THE RIEMANN ZETA FN $\zeta(s)$

Problems: $\zeta(s) = \sum_n 1/n^s = \prod_p (1-p^{-s})^{-1}$ $Re(s) > 1$

↳ extend to all s

↳ understand values / zeros

↳ generalize to $\sum_n a_n/n^s = \prod_p L_p(s)$

↳ many choices of a_n lead to interesting connections

↳ local ↔ global interplay

↳ examples: Dirichlet characters (primes in arithm progression, class number)

Elliptic Curves...

NEEDED INPUT: (FA, CA, NT/Alg)

Fourier Analysis: ① Fourier Transform

② Poisson Summation

Complex Analysis: ① terminology (zero, pole, residue, meromorphic cont, ...)

② logarithmic derivatives

③ Cauchy Residue Formula

↳ converts integrals to algebra (Taylor Series)

$$\frac{1}{2\pi i}\oint_\gamma f(z)\,dz = \sum_{z \in A} Res_z(f) \quad \partial A = \gamma$$

Number Theory: ① Basic group theory it do Dirichlet L-fns

② Elliptic curves for elliptic curve L-fns..

Basic Properties of $\zeta(s)$

Euler Product $\sum_{n=1}^{\infty} 1/n^s = \prod_{p \text{ prime}} (1-p^{-s})^{-1}$. $Re(s)>1$

↳ Proof: unique factorization of integers
  ↳ saw proof that "subtly" assumed this
  ↳ key ingredient: geometric series: $1+r+r^2+r^3+\dots = \frac{1}{1-r}$ $|r|<1$

Corr: Special Values

$\zeta(1)$, $\zeta(2)$ (see Exe 5.4.17 and Section 11.3.4)

ASIDE: Section 3.1.3

Complex Analysis Terminology

- **Zero/Pole/Order/Residue:** f has conv Taylor series at z_0 st
  $f(z) = a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \dots$
  ↳ $n \geq 1$ say zero of order n
  $n \leq -1$ say pole of order n with residue a_{-1}
- **Meromorphic/Analytic fn:** f is meromorphic at z_0 if the Taylor series converges for all z near z_0. In particular, there is a disk $D(z_0, r)$ and an integer n st $\forall z \in D(z_0, r)$, $f(z) = \sum_{n \geq n_0} a_n(z-z_0)^n$
  ↳ If meromorphic at each point in a disk, say meromorphic on the disk; if $n_0 \geq 0$ say f is analytic.
- **Meromorphic/Analytic Continuation:** Extending defn of fn:
  ↳ ex: $1+r+r^2+r^3+\dots$ initially makes sense only if $|r|<1$.
  But $\frac{1}{1-r}$ makes sense $\forall r \neq 1$ (or $\forall r$ and a pole at $r=1$): This is a meromorphic continuation of our original function.
- **Entire:** If a function is analytic $\forall z \in \mathbb{C}$, say entire

ASIDE: Liouville: Entire fast Contour integration ⇒ Fund Thm Alg

ANALYTIC / MEROMORPHIC CONTINUATION OF $\zeta(s)$

Gamma Fn: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \quad Re(s) > 0$

↳ $\Gamma(s+1) = s\Gamma(s)$ for $Re(s) > 1$ ($\int$ by parts)

↳ This is the fnal eq. of Γ, allows us to extend to $s \in \mathbb{C}$

↳ corr: $\Gamma(n+1) = n!$ (note $0! = 1$)

↳ aside: $\Gamma(1/2) = \sqrt{\pi}$

↳ occurs in prob (normalization constant of Gaussian)

ASIDE: Prove properties of $\Gamma(s)$, investigate occurances

THM 3.1.19 (Riemann): Analytic Cont of $\zeta(s)$

Define completed zeta function by $\xi(s) = \frac{1}{2} s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$

Then, though originally defined for $Re(s) > 1$, extends to all $s \in \mathbb{C}$ with only a simple pole of residue 1 at $s = 1$, and $\xi(s) = \xi(1-s)$

Proof: $\int_0^\infty x^{\frac{1}{2}s-1} e^{-n^2\pi x} dx = \frac{\Gamma(s/2)}{n^s \pi^{s/2}}$ (defn Γ-fn)

Sum over n with $Re(s) > 1$ so converge and can have $\sum \int = \int \sum$

(see Exe 11.4.12 for example where cannot interchange $\begin{matrix} & \uparrow & +1 \\ +1 & -1 & \\ -1 & & \rightarrow \end{matrix}$)

Setting $\omega(x) = \sum_{n=1}^\infty e^{-n^2\pi x}$, $\theta(x) = \sum_{n=-\infty}^\infty e^{-n^2\pi x}$, $\omega(x) = \frac{\theta(x)-1}{2}$

have $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1}\omega(x)dx$

$= \int_1^\infty x^{\frac{1}{2}s-1}\omega(x)dx + \int_1^\infty x^{-\frac{1}{2}s-1}\omega(\frac{1}{x})dx$

