# RANDOM MATRIX THEORY

**PROBLEMS:** Study eigenvalues of ensembles of matrices

↳ Density of eigenvalues

↳ Spacings b/w eigenvalues

**APPLICATIONS:** Physics: $H\Psi_n = E_n \Psi_n$

↳ sadly, H usually ∞-dim @ unknown entries!

Graph Theory: Eigenvalues Adjacency matrices

↳ building efficient, cheap networks

Number Theory: Zeros of L-fns

↳ density of primes, prime races, class numbers, ...

**NEEDED MATERIAL:** FA, P, LA, NT

FA: ① CLT (Thm 11.5.1), though could use Chebyshev (Exe 8.1.55)

P: ① Chebyshev's Thm (Exe 8.1.55):

② Moments of a distr: $\mu_k = \int x^k p(x)\,dx$

↳ when does knowing moments mean know distr?

LA: ① Eigenvalues (especially of real symm matrices)

② Diagonalization / Triangularization

③ $\mathrm{Tr}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$

④ $\mathrm{Tr}(A^k) = \sum_{i=1}^{N} \lambda_i(A)^k$

NT: ① Combinatorics: Catalan numbers / Moments of Gaussian

② L-fns and primes: $\zeta(s) = \sum \frac{1}{n^s} = \prod (1-p^{-s})^{-1}$ $\zeta(1), \zeta(2)$

# RANDOM MATRIX THEORY

History: 3 body problem → Uranium

Stat Mech: 

Approx $H$ by random $N \times N$, average

↳ hope ave is good approx for "generic"

## KEY INGREDIENTS:

All problems involve 3 key inputs:

① Normalization / Determining scale

↳ often CLT

② Trace Formula

↳ ex $\mathrm{Tr}(A^k) = \sum \lambda_i(A)^k$

connect evalues (which we want to know) with matrix elements (which we do know)

③ Averaging formulas

↳ often combinatorics (hard)

↳ Trace formula useless if can't do anything with matrix coeffs

↳ ave formulas better in RMT than NT

### 15.1.3. Random Matrix Ensembles

$p$ prob distr with mean 0, variance 1, finite moments

↳ $\int x p(x) dx = 0$ $\int x^2 p(x) dx = 1$ $\int |x|^k p(x) dx < \infty$

Consider real symm $N \times N$ matrices $A = (a_{ij})$

↳ have $\frac{N^2 - N}{2} + N = \frac{N^2 + N}{2}$ indep entries

$$\mathrm{Prob}(A)\,dA = \prod_{1 \le i \le j \le N} p(a_{ij})\,da_{ij}$$

↳ do 2x2 example

↳ Show $\int \mathrm{Prob}(A)\,dA = 1$ (still true if had $p_{ij}(a_{ij})$)

## 15.2. Eigenvalue Preliminaries

### 15.2.1 Eigenvalue Trace Formula

$\mathrm{Trace}(A) = a_{11} + \cdots + a_{NN}$

$\mathrm{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} a_{ji} = \sum_{i,j=1}^{N} a_{ij}^2$

$\mathrm{Tr}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}$

Key Fact: "Most important" Thm in Lin Alg

THM 15.2.4: EIGENVALUE TRACE LEMMA: $\mathrm{Tr}(A^k) = \sum_{i=1}^{N} \lambda_i(A)^k$

↳ Proof: $k=1$ from char poly: $\det(A - \lambda I) = 0$

$k \ge 2$: Triangularize $A \Rightarrow U^{-1} A U = T$, eigenvalues $A$ = those of $T$

↳ if $A$ real symm (our case): diagonalize

Idea behind moments: $N \times N$ matrix, know $\sum_{i=1}^{N} \lambda_i(A)^k$ $1 \le k \le N$

Then know the $\lambda_i(A)$

# 15.2.2. NORMALIZATIONS

Study primes on "right" scale
↳ Nicely + Pentium bug (twin primes hard)

Claim: $\lambda_i(A)$ is of size $\sqrt{N}$

"Proof": $Tr(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2$

$$\mathbb{E}[Tr(A^2)] = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}[a_{ij}^2]$$ as expectation's linear

Thus see it is of size $N^2 \cdot 1$, $N$ eigenvalues...

"better": $Tr(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2$

↳ CLT or Chebyshev says sum is like $N^2$

so $\sum \lambda_i(A)^2 \sim N^2$

$\Rightarrow \langle \lambda_i(A)^2 \rangle_{ave} \sim N$ so $\langle \lambda_i(A) \rangle_{ave} \sim \sqrt{N}$

↳ can't pass $\sqrt{\ }$ into ave...

