$\{n^k \alpha\}$ AND POISSONIAN BEHAVIOR

PROBLEM: Fix k and α, look at $n^k \alpha \bmod 1$ for $n \in \mathbb{N}$

What can you say about this sequence?

↳ Dense?

Equidistributed?

Spacings?

} DO EXAMPLES WITH MATHEMATICA CODE AFTER STUDENTS CONJ

Applications: • $k=2$ arises in special Hamiltonians (see refs in RSZ)

• At some scale, see same behavior in

↳ primes

↳ waiting at banks

Needed Material: FA, P, NT

FA: ① Fejér's Thm: f cont then $\forall \varepsilon\ \exists N_0$ s.t. $\forall N \geq N_0$ $|f(x) - T_N(x)| < \varepsilon$

where $T_N(x)$ is the "averaged" Fourier Series: hardest input

② $e^{2\pi i x} = e^{2\pi i (x \bmod 1)}$: This is why so useful

P: ① Point masses and induced probability distr

② Order statistics for uniform rand vars

NT: ① Dirichlet's Pidgeon-hole Principle

② Approx irrationals by rationals

CHAPTER 12: $\{n^k\alpha\}$ AND POISSONIAN BEHAVIOR

Questions: Dense, Equidistr, Spacings

Notation:
- Char fn: $\chi_{(a,b)}(x)$
- Wrapped unit interval and norm $\|x-y\| = \min_{n\in\mathbb{Z}} |x-y-n|$
- probability density: $\int_{\mathbb{R}} f(x)\,dx = 1$, $f(x) \geq 0$
  - ↳ X rv with density f means $\text{Prob}(X\in[a,b]) = \int_a^b f(x)\,dx$
  - ↳ key densities: uniform, standard exponential
  - ↳ mean: $\mathbb{E}[X] = \mu = \int x f(x)\,dx$ (if exists)
    - ↳ will do variance later (Chebyshev, CLT)

① DENSENESS

Pre-reqs:
- Dirichlet's Pidgeon-hole principle (Appendix A.4)
  - ↳ ex: A.4.2: $S \subset \{1,\dots,2n\}$ with $|S| = n+1$
    Then $\exists a,b\in S$ st $a|b$
    - ↳ proof: two elements in S have same odd part
- Exponent/Order of Approx: $\xi\in\mathbb{R}$ has approx order $\tau(\xi)$ if $\tau(\xi)$ smallest number st $\forall e > \tau(\xi)$ there are only finitely many solns to $|\xi - p/q| < 1/q^e$ (Section 5.5)
  - ↳ "many" properties in arithm dynamics governed by approx exponent

① DENSENESS (CONT)

THM (KRONECKER): $\alpha \notin \mathbb{Q} \implies n\alpha \bmod 1$ is dense

Proof: Dirichlet's Pidgeon-hole principle

wlog enough to show $\forall \varepsilon\ \exists N$ st $0 < N\alpha \leq \varepsilon$ (WHY)

$\alpha \notin \mathbb{Q} \to \exists\ \infty$ many p, q st $|\alpha - p/q| < 1/q^2$

↳ Proof: Q large, consider $\alpha \bmod 1, \ldots, (Q+1)\alpha \bmod 1$

two in same box: $|q_1\alpha - q_2\alpha - p| \leq \frac{1}{Q}$

so $\left|\alpha - \frac{p}{q}\right| < \frac{1}{Qq} < \frac{1}{q^2}$ ← why $<$ not $\leq$?

as $Q \to \infty$, must have ∞ many q

Thus if $1/q < \varepsilon$: $|\alpha - p/q| < \frac{1}{q^2} \to |q\alpha - p| < \frac{1}{q} < \varepsilon \ldots$

Remark: Proof aided by "linearity": can keep walking in blocks of N

Situation harder for $n^k\alpha \bmod 1$ when $k \geq 2$

As Equidistr $\implies$ Denseness, won't spend too much time here in general.