↳ Common technique: $\int_0^\infty = \int_1^\infty + \int_0^1$ and in $\int_0^1$ send $x \to x^{-1}$

Poisson Sum gives $\omega(\frac{1}{x}) = -\frac{1}{2} - \frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$

$\Rightarrow \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-1})\omega(x)dx$

↳ as ω decays RAPIDLY, integral makes sense for all s

↳ Fnl eq. for $\omega(x)$ from applying Poisson Sum to show $x^{\frac{1}{2}}\theta(x) = \theta(x^{-1}) = \sum_{n=-\infty}^\infty e^{-\pi n^2 x^{-1}} \quad (x > 0)$

3.2. Zeros of $\zeta(s)$ and Primes

Heuristic: why zeros give info on primes?

↳ Polynomials: $P(X) = A(X-r_1)\cdots(X-r_n)$
$$= A\left(X^n + a_{n-1}(r_1,\dots,r_n)X^{n-1} + \dots + a_0(r_1,\dots,r_n)\right)$$

Thus know zeros know coeffs and vice versa

ASIDE: Relations b/w roots and coeffs
Newton's identities + symmetric polynomials

Von Mangoldt fn: $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$

Set $\psi(x) = \sum_{n \le x} \Lambda(n)$

Complex Analysis Pre-Reqs

- Euler / Cotes / de Moivre: $e^{i\theta} = \cos\theta + i\sin\theta$
- $\frac{1}{2\pi i}\int_0^{2\pi} (re^{i\theta})^n re^{i\theta} i\,d\theta = \begin{cases} 1 & n = -1 \\ 0 & \text{otherwise} \end{cases}$
- $\frac{1}{2\pi i}\oint_\gamma z^n dz = \begin{cases} 1 & n = -1 \\ 0 & \text{otherwise} \end{cases}$ γ circle containing origin
- $\frac{1}{2\pi i}\int_{|z-z_0|=r} (z-z_0)^n dz = \begin{cases} 1 & n=-1 \\ 0 & \text{other} \end{cases}$
- $f(z)$ mero, for suff small r at most one pole in $D(z_0, r)$, namely z_0.
  Then $\frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)\,dz = a_{-1} = \mathrm{Res}_{z_0}(f)$
  ↳ Proof: justify $\int\sum = \sum\int$ with $f(z) = \sum_{n \ge n_0} a_n (z-z_0)^n$
  ↳ More generally, f meromorphic in A, $\frac{1}{2\pi i}\int_{\partial A} f(z)dz = \sum_{z\in A} \mathrm{Res}_z(f)$
    ↳ Similar to Green's Thm 

ZEROS OF $\zeta(s)$ AND PRIMES (CONT)

RESULTS FROM COMPLEX ANALYSIS (CONT)

- logarithmic derivative: $\frac{d}{ds} f(z) = f'(z)/f(z)$
  - ↳ VERY useful as converts a prod to sum (we like sums)
  - ↳ $\frac{1}{2\pi i} \oint_{|z-s|=r} \frac{f'(z)}{f(z)} dz = \sum_{n=1}^{N} \mathrm{ord}_f(z_n)$
  - ↳ $f(z) = \sum_{m \geq n} a_m (z-z_0)^m \Rightarrow \frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + h(z)$

    where $n = \mathrm{ord}_f(z_0)$ and $h(z) = \sum_{m=0}^{\infty} b_m (z-z_0)^m$

ASIDE: Prove results from Complex analysis

- $\oint \frac{(x/n)^s}{s} ds$ is basically $\begin{cases} 1 & n < x \\ 0 & \text{otherwise} \end{cases}$ (See Exe 3.2.32)

Exe 3.2.15: $\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$

$$\Rightarrow \oint \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = -\oint \sum_n \frac{\Lambda(n)}{n^s} \frac{x^s}{s} ds$$

|| shifting contours ||

$$x - \sum_{\rho : \zeta(\rho)=0} \frac{x^\rho}{\rho} = \sum_{n \leq x} \Lambda(n)$$

↳ See importance of Riemann Hypothesis: non-trivial ρ has $\mathrm{Re}(\rho) = \frac{1}{2}$

↳ makes error as "small" as possible ($\zeta(s) = \zeta(1-s)$)

↳ Can interpret as a CLT/square-root cancellation

↳ ASIDE: Hawkins primes and now RH is the CLT (though no Euler Product)

↳ PNT is "basically" $\mathrm{Re}(\rho) < 1$: See Exe 3.2.19 for proof

ASIDE: Product expansion, zeros on line, Li's constants + RH
Dirichlet characters / L-fns, Primes in arithm progression