Chebyshev's Thm: Prob distr $p$ with mean $\mu$, var $\sigma^2 < \infty$. Then for random variable $X$ with density $p$: $\text{Prob}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

Note: "natural" to do $k\sigma$ as "units" of $X, \mu$ are $\sigma$

Proof: $\text{Prob}(|X-\mu| \geq k\sigma) = \int_{|x-\mu| \geq k\sigma} p(x)\,dx$

$$= \int_{\left|\frac{x-\mu}{k\sigma}\right| \geq 1} \left(\frac{x-\mu}{k\sigma}\right)^2 p(x)\,dx$$

extend integral $\leq \frac{1}{k^2\sigma^2} \int (x-\mu)^2 p(x)\,dx = \frac{1}{k^2}$

ASIDE

(Note: Can also use CLT, Thm 8.4.1. Chebyshev is weaker, but applies to more densities: like divide + conquer vs Newton's method)

# 15.2.3. EIGENVALUE DISTRIBUTION

$A$ is $N \times N$, eigenvalues $\lambda_i(A)$

Define $\mu_{A,N}(x)\,dx = \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right) dx$

↳ $\sqrt{N}$ is from "natural" scale, to have limit as $N \to \infty$

2 is to show off (semi-circle vs semi-ellipse)

# Normalized evalues in $[a,b]$ is $\int_a^b \mu_{A,N}(x)\,dx$

Denote $k^{th}$ moment by $\mathbb{E}[x^k]_A$ or $M_{N,k}(A)$

IMPORTANT

$$M_{N,k}(A) = \frac{\mathrm{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}$$

Proof: $M_{N,k}(A) = \int x^k \mu_{A,N}(x)\,dx$

do algebra...

# 15.3. Semi-Circle Law

Idea: $A \to \mu_{A,N}(x)\,dx \to \{M_{N,k}(A)\}_{k=1}^{\infty}$

Study $\mathbb{E}[M_{N,k}(A)]$ (expectation over $A$)

on average these equal moments of semi-circle

want to claim that as $N \to \infty$ "most" $\mu_{A,N}$ are "close" to semi-circle

Caveats: • Exe 15.3.2: non-uniqueness of moments

• Remark 15.3.3: $\{a_{n,2N}\}_{n=1}^{2N}$ st half $+1$, half $-1$

then average 0, none close

need to control variances

Idea of Proof of Wigner's Semicircle Law when $k=2$

$$\text{Let } M_{N,2} = \int_A M_{N,k}(A)\, \text{Prob}(A)\, dA$$

$$= \frac{1}{2^2 N^{\frac{2}{2}+1}} \int_A \text{Tr}(A^2)\, \text{Prob}(A)\, dA$$

$$= \frac{1}{4N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{a_{11}=-\infty}^{\infty} \cdots \int_{a_{NN}=-\infty}^{\infty} a_{ij}^2\, p(a_{11}) \cdots p(a_{NN})\, da_{11} \cdots da_{NN}$$

as finite sums, $\Sigma\Sigma\int\int = \int\int\Sigma\Sigma$

integrals factor $\int a_{ij}^2 p(a_{ij})\, da_{ij}\ \int \cdots \int p(a_{kl})\, da_{kl} = 1$

assuming variance $=1$, mean $=0$

$$= \frac{N^2}{4N^2} = \frac{1}{4}$$

$k \geq 3$: Combinatorics

note have integrals of poly in matrix elements

# CHAPTER 15: ODDS AND ENDS

Show plots of semi-circle law

↳ especially @ Cauchy distr, violating finite moments

↳ very good distr to test universalities

Mention GOE conj for spacings:

$$V_{A,N}(s)\,ds = \frac{1}{N-1}\sum_{i=2}^{N} \delta\left(s - \frac{\lambda_i(A) - \lambda_{i-1}(A)}{2\sqrt{N}}\right) ds$$

Show pictures of spacings

Talk about d-regular graphs

ASIDE

THM 15.5.9: McKay's Law

d-regular graphs. As $N \to \infty$ $\mu_{A(G),N}(x)$ converges to

Kesten's Measure: $f(x) = \begin{cases} \frac{d}{2\pi(d^2 - x^2)}\sqrt{4(d-1) - x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{other} \end{cases}$

Note: as $d \to \infty$, if change scale becomes semi-circle

Conj spacings are GOE even though density of states is not semi-circle

↳ distr of spacings more "fundamental"

Number Theory: Mention zeros of L-fns

↳ GUE, Mont-Odlyzko Law

# CHAPTER 16: RMT: EIGENVALUE DENSITIES

## Semi-circle Law (Wigner)

$p$ mean 0, variance 1, finite higher moment. Then as $N \to \infty$
The typical $\mu_{A,N}(x)$ converges to the semi-circle density

$$P(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

↳ Can prove @ weaker assumps
↳ Will sketch the proof using Method of Moments
(other techniques more powerful but "harder")