Idea for $k=2$, assuming α has approx exponent $4+h$ for some $h>0$

↳ now given $x \in [0,1]$ and $\varepsilon > 0$, show $\exists N$ st $\|N^2\alpha - x\| < \varepsilon$

Let $\frac{1}{q} < \frac{\varepsilon}{100}$ and choose p, q st $\alpha - \frac{p}{q} = \frac{\delta}{q^4}$ $\quad 0 \leq \delta \leq 1$, (wlog $\delta > 0$)

Thus $(qm)^2\alpha - pqm^2 = (\delta m^2)\frac{1}{q^2}$ and can ~~assume~~ choose m st $1 \leq \delta m^2 \leq 4$

As $\frac{x}{\delta m^2} \in [0,1]$, choose n st $\frac{n}{q}$ is within $\frac{1}{q}$ of $\sqrt{\frac{x}{\delta m^2}}$ (and smaller)

Gives $(qmn)^2\alpha - \underbrace{pqm^2n^2}_{\text{int}} = \underbrace{x}_{\text{desired}} - \underbrace{\frac{(\delta m^2)(2n\theta + \theta^2)}{q^2}}_{< 12/q < \varepsilon}$ $\quad \theta \leq 1,\ n \leq q$

Aside: Algebraic Structure of Numbers

How well can an irrational be approx by rationals?

↳ Can generalize: theory of heights, ell curves, ...

Measure cost by size of denom

<u>Thm</u> (Hurwitz, Thm 7.9.4): $\forall \alpha \notin \mathbb{Q}$ $\exists \infty$ many p, q (rel prime) st $|\alpha - p/q| \le \frac{1}{\sqrt{5}} \frac{1}{q^2}$. Taking $\alpha = \frac{1+\sqrt{5}}{2}$ we see this is sharp (ie, golden mean "most" irrational)

Comments: • Continued fraction expansions (Chapter 7)

• $\gamma = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \dots]$

• Note Dirichlet easily gives $\frac{1}{q^2}$: save $\frac{1}{\sqrt{5}}$

<u>Thm</u> (Liouville): $\alpha \in \mathbb{R}$ algebraic of deg d. Then $\tau(\alpha) \le d$ where $\tau(\alpha)$ is the approx exponent (Thm 5.6.1)

Proof: $f(x) = \sum_{k=0}^{d} a_k (x - \alpha)^k$ and $a_0 = 0$ since $f(\alpha) = 0$

Choose p/q "close" to α, say within 1 unit

$$\left|f\left(\frac{p}{q}\right)\right| = \left|\frac{\text{int}}{q^d}\right| \le \left|\frac{p}{q} - \alpha\right| \cdot \sum_{k=1}^{d} |a_k| \cdot 1^{k-1} \le A \left|\frac{p}{q} - \alpha\right|$$

If ~~$\tau(\alpha)$~~ $\tau(\alpha) > d$ then ∞ many p, q st $\left|\frac{p}{q} - \alpha\right| < 1/q^{d+\epsilon}$

Get $\left|f\left(\frac{p}{q}\right)\right| = \left|\frac{\text{int}}{q^d}\right| \le \frac{A}{q^{d+\epsilon}}$ q large $\Rightarrow$ int $= 0$

Thus infinitely many rational roots!

<u>Corr</u>: Liouville Numbers transcendental! (Thm 5.6.4)

↳ $\sum 10^{-m!}$ (or my generalization)

<u>Comment</u>: Roth's Thm (Thm 5.7.1, Chapter 6) that $\tau(\alpha) = 2$ if $\alpha \notin \mathbb{Q}$ is algebraic of deg $d \ge 2$.

Other: $\tau(e)$, $\tau(\pi)$, Thm 5.5.9 ("small measure" with $\tau(\alpha) \ge 2+\epsilon$)

ASIDE (CONT)

Algebraic / Transcendental numbers: Chapter 5

↳ countable and uncountable sets

↳ Cantor's Thm ("most" transcendental, Thm 5.3.24)

↳ Continuum Hypothesis

↳ Gödel: ZF consistent ⇒ ZF + Continuum consistent

Cohen (my math grandfather): ZF ⇒ ZF + not Continuum!

