Sequences

(2.2.1) **Sequence**. A sequence is a function whose domain is **N**.

(2.2.3) **Convergence of a Sequence.** A sequence (a_n) converges to a real number a if, for every $\varepsilon > 0$, there exists an $N \in$ **N** such that whenever $n \geq N$, it follows that $|a_n - a| < \varepsilon$.

(2.2.3B) **Convergence of a Sequence, Topological Characterization.** A sequence (a_n) converges to a if, given any ε-neighborhood $V_\varepsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\varepsilon(a)$ $\Leftrightarrow$ every $V_\varepsilon(a)$ contains all but a finite number of terms of (a_n).

(2.2.7) **Uniqueness of Limits.** The limit of a sequence, when it exists, must be unique.

(2.2.9) **Divergence**. A sequence that does not converge is said to diverge.

(2.3.1) **Bounded**. A sequence (x_n) is bounded if $\exists$ $M > 0$ such that $|xn| < M$ for all $n \in$ **N**.

(2.3.2) Every convergent sequence is bounded.

(2.3.3) **Algebraic Limit Theorem**. Let $\lim a_n = a$ and $\lim b_n = b$. Then, (i) $\lim(ca_n) = ca$ for all $c \in$ **R**; (ii) $\lim(a_n + b_n) = a + b$; (iii) $\lim(a_n b_n) = ab$; (iv) $\lim(a_n/b_n) = a/b$ provided $b \neq 0$.

(2.3.4) **Order Limit Theorem**. Assume $\lim a_n = a$ and $\lim b_n = b$. Then, (i) if $a_n \geq 0$ for all $n \in$ **N**, then $a \geq 0$; (ii) if $a_n \geq b_n$ for every $n \in$ **N**, then $a \geq b$; (iii) If there exists $c \in$ **R** for which $c \leq a_n$ for all $n \in$ N, then $c \leq a$.

(2.4.3) **Convergence of a Series.** Let (b_n) be a sequence, and define the corresponding sequence of partial sums (s_m) of the series Σb_n where $s_m = b_1 + b_2 + \ldots + b_m$. The series Σb_n converges to B if the sequence (s_m) converges to B. Thus, $\Sigma b_n = B$.

(2.4.6) **Cauchy Condensation Test.** Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in$ **N**. Then, the series Σb_n converges if and only if the series $\Sigma 2^n b_{2n} = b_1 + 2b_2 + 4b_4 + 8b_8 + \ldots$ converges.

(2.4.7) The series $\Sigma 1/n^p$ converges if and only if $p > 1$.

(2.5.1) **Subsequences.** Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \ldots.$ be an increasing sequence of natural numbers. Then the sequence $(a_{n1}, a_{n2}, a_{n3}, \ldots)$ is called a subsequence of (a_n) and is denoted by (a_{nk}), where $k \in$ **N** indexes the subsequence.

(2.5.2) Subsequence of a convergent sequence converge to the same limit as the original sequence.

(2.5.5) **Bolzano-Weierstrass Theorem**. Every bounded sequence contains a convergent subsequence.

(2.6.1) **Cauchy Sequence**. A sequence (a_n) is called a Cauchy sequence if, for every $\varepsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $m, n \geq N$, it follows that $|a_n - a_m| < \varepsilon$.

(2.6.2) Every convergent sequence is a Cauchy sequence.

(2.6.3) Cauchy sequences are bounded.

(2.6.4) **Cauchy Criterion.** A sequence converges if and only if it is a Cauchy sequence.

Series

(2.7.1) **Algebraic Limit Theorem for Series.** If $\Sigma a_k = A$ and $\Sigma b_k = B$, then (i) $\Sigma ca_k = cA$ for all $c \in \mathbf{R}$ and (ii) $\Sigma(a_k + b_k) = A + B$.

(2.7.2) **Cauchy Criterion for Series.** The series Σa_k converges if and only if, given $\varepsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$, it follows that $|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon$.

(2.7.3) If the series Σa_k converges, then the sequence (a_k) converges to 0.

