MATH 406: TAKE AWAYS

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ABSTRACT. Below we summarize some items to take away from the class. In particular, what are one time tricks and methods, and what are general techniques to solve a variety of problems, as well as what have we used from various classes.

1. CALCULUS I AND II (MATH 103 AND 104)

We used a variety of results and techniques from 103 and 104:

(1) Standard integration theory: integrations of cosines and sines, integration by parts (for example, moments of the Gaussian).
(2) Definition of the derivative (this was used in particular for the strange function which is $\exp(-1/x^2)$ for $x \neq 0$ and 0 when $x = 0$). This function has all derivatives zero at $x = 0$, but is non-zero for $x \neq 0$. Thus the Taylor series does not converge in a neighborhood of positive length containing the origin. This function shows how different real analysis is from Complex analysis. Explicitly, here we have an infinitely differentiable function which is not equal to its Taylor series in a neighborhood of $x = 0$; if a complex function is differentiable once it is infinitely differentiable and it equals its derivative in a neighborhood of that point.
(3) Ratio, root and comparison tests to determine if a series or integral converges (ranging from series that arise in the circle method to the integral defining the Gamma function); we frequently used the geometric series formula.
(4) Taylor series expansions.

2. MULTIVARIABLE CALCULUS

(1) We frequently used the Fubini Theorem (or Fubini-Tonelli) to justify interchanging two integrals (or an integral and a sum). Doing such interchanges is one of the most frequent tricks in mathematics.
(2) Path / line integrals and Green's theorem in the plane to motivate the Residue Theorem from Complex Analysis.

3. LINEAR ALGEBRA

Obviously linear algebra was very important in the random matrix theory section, ranging from:

(1) Eigenvalues of real symmetric matrices: these are real, and an $N \times N$ matrix has N linearly independent eigenvectors.
(2) Eigenvalue Trace Lemma: allows us to pass from knowledge of the matrix elements to knowledge of the eigenvalues.

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(3) The concept of a vector space and a basis: We had to deal with infinite dimensional spaces when we did Fourier analysis, but fortunately were able to make do with finite dimensional spaces for the circle method.

4. ANALYSIS

(1) General continuity properties, in particular some of the $\epsilon - \delta$ arguments to bound quantities.
(2) Most important, however, was probably when we can interchange operations, typically interchanging integrals, sums, or an infinite sum and a derivative. We proved that we can differentiate some infinite sums term by term (for the geometric series sum, this can be done by noting the tail is another geometric series; in general this is proved by estimating the contribution from the tail of the sum).
(3) Proofs by Induction.
(4) Dirichlet's Pidgeonhole principle: this was very useful in studying $n^k\alpha \bmod 1$, and gave us very good rational approximations to irrationals.

5. NUMBER THEORY

(1) Elementary functions: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, $\phi(q)$ is the number of positive integers at most q that are relatively prime to q, ...
(2) The Prime Number Theorem or the Siegel-Walfisz Theorem: we used these frequently in analyzing prime sums.
(3) Partial summation: allows us to pass from one known sum to another. For example, knowing $\sum_{p\le x} \log p \sim x$ we can then evaluate $\sum_{p\le x} 1$.
(4) Dirichlet's Pidgeonhole principle: this was very useful in studying $n^k\alpha \bmod 1$, and gave us very good rational approximations to irrationals.
(5) Unique factorization of the integers: this was crucial in proving $\zeta(s) = \sum 1/n^s$ also equals $\prod_p (1-p^{-1})^{-1}$; as we knwo where the integers are, the hope is that we can use this knowledge to deduce information about the primes.

6. PROBABLITY THEORY

(1) A probability density is a non-negative function $p(x)$ that integrates to 1. If it is nice, it is uniquely determined by its moments $\int x^k p(x)dx$. This was one of the key ingredients in the random matrix theory unit.
(2) Probabilities of independent events are the products of the probabilities. One instance this was used was in assigning probabilities to random matrix ensembles.

