Cramérs Conjecture

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Abstract

In this paper we prove the conjecture on prime gaps by Cramér.

1 Background

"How thoroughly it is ingrained in mathematical science that every real advance goes hand in hand with the invention of sharper tools and simpler methods which, at the same time, assist in understanding earlier theories and in casting aside some more complicated developments." – David Hilbert

Harald Cramér gave some of the most influential insights into prime differences. Cramér showed that assuming the truth of Riemann Hypothesis, one has $p_{n+1} - p_n = O(\sqrt{p_n} \log p_n)$. He subsequently conjectured in 1937 that $p_{n+1} - p_n = O((\log p_n)^2)$, and more specifically that

$$\limsup_{n\to\infty} (p_{n+1} - p_n)/(\log p_n)^2 = 1$$

which is a much tighter bound, than that implied by the Riemann Hypothesis. Cramér's approach was based on statistical and probabilistic grounds. These new insights, brought in a set of new problems. It became clear, that it was important to understand the extreme cases, along with the average (which had been explored through the prime number theorem). In 1931, Westzynthius proved

$$\limsup_{n\to\infty} (p_{n+1} - p_n)/\log p_n = \infty$$

This expression $(p_{n+1} - p_n)/\log p_n$ is now known as the merit of a prime gap. As of August 2009 the largest known merit is 35.31. So we have a long way to go before this merit reaches infinity. This also indicates the possibility, that extreme cases in the study of prime gaps, are relatively rare. Recently, in 2005, another spectacular breakthrough was made, when D. A. Goldston, J. Pintz and C. Y. Yıldırım together showed,

$$\liminf_{n\to\infty} (p_{n+1} - p_n)/\log p_n = 0$$

So, a few substantial breakthroughs were made in the theory of prime differences, but still majority of the area remains unexplored. Most cases of explorations in this unchartered territory was made using heuristic and probabilistic arguments, as was done by Cramér himself.

2 Cramér's Conjecture

Theorem (Cramér, 1937)**.** *In the big-oh notation we have,*

$$d_n = O((\log p_n)^2)$$

where, p_n is the n^{th} prime and $d_n = p_{n+1} - p_n$

Proof. We can see easily, that,

$$\int_{p_n}^{p_{n+1}} \frac{\mathrm{d}x}{\log x} \leq \frac{d_n}{\log p_n} \tag{1}$$

ie.,

$$Li(p_{n+1}) - Li(p_n) \leq \frac{d_n}{\log p_n}$$

where, $Li(x)$ is the logarithmic integral function.

From which we get,

$$\int_{p_n}^{p_{n+1}} \frac{\mathrm{d}x}{(\log x)^2} = Li(p_{n+1}) - Li(p_n) - \frac{p_{n+1}}{\log p_{n+1}} + \frac{p_n}{\log p_n}$$
$$\int_{p_n}^{p_{n+1}} \frac{\mathrm{d}x}{(\log x)^2} \leq \frac{d_n}{\log p_n} - \frac{p_{n+1}}{\log p_{n+1}} + \frac{p_n}{\log p_n}$$
$$\int_{p_n}^{p_{n+1}} \frac{\mathrm{d}x}{(\log x)^2} \leq \frac{p_{n+1}}{\log p_n} - \frac{p_{n+1}}{\log p_{n+1}}$$
$$\int_{p_n}^{p_{n+1}} \frac{\mathrm{d}x}{(\log x)^2} \leq \frac{p_{n+1}}{\log p_{n+1}} \left(\frac{\log p_{n+1}}{\log p_n} - 1 \right)$$

But we have,

$$\frac{d_n}{(\log p_{n+1})^2} \leq \int_{p_n}^{p_{n+1}} \frac{\mathrm{d}x}{(\log x)^2} \tag{2}$$

Hence, we get

$$\limsup_{n\to\infty} \frac{d_n}{(\log p_{n+1})^2} \leq \limsup_{n\to\infty} \frac{p_{n+1}}{\log p_{n+1}} \left(\frac{\log p_{n+1}}{\log p_n} - 1 \right) \tag{3}$$

Now, we have $p_n^{\frac{1}{n+1}} > p_n^{\frac{1}{p_n}}$. Also, as $n \to \infty$, $\lim_{n\to\infty} p_n^{\frac{1}{p_n}} = 1$. Moreover, we know from the prime number theorem, $\lim_{n\to\infty} p_{n+1}/p_n = 1$. Hence, we have

$$\liminf_{n\to\infty} p_n^{\frac{1}{n+1}} \geq \lim_{n\to\infty} p_n^{\frac{1}{p_n}} = 1$$

Which gives,

$$\liminf_{n\to\infty} p_n^{\frac{1}{n+1}} \geq \lim_{n\to\infty} \frac{p_{n+1}}{p_n}$$
$$\liminf_{n\to\infty} \exp\left(\frac{\log p_n}{n+1} \right) \geq \lim_{n\to\infty} \exp\left(\log \frac{p_{n+1}}{p_n} \right) \tag{4}$$

From which we get,

$$\liminf_{n\to\infty} \exp\left(\frac{p_{n+1}}{\log p_{n+1} \log p_n} \quad \frac{\log p_n}{n+1} \right) \geq \limsup_{n\to\infty} \exp\left(\frac{p_{n+1}}{\log p_{n+1} \log p_n} \log \frac{p_{n+1}}{p_n} \right)$$

ie.,

$$\liminf_{n\to\infty} \exp\left(\frac{p_{n+1}}{(n+1)\log p_{n+1}} \right) \geq \limsup_{n\to\infty} \exp\left(\frac{p_{n+1}}{\log p_{n+1}} \left(\frac{\log p_{n+1}}{\log p_n} - 1 \right) \right) \tag{5}$$

From the prime number theorem we know that $\lim_{n\to\infty} p_n/(n \log p_n) = 1$,

$$\exp(1) \geq \limsup_{n\to\infty} \exp\left(\frac{p_{n+1}}{\log p_{n+1}} \left(\frac{\log p_{n+1}}{\log p_n} - 1 \right) \right)$$

Taking logarithm we get,

$$1 \geq \limsup_{n\to\infty} \frac{p_{n+1}}{\log p_{n+1}} \left(\frac{\log p_{n+1}}{\log p_n} - 1 \right) \tag{6}$$

Hence, we get

$$\limsup_{n\to\infty} \frac{d_n}{(\log p_{n+1})^2} \leq \limsup_{n\to\infty} \frac{p_{n+1}}{\log p_{n+1}} \left(\frac{\log p_{n+1}}{\log p_n} - 1 \right) \leq 1 \tag{7}$$

Applying Bertrand's postulate we get,

$$\limsup_{n\to\infty} \frac{d_n}{(\log(2p_n))^2} \leq 1$$

Which gives us our required result, i.e.,

$$d_n = O((\log p_n)^2)$$

□

3 Conclusion

"Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate." – Leonhard Euler

Every research paper is incomplete, in the sense that, there's a lot more that is left to be written. I hope the reader has understood what I wanted to express, regarding the usefulness of the tools presented in this paper.

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