(2.7.4) **Comparison Test.** Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k \ \forall\ k \in \mathbf{N}$. Then, (i) if Σb_k converges, then Σa_k converges and (ii) if Σa_k diverges, then Σb_k diverges.

(2.7.6) **Absolute Convergence Test.** If the series $\Sigma |a_n|$ converges, then Σa_n converges as well.
jl12

(2.7.7) **Alternating Series Test.** Let (a_n) be a sequence satisfying (i) $a_1 \geq a_2 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots$ and (ii) (a_n) converges to 0. Then, the alternating series $\Sigma(-1)^{n+1}a_n$ converges.

Topology of The Reals

(3.2.1) **Open**. A set O is open if for all points $a \in O$, there exists a $V_\varepsilon(a) \subseteq O$.

(3.2.3) (i) The union of an arbitrary collection of open sets is open. (ii) The intersection of a finite collection of open sets is open.

(3.2.4) **Limit Point.** A point x is a limit point of a set A if every $V_\varepsilon(x)$ intersects the set A at some point other than x.

(3.2.5) A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbf{N}$.

(3.2.6) **Isolated Point**. A point $a \in A$ is an isolated point of A if it is not a limit point of A.

(3.2.7) **Closed**. A set $F \subseteq \mathbf{R}$ is closed if it contains its limit points.

(3.2.8) A set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

(3.2.10) **Density of Q in R**. For every $y \in \mathbf{R}$, there exists a sequence of rational numbers that converges to y.

(3.2.11) **Closure**. Given a set $A \subseteq \mathbf{R}$, let L be the set of all limit points of A. The closure of A is defined to be $\text{Cl}(A) = A \cup L$.

(3.2.12) For any $A \subseteq \mathbf{R}$, the closure of A is a closed set and is the smallest closed set containing A.

(3.2.13) A set O is open if and only if O^c is closed.

(3.2.14) (i) The union of a finite collection of closed sets is closed. (ii) The intersection of an arbitrary collection of closed sets is closed.

(3.3.1) **Compactness**. A set $K \subseteq \mathbf{R}$ is compact if every sequence in K has a subsequence that converges to a limit that is also in K.

(3.3.3) **Bounded**. A set $A \subseteq \mathbf{R}$ is bounded if there exists $M > 0$ such that $|a| < M$ for all $a \in A$.

(3.3.4) **Characterization of Compactness in R**. A set $K \subseteq \mathbf{R}$ is compact if and only if it is closed and bounded.

(3.3.5) If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ is a nested sequence of nonempty compact sets, then the intersection $\cap K_n$ is not empty.

(3.3.6) **Open Cover.** Let $A \subseteq \mathbf{R}$. An open cover for A is a (possibly infinite) collection of open sets $\{O_d : d \in D\}$ whose union contains the set A. A finite subcover isa a finite sub collection of open sets from the original open cover whose union still manages to completely contain A.

(3.3.8) **Heine-Borel Theorem**. Let K be a subset of $\mathbf{R}$. All of the following statements are equivalent in the sense that any one of them implies the two others: (i) K is compact; (ii) K is closed and bounded; (iii) Every open over for K has a finite subcover.

Functional Limits

(4.2.1) **Functional Limit.** Let f be defined on A, and let c be the limit point of A. Then, $\lim_{x \to c} f(x) = L$ provided that for all   > 0, ∃   > 0 such that whenever $0 < |x - c| <$ it follows that $|f(x) - L| <$.

(4.2.1B) **Functional Limit - Topological Characterization**. Let c be the limit point of the domain of f. We say $\lim_{x \to c} f(x) = L$ provided that, for every $V\ (L)$ of L, there exists a $V\ (c)$ such that for all $x \in V\ (c)$ it follows that $f(x) \in V\ (L)$.

(4.2.3) **Sequential Criterion for Functional Limits**. Given a function f defined on A and a limit point c of A, then $\lim_{x \to c} f(x) = L \Leftrightarrow$ for all sequences $(x_n) \subseteq A$ satisfying $x_n \neq x$ and $(x_n) \to c$, then $f(x_n) \to L$.

