sum_{n \le x} \Lambda(n)$

Complex Analysis Pre-Reqs

- Euler / Cotes / de Moivre: $e^{i\theta} = \cos\theta + i\sin\theta$
- $\frac{1}{2\pi i}\int_0^{2\pi} (re^{i\theta})^n re^{i\theta} i\, d\theta = \begin{cases} 1 & n=-1 \\ 0 & \text{otherwise} \end{cases}$
- $\frac{1}{2\pi i}\oint_\gamma z^n dz = \begin{cases} 1 & n=-1 \\ 0 & \text{otherwise} \end{cases}$ γ circle containing origin
- $\frac{1}{2\pi i}\int_{|z-z_0|=r} (z-z_0)^n dz = \begin{cases} 1 & n=-1 \\ 0 & \text{other} \end{cases}$
- $f(z)$ mero, for suff small r at most one pole in $D(z_0, r)$, namely z_0

Then $\frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)dz = a_{-1} = \text{Res}_{z_0}(f)$

↳ Proof: justify $\int\sum = \sum\int$ with $f(z) = \sum_{n \ge n_0} a_n (z-z_0)^n$

↳ More generally, f meromorphic in A, $\frac{1}{2\pi i}\int_{\partial A} f(z)dz = \sum_{z \in A} \text{Res}_z(f)$

↳ Similar to Green's thm 

ZEROS OF $\zeta(s)$ AND PRIMES (CONT)

RESULTS FROM COMPLEX ANALYSIS (CONT)

- logarithmic derivative: $\frac{d}{ds} f(z) = f'(z)/f(z)$
  - ↳ VERY useful as converts a prod to sum (we like sums)
  - ↳ $\frac{1}{2\pi i} \oint_{|z-s|=r} \frac{f'(z)}{f(z)} dz = \sum_{n=1}^{N} \operatorname{ord}_f(z_n)$
    - ↳ $f(z) = \sum_{m \geq n} a_m (z-z_0)^m \Rightarrow \frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + h(z)$

      where $n = \operatorname{ord}_f(z_0)$ and $h(z) = \sum_{m=0}^{\infty} b_m (z-z_0)^m$

ASIDE: Prove results from Complex analysis

- $\oint \frac{(x/n)^s}{s} ds$ is basically $\begin{cases} 1 & n < x \\ 0 & \text{otherwise} \end{cases}$ (See Exe 3.2.32)

Exe 3.2.15: $\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$

$$\Rightarrow \oint \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = -\oint \sum_n \frac{\Lambda(n)}{n^s} \frac{x^s}{s} ds$$

|| shifting contours ||

$$x - \sum_{\rho : \zeta(\rho)=0} \frac{x^\rho}{\rho} = \sum_{n \leq x} \Lambda(n)$$

↳ See importance of Riemann Hypothesis: non-trivial ρ has $\operatorname{Re}(\rho) = \frac{1}{2}$

↳ makes error as "small" as possible ($\zeta(s) = \zeta(1-s)$)

↳ Can interpret as a CLT/square-root cancellation

↳ ASIDE: Hawkins primes and now RH is the CLT (though no Euler Product)

↳ PNT is "basically" $\operatorname{Re}(\rho) < 1$: see Exe 3.2.19 for proof

$$\boxed{\begin{array}{l}\text{ASIDE: Product expansion, zeros on line, Li's constants + RH} \\ \text{Dirichlet characters/L-fns, Primes in arithm progression}\end{array}}$$

Fourier Analysis for Poisson Summation

THM 11.4.6 (Poisson Summation)

Assume f is twice differentiable and f, f', f'' decay like $x^{-(1+\eta)}$ for some $\eta > 0$ (means $\exists x_0, C$ st $\forall |x| > x_0$, $|g(x)| \le C|x|^{-(1+\eta)}$)

Then $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$ is Fourier Transform of f

↳ aside: "many" normalizations for Fourier Transform. This is the "correct" one as makes Poisson Sum nice

↳ aside: do not need many properties of Fourier Transform.

<u>ASIDE</u>: For more, see Sections 11.4.1, 11.4.3, 11.5 (CLT) and exercise 11.6.4 (applications to PDEs)

↳ Key ingredient in proof of Poisson Sum:

THM 11.3.8 (Dirichlet)

Suppose f is periodic with period 1, $|f(x)|$ is bounded and f is differentiable at x_0. Then $\lim_{N\to\infty} S_N(x_0) = f(x_0)$, where $S_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}$

↳ <u>ASIDE</u>: Proof of this / general convergence questions of Fourier Series. Need differentiability assumption as Dirichlet kernel D_N is not an approx to the identity (Fejer kernel was).
Remember $S_N(x_0) = (f * D_N)(x_0)$.
See Exe 11.6.1 for what can happen at discontinuities

PROOF OF POISSON SUMMATION

Natural to study $F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ and $F(0)$

Clearly F is periodic with period 1, To use Dirichlet's Thm need to show F bounded and differentiable for all x.

ASIDE: If only knew F cont, could we use Fejér's Thm and get something?

↳ caveat: Exe 11.4.7 shows why we need some conds on f, f', f'' to ensure F is cont and diff

↳ ex: $f(x) = \begin{cases} n^6 \left| \frac{1}{n^4} - |n-x| \right| & |n-x| \le \frac{1}{n^4} \\ 0 & \text{otherwise} \end{cases}$

↳ for this f, $F(0)$ (or $F(n)$) does not exist

LEMMA 11.4.8: g decays like $x^{-(1+\eta)}$ Then $G(x) = \sum_{n=-\infty}^{\infty} g(x+n)$ converges for all x and is continuous

Proof: $|G(x)| \le \sum_{|n| \le N} |g(x+n)| + \sum_{|n|>N} |g(x+n)|$

$\le \text{Const}(N) + C \sum_{|n|>N} \left(\frac{1}{x+n}\right)^{1+\eta}$

as $\eta > 0$, sum converges
(N large st $x+n \neq 0$)

must show continuous