7. FOURIER ANALYSIS

(1) A nice function can be uniformly approximated by a trigonometric polynomial (Fejer's theorem). One great use of this is in $n^k\alpha \bmod 1$, as trig functions are particularly nice to work with.
(2) Poisson summation: for nice functions, $\sum f(n) = \sum \widehat{f}(n)$. This was very useful in proving the functional equation of the Riemann zeta function.

8. General Techniques

(1) One of the most common features we've seen in a variety of problems is that we can reduce a problem to the determination of an integral, ranging from the equidistribution of $n^k\alpha \bmod 1$ (for $n \in \{1, \dots, N\}$) to writing N as a sum of three primes to writing $\zeta(s)$ as an integral of the theta function. That's the good news. The bad news is that we must evaluate these integrals as a function of some parameter, say N, and typically it is *not* possible to write down a closed form answer. Thus we need to develop techniques to approximate these integrals and determine some good control of the N dependence.

(2) Another common theme is that we try to do as little work as possible to get as good of an estimate as needed. A terrific example is in the circle method, where we bound the contribution to $F_N(a/q)$ from primes $p \equiv r \bmod q$ with r and q sharing a common factor. The main term is of size $N/\log^D N$ for some D; thus trivially bounding the error term by $q^2 \log q \ll \log^{2B+\epsilon}$ is fine as this is so much smaller than the main term. Of course, in many problems the trivial counting or error estimation does not suffice, and thus one must then be more clever.

(3) We've also seen on a few problems how the way the problem is formulated can influence how one attempts to solve it. Examples include the graph coloring problem from the HW (vertices are 2 through N and are connected if they share a divisor; the HW problem was to show the coloring number is at least 13, which can be done by looking at powers of 2, but it's actually at least 5000, from looking at even numbers) and the first problem from the midterm (for each $n > 1$ finding an $m > 1$ such that nm only has 0s and 1s base 10; one proof is similar to the pidgeonhole problem of a subset of $\{a_1, \dots, a_n\}$ has a sum divisible by n). It is amazing how often one can get trapped at looking at a problem in a certain way; this is something to be aware of.

(4) Certain functions become natural choices in studying certain problems. For example, for $n^k\alpha \bmod 1$ we use the exponential function. The reason this is so useful is that $\exp(2\pi i n^k \alpha)$ $= \exp(2\pi i (n^k\alpha \bmod 1))$. Thus we may drop the difficult modulo 1 condition and sum more easily. Depending on the problem, different functions and expansions will be more useful than others. The ease at which the exponential function handles the modulo 1 condition suggests the usefulness of applying Fourier analysis.

(5) L^2-norms: in the Circle Method we had the generating function $F_N(x) = \sum_{p \le N} \log p \cdot \exp(2\pi i p x)$. We are able to get a very good bound for $\int_0^1 |F_N(x)|^2 dx$ as $|F_N(x)|^2 = F_N(x)F_N(-x)$, and the only terms that survive the integration are when we have reinforcement. More generally, it is often easy (or at least easier) to get reasonable estimates for quantities such as $\int |F(x)|^{2^k} dx$.

(6) Adding zero / multiplying by one: The difficult part of these methods is figuring out how to 'do nothing' in an intelligent way. The first example you might remember is proving the

product rule from calculus. Let $A(x) = f(x)g(x)$. Then

$$\begin{aligned}
A'(x) &= \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)\textbf{-f(x)g(x+h) + f(x)g(x+h)}-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h} + \frac{f(x)g(x+h)-f(x)g(x)}{h}\right] \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} g(x+h) + \lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \lim_{h\to 0} g(x+h) + f(x)\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

(7) Replacing $\prod a_n$ with $\exp\left(\log \prod a_n\right)$, as this converts the product to a sum, and we have a much better understanding of sums. We used this to provide a good lower bound for the singular series $\mathfrak{S}(N) = \prod_{p|N}\left(1 - \frac{1}{(p-1)^2}\right)$ in the Circle Method (writing odd numbers as the sum of three primes). We also used it to get a good lower bound for $\phi(q)$, which allowed us to see that $q/\log\log q \ll \phi(q) \ll q-1$.

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