(4.2.4) **Algebraic Limit Theorem for Functional Limits**. Let f and g be functions defined on domain $A \subseteq \mathbf{R}$, and assume that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point $c \in A$. Then, (i) $\lim_{x \to c} kf(x) = kL$ for all $k \in \mathbf{R}$, (ii) $\lim_{x \to c}(f(x) + g(x)) = L + M$; $\lim_{x \to c}(f(x)g(x)) = LM$ and (iv) $\lim(f(x)/g(x)) = L/M$, provided $M \neq 0$.

(4.2.5) **Divergence Criterion for Functional Limits**. Let f be a function defined on A, and c be a limit point of A. If there exists two sequences (x_n), (y_n) with $x_n \neq c$ and $y_n \neq c$ and $\lim_{x \to c} x_n = \lim_{x \to c} y_n = c$ but $\lim_{x \to c} f(x_n) \neq \lim_{x \to c} f(y_n)$, then $\lim f(x)$ does not exist.

(4.3.1) **Continuity**. A function f is continuous at a point $c \in A$ if, for all $\quad > 0$, there exists a $\quad > 0$ such that whenever $|x - c| < \quad$ it follows that $|f(x) - f(c)| < \quad$. If f is continuous at every point in the domain A, then f is continuous on A.

(4.3.2) **Characterizations of Continuity**. Let f, defined on A, and $c \in A$. The function f is continuous at c if and only if any one of the following conditions is met: (i) For all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$; (ii) For all $V\ (f(c))$, there exists a $V\ (c)$ such that $x \in V\ (c)$ implies $f(x) \in V\ (f(c))$; (iii) If $(x_n) \to c$, then $f(x_n) \to f(c)$; If c is a limit point of A, then the above conditions are equivalent to (iv) $\lim_{x \to c} f(x) = f(c)$.

(4.3.3) **Criterion for Discontinuity**. Let f, defined on A, and $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.

(4.3.4) **Algebraic Continuity Theorem**. Assume f, g defined on A, continuous at a point $c \in A$. Then, (i) $kf(x)$ is continuous at $c\ \forall\ k \in \mathbf{R}$; (ii) $f(x) + g(x)$ is continuous at c; (iii) $f(x)g(x)$ is continuous at c; and (iv) $f(x)/g(x)$ is continuous at c, provided the quotient is defined.

(*) All polynomials are continuous on **R**.

(4.3.9) **Compositions of Continuous Functions**. Given f defined on A and g defined on B, and assume the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \cdot f(x) = g(f(x))$ is defined on A. If f is continuous at $c \in A$, and g is continuous at $f(c) \in B$, then $g(f(x))$ is continuous at c.

(4.4.1) **Preservation of Compact Sets**. Let f defined on A be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.

(4.4.2) **Extreme Value Theorem**. If f, defined on K compact, is continuous on $K \subseteq \mathbf{R}$, then f attains a maximum and a minimum value. In other words, there exists $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.

(4.4.4) **Uniform Continuity**. A function f defined on A is uniformly continuous on A if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

(4.4.5) **Sequential Criterion for Absence of Uniform Continuity**. A function defined on A fails to be uniformly continuous on A if and only if there exists a particular $\varepsilon_0 > 0$ and two sequences (x_n), (y_n) in A satisfying $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

(4.4.7) **Uniform Continuity on Compact Sets**. A function that is continuous on a compact set K is uniformly continuous on K.

(4.5.1) **Intermediate Value Theorem**. Let f be defined on $[a, b]$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ such that $f(c) = L$.

(4.5.3) **Intermediate Value Property**. A function f has the intermediate value property on an interval $[a, b]$ if for all $x < y$ in $[a, b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x, y)$ where $f(c) = L$.