Note: Moments of semi-circle are $C(k) = \begin{cases} 2\frac{(2m-1)!!}{(2m+2)!!} & k=2m \\ 0 & k=2m+1 \end{cases}$

↳ Proof is change of variable: $\sqrt{1-x^2}\,dx \to \cos^2\theta\, d\theta$
finish with Mathematical Induction (Appendix A.1)

↳ $k!!$ means $k(k-2)(k-4)(k-6)\cdots$ to 2 or 1

**aside**: investigate combinatorial aspects
arises in Toeplitz matrices
↳ part of matching $2m$ objects in $m$ pairs

## Moment Preliminaries

$$M_{N,k}(A) = \int x^k \mu_{A,N}(x)\,dx$$
$$= \frac{1}{N}\sum\left(\frac{\lambda_i(A)}{2\sqrt{N}}\right)^k$$
$$= \frac{1}{2^k N^{\frac{k}{2}+1}}\,\mathrm{Tr}(A^k)$$

Thus need $\int \mathrm{Tr}(A^k)\,\mathrm{Prob}(A)\,dA$: as $N \to \infty$ terms
$o(N^{\frac{k}{2}+1})$ negligible

# WIGNER'S SEMI-CIRCLE LAW

$$M_{N,k} = \mathbb{E}[M_{N,k}(A)]_A = \int M_{N,k}(A)\, \text{Prob}(A)\, dA$$

$$= \frac{1}{2^k N^{\frac{k}{2}+1}} \int \cdots \int a_{i_1 i_2} \cdots a_{i_k i_1} \prod_{1 \le i \le j \le N} p(a_{ij})\, da_{ij}$$

note $a_{ij} = a_{ji}$ as $A$ is real symm

## Observations

Write $a_{i_1 i_2} \cdots a_{i_k i_1}$ as $a_{x_1 y_1}^{r_1} \cdots a_{x_\ell y_\ell}^{r_\ell}$
with $\{x_i, y_i\} \ne \{x_j, y_j\}$ if $i \ne j$

Then $\int \cdots \int a_{i_1 i_2} \cdot a_{i_k i_1} \text{Prob}(A)\, dA$

$$= \int \cdots \int a_{x_1 y_1}^{r_1} \cdots a_{x_\ell y_\ell}^{r_\ell} \text{Prob}(A)\, dA$$

$$= P_{r_1} \cdots P_{r_\ell} \quad \text{where } P_r = \int x^r p(x)\, dx$$

Exercise: do first three moments / read in book

**Lemma**: If an $a$-tuple has an $r_j = 1$ then it vanishes

Proof: $P_1 = 0$

Thus each $r_j \geq 2$, and everything is <u>at least</u> paired.
Will see later no contribution if a triple or more.

Do third moment: triple gives $N p_3$, divide by $N^{5/2}$ and thus negligible in the limit.

# Wigner's Semicircle Law (cont)

**Lemma 16.1.12**: $M_{N;k} = 0$ for $k$ odd

Proof: Trivial if density $p$ is even: $p(x) = p(-x)$

↳ odd moments of $p$ vanish

$k$ odd means in $P_{r_1} \cdots P_{r_\ell}$ at least one $r$ is odd

Counting argument for general $P$

↳ dividing by $N^{\frac{k}{2}+1} = N^{m+3/2}$ if $k = 2m+1$

proof completed by showing $O(N^{m+1})$ matchings

where no ~~$a_{ij}$~~ $a$-term by itself (no $P_1$).

Thus in our $a_{x_1y_1}^{r_1} \cdots a_{x_\ell y_\ell}^{r_\ell}$ each $r \geq 2$ and one is odd (and thus $\geq 3$)

have $k = 2m+1$ indices initially free

↳ how many result in matchings in pairs?...

As higher moments bounded, each term at most $B_k$

Aside on Combinatorics

Don't really need

Suffices to note The number of matchings depends on $k = 2m+1$ and NOT on $N$.

Harder: # of solns to $r_1^2 + \cdots + r_\ell^2 = 2m+1$ Leads to Circle Method

Combinatorics: $r_1 + \cdots + r_\ell = 2m+1$

each $r_i \geq 2$, one is $\geq 3$ (as odd)

↳ Soln: Cookie Problem (Pages 11-13)

10 cookies, 5 people $\rightarrow \binom{10+5-1}{5-1}$

all 2's and a 3 for $r_j$ gives $\ell = m$

Could do $r_1 = 2m+1$ and $\ell = 1$

Fixed $\ell$ count/solve $r_1 + \cdots + r_\ell = 2m+1$

↳ so $\tilde{r}_1 + \cdots + \tilde{r}_\ell = 2(m-\ell)$

number solns is $\binom{2(m-\ell)+\ell-1}{\ell-1} = \binom{2m-\ell-1}{\ell-1}$

Sum over $\ell = 1$ to $m$

↳ all that matters is answer depends ONLY on $k = 2m+1$ and NOT on $N$.