(ie, Continuum Hyp indep standard axioms set theory)

Paradoxes: Russell (Section 5.1)

Irrationality of e easy (Thm 5.4.5) (Due to Euler) (1737)

Transcendence of e harder but "elementary" (Hermite 1873, Thm 5.4.6)

Good exercise to show $\pi^2 \notin \mathbb{Q}$ (Exercise 5.4.17)

↳ consequence: infinitely many primes! (See Exe 3.1.7, 3.3.28)

idea: $\zeta(2) = \pi^2/6 = \prod_p (1 - 1/p^2)^{-1}$

Can we get information on $\pi(x)$ from this?

Some suggested exercises:

5.5.2, 5.5.5, 5.6.7, 5.6.8

Harder: 5.5.6, 5.6.5, 5.6.9

Very hard: 5.5.13

② EQUIDISTRIBUTION OF $n^k \alpha \bmod 1$

Will do for $k=1$, see book for general k

If $x_n = n\alpha \bmod 1 = n\alpha - [n\alpha]$, have $e(x_n) = e(n\alpha)$

where $e(z) = e^{2\pi i z}$: See utility FA

as can "drop" the modulo 1.

<u>THM</u> (WEYL, THM $\begin{cases} 12.3.5 \\ 12.3.2 \end{cases}$): $\alpha \notin \mathbb{Q} \Rightarrow n^k \alpha \bmod 1$ is equidistr

<u>Proof ($k=1$)</u>: Must show $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=1}^{N} \chi_{a,b}(x_n) = b-a$

where $\chi_{a,b}$ is char fn of (a,b) and $x_n = n\alpha \bmod 1$,

↳ Caveat: book looks at $\{x_n\}_{n=-\infty}^{\infty}$: typo

doesn't change anything

Geometric series: $\frac{1}{N} \sum\limits_{n=1}^{N} e_m(x_n) = \begin{cases} 1 & m=0 \\ \frac{1}{N}\left(\frac{e_m(\alpha) - e_m((N+1)\alpha)}{1 - e_m(\alpha)}\right) & \text{o/w} \end{cases}$

with $e_m(z) = e(mz) = e^{2\pi i m z}$

As $\alpha \notin \mathbb{Q}$, for m fixed have $|1 - e_m(\alpha)| > 0$ (this is where use $\alpha \notin \mathbb{Q}$)

$\Longrightarrow P(x)$ any finite trig poly over a symmetric range, so

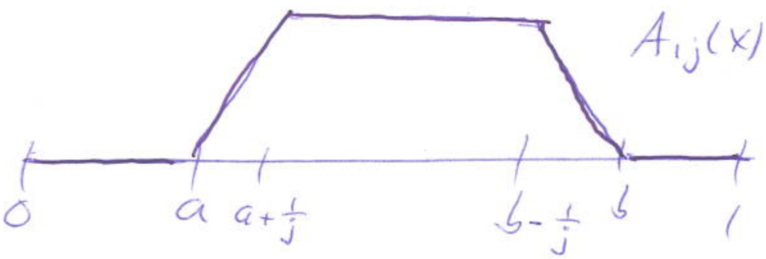
$P(x) = \sum\limits_{m=-M}^{M} a_m e_m(x)$, then $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=1}^{N} P(x_n) = a_0 = \int_0^1 P(x)\,dx$

↳ key fact: M fixed, let $C = \max\limits_{\substack{|m| \le M \\ m \ne 0}} \frac{1}{|1 - e_m(\alpha)|}$

② EQUIDISTR (CONT)

Approx step fn by cont fn
Approx cont fn by trig poly

} common analysis technique



$0 \le A_{1j}(x) \le \chi_{a,b}(x) \le A_{2j}(x)$

Use Fejer: For each j, given ε $\exists$ symm trig poly P_{1j}, P_{2j} st $|P_{ij}(x) - A_{ij}(x)| \le \varepsilon$.

$$\Longrightarrow \frac{1}{N}\sum_{n=1}^{N} A_{ij}(x_n) \text{ and } \frac{1}{N}\sum_{n=1}^{N} P_{ij}(x_n)$$

differ by at most ε, and know the P_{ij}-sums tend to $\int_0^1 P_{ij}(x)dx$ as $N \to \infty$.