Sequences of Functions

(6.2.1) **Pointwise Convergence:** For each $n \in N$, let f_n be a function defined on set $A \subseteq \mathbf{R}$. The sequence of functions converges pointwise on A to a function f if, (1) for all $x \in A$, the sequence pf real numbers $f_n(x)$ converges to $f(x)$ $\Leftrightarrow$ for every > 0 and $x \in A$, there exists an N such that $|f_n(x) - f(x)| <$ $\forall\ n \geq N$.

(6.2.3) **Uniform Convergence:** Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbf{R}$. Then, (f_n) converges uniformly on A to a limit function f defined on A if, for every > 0, there exists an $N \in \mathbf{N}$ such that $|f_n(x) - f(x)| <$ whenever $n \geq N$ and $x \in A$.

(6.2.5) **Cauchy Criterion for Uniform Convergence:** A sequence of functions (fn) defined on a set A $\subseteq$ R converges uniformly on A if and only if for every $\varepsilon > 0$, there exists an N $\in$ N such that |fn(x) - fm(x)| < ε whenever m, n $\geq$ N and x $\in$ A.

(6.2.6) **Continuous Limit Theorem:** Let (f_n) be a sequence of functions defined on $A \subseteq \mathbf{R}$ that converges uniformly on A to a function f. If each fn is continuous at c $\in$ A, then f is continuous at c.

(6.3.1) **Differentiable Limit Theorem:** Let $f_n \to f$ pointwise on the closed interval [a, b], and assume that each f_n is differentiable. If (f_n') converges uniformly on [a, b] to a function f, then the function f is differentiable and $f' = g$.

(6.3.2) **Weaker Differentiability Limit Theorem:** Let (f_n) be a sequence of differentiable functions defined on the closed interval [a, b], and assume (f_n') converges uniformly on [a, b]. If there exists a point $x_0 \in [a, b]$ where $f_n(x_0)$ is convergent, then (f_n) converges uniformly on [a, b].

(6.3.3) **Stronger Differentiable Limit Theorem:** Let (f_n) be a sequence of differentiable functions defined on the closed interval [a, b], and (f_n') converges uniformly to a function g on [a, b]. If there exists a point $x_0 \in [a, b]$ where $f_n(x_0)$ is convergent, then (f_n) converges uniformly. Moreover, the limit function $f = \lim f_n$ is differentiable and satisfies $f' = g$.

Series of Functions

(6.4.1) **Convergence of Series of Functions**: For each $n \in \mathbf{N}$, let f_n and f be functions defined on a set $A \subseteq \mathbf{R}$. The infinite series $\Sigma f_n(x)$ ***converges pointwise*** on A to $f(x)$ if the sequence $s_k(x)$ of partial sums defined by $s_k(x) = f_1(x) + f_2(x) + \ldots + f_k(x)$ converges pointwise to $f(x)$. The series ***converges uniformly*** on A to f if the sequences $s_k(x)$ converges uniformly on A to $f(x)$.

(*) If have series in which functions f_n are continuous, then by the Algebraic Continuity Theorem the partial sums will be continuous as well.

(6.4.2) **Term by Term Continuity Theorem.** Let f_n be continuous functions defined on a set $A \subseteq \mathbf{R}$, and assume that Σf_n converges uniformly to a function f. Then, f is continuous on A. **Proof idea:** Apply Continuous Limit Theorem (6.2.6) to partial sums $s_k = f_1 + f_2 + \ldots + f_k$.

(6.4.3) **Term by Term Differentiability Theorem**. Let f_n be differentiable functions defined on an interval A, and assume that $\Sigma f_n'(x)$ converges uniformly to a limit $g(x)$ in A. If there exists a point $x_0 \in$ [a, b] where $\Sigma f_n(x_0)$ converges, then the series $\Sigma f_n(x)$ converges uniformly to a differentiable function $f(x)$ satisfying $f'(x) = g(x)$ on A. In other words, $f(x) = \Sigma f_n(x)$ and $f'(x) = \Sigma f_n'(x)$. **Proof idea:** Apply the Stronger Differentiable Limit Theorem to the partial sums $s_k = f_1 + f_2 + \ldots + f_k$. and observe that the Algebraic Differentiability Theorem (5.2.4) implies that $s_k' = f_1' + f_2' + \ldots + f_k'$