# Wigner's Semicircle Law (cont)

↳ some number (depending on $k = 2m+1$) matchings, say $M_k$

initially have $N^k = N^{2m+1}$ free variables, $i_1, \dots, i_k$

Every time an $a_{i_j i_{j+1}}$ is paired with an $a_{i_n i_{n+1}}$ lose at least one degree of freedom and gain at most a factor of 2

↳ if adjacent: $a_{i_j i_{j+1}} \text{———} a_{i_{j+1} i_{j+2}}$

Then $i_{j+1} = i_j$

↳ if not adjacent

$\cdot\, a_{i_{n-1}, i_n}$

$a_{i_j, i_{j+1}} \text{———} a_{i_n, i_{n+1}} \quad \begin{cases} i_n = i_j \text{ or } i_{j+1} \\ i_{n+1} = i_{j+1} \text{ or } i_j \end{cases}$

$\cdot\, a_{i_{n+1}, i_{n+2}}$

Thus lose the freedom to choose $i_n, i_{n+1}$ but can "flip" and so have a factor of 2.

Thus number of "free" variables at most $\frac{k}{2} = m + \frac{1}{2}$ (is at most $m$ as an integer and have a triple) but can multiply by $2^{k/2}$ (or $2^m$) from "flips"

Thus $|M_{N,k}| \leq \cancel{B_k} \cdot \underbrace{M_k}_{\text{\# matchings}} \cdot \underbrace{B_k}_{\text{max contr}} \cdot 2^{k/2} \cdot \underbrace{N^{m+1/2}}_{\text{choices for free vars}} \Big/ 2^k N^{\frac{k}{2}+1}$

$$\leq \frac{M_k B_k}{2^{k/2}} \cdot \frac{1}{N} \xrightarrow[N \to \infty]{} 0$$

# WIGNER'S SEMICIRCLE LAW

**Lemma 16.1.14**: k even, as $N \to \infty$ no contribution unless all $r = 2$

Main term thus $k = 2m$ and all $r_j = 2$

$$M_{N,2m} = \frac{1}{2^m N^{1+m}} \sum_{1 \le i_1 \le \cdots \le i_{2m} \le N}^{*} 1$$

where $*$ means restrict to $(i_1, \ldots, i_{2m})$ matched in pairs

**ASIDE**

As this is a main term, estimates are not enough

Must do combinatorics

The soln involves the Catalan numbers $C_k = \frac{1}{k+1}\binom{2k}{k}$

↳ see [Leh] for details

## Non-Semi-circle Behavior

① Band matrices (especially diagonal!)

② Toeplitz and Palindromic Toeplitz (student papers)

↳ 4th moment first "real" moment for sym, even density

↳ Stories of results

③ Truncated Cauchy

**ASIDE**: Lots of Research Projects

# ASIDE

Just scratched the surface on rich, fascinating subject

Combinatorics essential: lots of times "believe" two expressions are equal but need the "aha" inspiration to count.

↳ ex: telescoping series

ex: matching coeff: $\sum_{k=0}^{n} \binom{n}{k}^2$ from $(x+y)^{2n}$

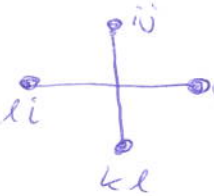
## Real Symmetric (Palindromic) Toeplitz Matrices

↳ Can view probabilistically or number theoretically

↳ I prefer the latter and see Diophantine Eqs

Comes down to counting solns

4th Moment




$\Rightarrow \frac{1}{N^3} \sum_{\substack{i,j,k,l=1 \\ i=j+l-k}}^{N} \cdots \sum 1$

↳ number of tuples is $\frac{2}{3}N^3 + \frac{1}{3}N$

↳ need value, though easy to see at most $\left(1-\frac{1}{27}\right)N^3$

↳ each index in $\{1, \ldots, N\}$: trouble if $j, l \geq \frac{2}{3}N$, $k < \frac{1}{3}N$

## Chapter 17: Spacings: GOE

↳ more probability, elementary diff eq: $y' = ay$

linear alg (o-thog rotations)

↳ lots of open problems

↳ Key idea behind GOE: nature doesn't care which axes choose

$P(A) = \prod_{i \leq j} p(a_{ij})$ and $P(QAQ^T) = P(A)$ $\forall$ o-thog $Q$

↳ rotate by $\varepsilon$ and then coeff of $\varepsilon$ term must vanish