These integrals are just $b-a \pm 1/j$ ∎

③ POISSONIAN BEHAVIOR

Probability review

- Dirac Delta Functional: "point mass": $\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$
- Seq $\{x_n\}$ induces a discrete measure

$$\mu_N(x)\,dx = \frac{1}{N}\sum_{n=1}^{N}\delta(x-x_n)\,dx$$

or $$\int_{-\infty}^{\infty} f(x)\mu_N(x)\,dx = \frac{1}{N}\sum_{n=1}^{N} f(x_n)$$

↳ will see again in RMT

BIG APPLICATION: MONTE CARLO INTEGRATION

↳ Called one of most influential papers in 20th cent

allows approx multi-dim integration

↳ See Exercise 12.5.2

THM 12.7.3: X_n iidrv Unif$(0,1)$ Then adj spacings (normalized) Converges to standard exponential.

Proof: "unfold" to have unit mean spacing (KEY CONCEPT)

↳ why twin primes so hard (Nicely + Pentium Bug)

Let $\{Y_n\}_{n=1}^{N}$ be $\{X_n\}_{n=1}^{N}$ in increasing order, $Z_n = NY_n$

Study $Y_{n+1} - Y_n$ or $Z_{n+1} - Z_n$

↳ For y's: as ave spacing size $1/N$, natural to look at diff of size t/N and send $N\to\infty$

③ POISSONIAN BEHAVIOR (CONT)

By symm of wrapped around interval, all look same

(1) Calc prob <u>all</u> $N-1$ other X_n's are at least $\frac{t}{N}$ units to the right of X_1: $P_N(t) = \left(1 - \frac{t}{N}\right)^{N-1}$

(2) Calc prob <u>all</u> $N-1$ other X_n's are at least $\frac{t+\Delta t}{N}$ units to the right of X_1: $P_N(t+\Delta t) = \left(1 - \frac{t+\Delta t}{N}\right)^{N-1}$

Thus prob ~~next~~ spacing is b/w t/N and $\frac{t+\Delta t}{N}$ is

$$\left(1-\frac{t}{N}\right)^{N-1} - \left(1-\frac{t+\Delta t}{N}\right)^{N-1}$$

As $N \to \infty$ this goes to $e^{-t} - e^{-(t+\Delta t)}$

$$= e^{-t}\left(1 - e^{-\Delta t}\right)$$

$$= e^{-t}\Delta t + O((\Delta t)^2)$$

Thus converges to standard exponential.

(See also Remark 12.7.4)

<u>Remark</u>: These are order statistics, quite useful

↳ Miller–Nigrini and Benford

↳ Median (Exe 12.7.8) (and Exe 12.7.9)

↑ great project for prob/stat minded person

Call behavior of iidrv Uniform(0,1) order statistics Poissonian

③ POISSONIAN BEHAVIOR (CONT)

Section 12.8: What do we know about $\{n^k \alpha\}$?

↳ "Know" much, can "prove" only a little

- $k=1$: $\alpha \notin \mathbb{Q} \Rightarrow$ at most 3 spacings!
  ↳ Good Challenge Problem: Exe 12.6.3!

CONJ: $k \geq 2$, for almost all $\alpha \notin \mathbb{Q}$ in sense of measure, Then $n^k \alpha \bmod 1$ is Poissonian as $N \to \infty$.

↳ many different notions of what a "generic" element is

See Appendix A.5 for a brief introduction to measure Theory (or Lebesgue Theory).

↳ contrast with Thm 5.5.9

↳ See book: know Poissonian or not-Poissonian along some subseq depending on how well α approx by rationals (See proof of Thm 12.8.5)

GOING FURTHER

- Several research projects given in Section 12.9
- Project 12.9.4 (might still be open, should be doable) may require basic knowledge of continued fractions (Chap 7)