(6.4.4) **Cauchy Criterion for Uniform Convergence of a Series**. A series Σf_n converges uniformly on $A \subseteq \mathbf{R}$ if and only if for every $\quad > 0$, there exists an $N \in \mathbf{N}$ such that $|f_{m+1}(x) + f_{m+2}(x) + \ldots + f_n(x)| <$ whenever $n > m \geq N$ and $x \in A$.

(6.4.5) **Weierstrass M-Test**. For each $n \in \mathbf{N}$, let f_n be a function defined on a set $A \subseteq \mathbf{R}$ and let $M_n > 0$ be a real number satisfying $|f_n(x)| \leq M_n$ for all $x \in A$. If ΣM_n converges, then Σf_n converges uniformly on A. **Proof idea**: Cauchy Criterion and the triangle inequality.

Power Series: functions of the form $f(x) = \Sigma a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

(6.5.1) If a power series $\Sigma a_n x^n$ converges at some point $x_0 \in \mathbf{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$. **Proof Idea:** Since the series converges, then the sequence of terms is bounded (converges to 0). Using the hypothesis (if $x \in \mathbf{R} : |x| < |x_0|$), find series of $M|x/x_0|^n$ to be geometric with ratio $|x/x_0| < 1$, so converges and thus by Comparison Test, converges absolutely.

(*) Implies that the set of points for which a given power series converges must necessarily be {0}, **R**, or a bounded interval centered around $x = 0$. R is referred to as the radius of convergence of a power series.

(6.5.2) If a power series $\Sigma a_n x^n$ converges absolutely at a point x0, then it converges uniformly on the closed interval [-c, c] where c = |x0|. **Proof Idea**: Application of the Weierstrass M-Test.

(*) if the power series g(x) = $\Sigma a_n x^n$ converges conditionally at x = R, then it is possible for it to diverge when x = -R. Sample with R = 1: $\Sigma(-1)^n x^n/n$.

(6.5.3) **Abel's Lemma.** Let b_n satisfy $b_1 \geq b_2 \geq b_3 \geq \ldots \geq 0$, and let Σa_n be a series for which the partial sums are bounded. In other words, assume that there exists $A > 0$ such that $|a_1 + a_2 + \ldots + a_n| \leq A$ for all $n \in \mathbf{N}$. Then for all $n \in \mathbf{N}$, $|a_1 b_1 + a_2 b_2 + \ldots + a_n b_n| \leq A b_1$.

(6.5.4) **Abel's Theorem**. Let $g(x) = \Sigma a_n x^n$ be a power series that converges at the point $x = R > 0$. Then the series converges uniformly on the interval $[0, R]$. (Similar result for $x = -R$.)

(6.5.5) If a power series converges pointwise on the set $A \subseteq R$, then it converges uniformly on any compact set $K \subseteq A$. **Proof idea**: Apply Abel's Theorem (6.5.4) to the max and min of the compact set K.

(*) Power series is continuous at every point at which it converges.

(6.5.6) If $\Sigma a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\Sigma n a_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on compact sets contained in $(-R, R)$.

(*) Series can converge at endpoint, but differentiated series can diverge. Ex: $\Sigma x^n/n$ at $x = -1$.

(6.5.7) Assume $f(x) = \Sigma a_n x^n$ converges on an interval $A \subseteq \mathbf{R}$. Then, the function f is continuous on A and differentiable on any open interval $(-R, R) \subseteq A$. Moreover, the derivative is given by $f'(x) = \Sigma n a_n x^{n-1}$ and f is infinitely differentiable on $(-R, R)$, and the successive derivatives can be obtained via term by term differentiation of the appropriate series.

Results from psets:

4W:

- The limit of a sequence, if it exists, must be unique. First, assume lim $a_n = a$ and lim $a_n = b$, and proceed to show that $a = b$.
- (Reverse Triangle Inequality): $|a + b| \leq |a| + |b| \Rightarrow$ Inverse Triangle Inequality: $|a - b| \geq ||a| - |b||$.
- For sequences (x_n), (y_n):
  - (x_n) and (yn) divergent but $(x_n + y_n)$ convergent; $x_n = n$, $y_n = -n$.
  - (x_n) convergent and (y_n) convergent, and $(x_n + y_n)$ converges; impossible by the ALT
  - (b_n) convergent with $b_n \neq 0\ \forall\ n$: $(1/b_n)$ convergent; $b_n = 1/n$
  - unbounded (a_n) and convergent (b_n) and $(a_n - b_n)$ bounded; impossible
  - (a_n), (b_n) such that (a_nb_n) converges but (b_n) does not; $(a_n) = 0$, $(b_n) = n$.

4F:

- (Squeeze Theorem): If $x_n \leq y_n \leq z_n\ \forall\ n \in \mathbf{N}$ and lim x_n = lim z_n = L, then lim y_n = L.
- (Cesaro Means): If (x_n) is a convergent sequence, then the sequence given by the averages $y_n = n^{-1}(x_1 + x_2 + \ldots + x_n)$ also converges to the same limit. Note: it is possible for (y_n) of averages to converge even if (x_n) does not. Example: $x_n = (-1)^n$
- (Limit Superior): lim sup $a_n = \lim_{n\to\infty} y_n$ where $y_n = \sup\{a_k : k \geq n\}$
  - y_k converges
  - lim inf $a_n = \lim_{n\to\infty} x_n$ where $x_n = \inf\{a_k : k \geq n\}$
  - lim inf $a_n \leq$ lim sup a_n for every bounded sequence.
    - Strict inequality when lim inf a_n = -1 lim sup a_n = 1.
  - lim inf a_n = lim sup a_n if and only if lim an exists, and all three values are equal.

5W:

- For (a_n), (b_n) Cauchy, we have that:
  - $c_n = |a_n - b_n|$ is Cauchy while $c_n = (-1)^n a_n$ is not Cauchy.

5F:

- (Infinite product) $\prod b_n = b_1b_2b_3\ldots$
  - Understood in terms of sequence of partial products $p_m = \prod b_n = b_1b_2\ldots b_m$
  - The sequence of partial products converges if and only if $\sum a_n$ converges.
- If $a_n > 0$ and lim $(na_n) = L \neq 0$, then $\sum a_n$ diverges.
- Assume that $a_n > 0$ and lim n^2a_n exists. Then $\sum a_n$ converges.
- For sequence (a_n):
  - If $\sum a_n$ converges absolutely, then $\sum a_n^2$ converges absolutely

    - FALSE: If Σa_n converges and (b_n) converges, then $\Sigma a_n b_n$ converges. Counterexample: $a_n = b_n = (-1)^n(\sqrt{n})^{-1}$
    - If Σa_n converges conditionally, then $\Sigma n^2 a_n$ diverges.
- (Ratio Test): Given series Σa_n with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies lim $|a_{n+1}/a_n| = r < 1$, then the series converges absolutely.

6W:

6F:

7W:

7F:

- Lipschitz Condition
    - A function f is called Lipschitz if there exists a bound $M > 0$ such that $|(f(x) - f(y))/(x - y)| \leq M$ for all $x \neq y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.
    - If f defined on A is Lipschitz, then it is uniformly continuous on A.
- Inverse function + Topological Characterization of Continuity
    - Let g be defined on all of **R**. If $B \subseteq$ **R**, define the set $g^{-1}(B)$ by $g^{-1}(B) = \{x \in \mathbf{R} : g(x) \in B\}$
        - g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq$ **R** is an open set
        - if f is a continuous function defined on **R**,
            - $g^{-1}(K)$ is not necessarily compact whenever K is compact
            - $g^{-1}(F)$ is closed whenever F is closed

8W:

8F:

9W:

9F: