Math 317: Operations Research
First Lecture

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http://www.williams.edu/Mathematics/sjmiller/public_html/317

Williams College

Introduction and Objectives

Introduction / Objectives

Main Topic: Optimization: Linear Programming.

Objectives

- Obviously learn linear programming.
- Emphasize techniques / asking the right questions.
- Model problems and analyze model.
- Elegant solutions vs brute force.
- Writing textbook for AMS.

Types of Problems

- Diet problem.
- Banking (asset allocation).
- Scheduling (movies, airlines, TSP, MLB).
- Elimination numbers.
- Sphere packing....

My (applied) experiences

- Marketing: parameters for linear programming (SilverScreener).
- Data integrity: detecting fraud with Benford's Law (IRS, Iranian elections).
- Sabermetrics: Pythagorean Won-Loss Theorem.

Course Mechanics

Grading / Administrative

- HW: 15%. Midterm 40%. Final/Project 40%. Class Participation 5%. May change a bit. A large portion of work/grade from a group project: you'll give a talk, prepare a well-crafted manuscript, and respectfully listen to reports of others.

- Pre-reqs: linear algebra (analysis, stats, programming a plus).

Office hours / feedback

- TBD and when I'm in my office (schedule online).
- Feedback ephsmath@gmail.com, password first 8 Fibonacci numbers (011235813).

Other

- Webpage: numerous handouts, additional comments each day (mix of review and optional advanced material).
- Opportunity to help write a book.
- PREPARE FOR CLASS! Must do readings before each class.

Other: Advice from Jeff Miller

- Party less than the person next to you.

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Happy to do practice interviews, adjust deadlines....

Linear algebra textbooks online: `http://joshua.smcvt.edu/linalg.html/book.pdf`

Useful links

LaTeX and Mathematica Tutorials and Templates

http://web.williams.edu/Mathematics/sjmiller/public_html/math/handouts/latex.htm

Has templates for using LaTeX for papers, talks, posters, and a Mathematica tutorial.

Also videos on each.

Examples / Jobs

Alabama vs Auburn: 2013

https://www.youtube.com/watch?v=sLO2SmM9gPw

Log ruler (and WCMA)

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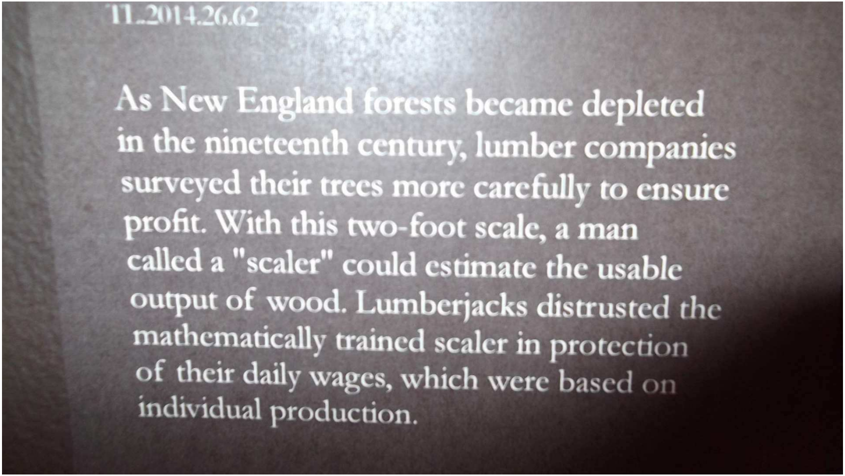

Scheduling: Baseball Tournaments, Swim Lessons

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Inefficiencies from Location

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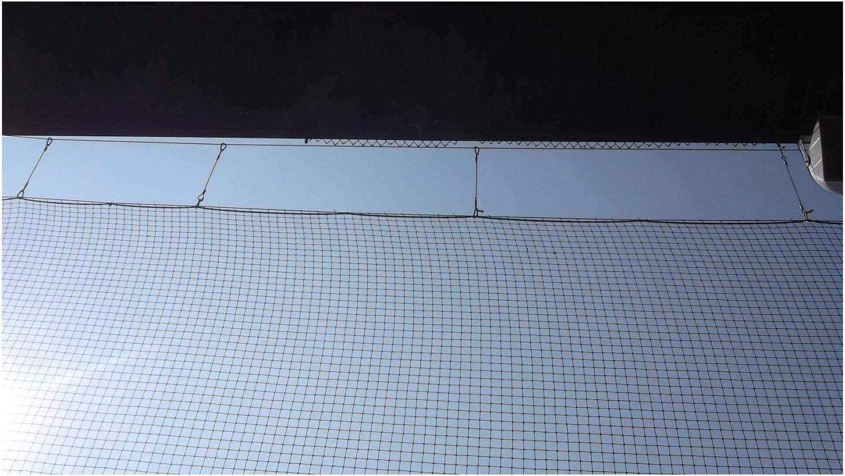

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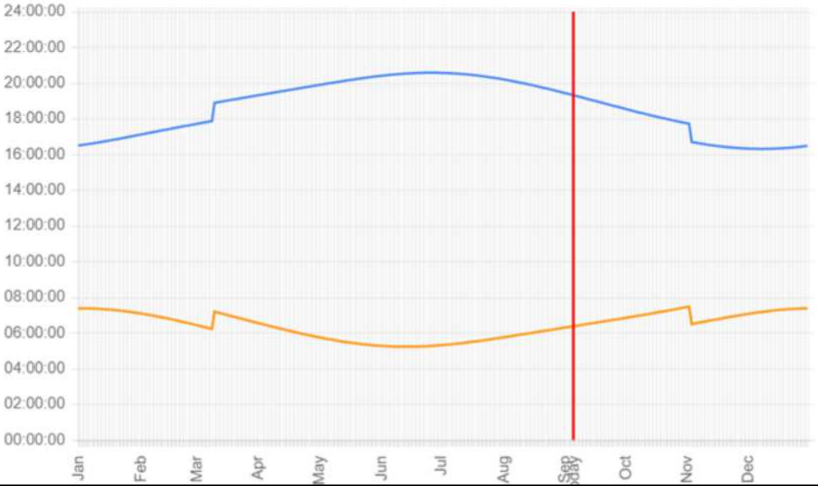
Maps

Year distribution of sunrise and sunset times in North Adams, MA – 2019

https : // sunrise – sunset.org / us / north – adams – ma



Maps

Who America is rooting for in the Super Bowl:

Pascal's Triangle

Pascal's Triangle

Video on Pascal's Triangle

https://www.youtube.com/watch?v=tt4_4YajqRM

Fast Multiplication

Cost of Standard Polynomial Evaluation

Multiplication far more expensive than addition....

$f(x) = 3x^5 - 8x^4 + 7x^3 + 6x^2 - 9x + 2$: Cost is $5 + 4 + 3 + 2 + 1 + 0 = 15$ multiplications.

These are triangle numbers: degree d have $d(d+1)/2$.

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Thus $2S(d) = d \cdot (d+1)$ and claim follows.

Horner's Algorithm

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Cost is degree d multiplications!

Useful also in fractal plotting.... Shows can often do common tasks faster.

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x^{16} \cdot x^{16} &= x^{32} \\
x^{32} \cdot x^{32} &= x^{64}
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Recap

Horner takes us from order d^2 to order d.

Fast multiplication takes us to order $\log_2 d$, but only for special polynomials; these though are the ones used in RSA!

Euclidean Algorithm

Preliminaries

Input x, y with $y > x$.

Goals: find $\gcd(x, y)$, find a, b so that $ax + by = \gcd(x, y)$.

Lot of ways to go: non-constructive proofs of a, b but need values; Euclidean algorithm is *very* fast.

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Continue until....

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Note $\gcd(r_0, r_1) = \gcd(r_1, r_2) = \gcd(r_2, r_3), \ldots$.

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Can 'climb upwards' to get a, b such that $ax + by = \gcd(x, y)$.

Fermat's little Theorem

Euler totient function

$\phi(n)$ is the number of integers from 1 to n relatively prime to n.

$\phi(p) = p - 1$ and $\phi(pq) = (p-1)(q-1)$ if p, q distinct primes.

Do not need, but $\phi(mn) = \phi(m)\phi(n)$ if $\gcd(m, n) = 1$, and $\phi(p^k) = p^k - p^{k-1}$.

A lot of group theory lurking in the background, only doing what absolutely need.

Fermat's little Theorem

Fermat's little Theorem (FlT)

Let a be relatively prime to n. Then $a^{\phi(n)} = 1 \mod n$.

Special cases: $a^{p-1} = 1 \mod p$, $a^{(p-1)(q-1)} = 1 \mod pq$.

Will only prove these two cases....

Proof of Fermat's little Theorem: $n = p$

Proof: Let $n = p$, let $\gcd(a, p) = 1$.

Consider $1, 2, \ldots, p-1$ and $a, 2a, \ldots, (p-1)a$.

Claim both sets are all residues modulo p.

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Thus $(p - 1)! = (p - 1)!a^{p-1} \bmod p$, so $a^{p-1} = 1 \bmod p$. $\square$

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Note: General case: $x_1, \ldots, x_{\phi(n)}$ and $ax_1, \ldots, ax_{\phi(n)}$.

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Thus $a^{(p-1)(q-1)}$ is $1 \mod p$ and is $1 \bmod q$.

$a^{(p-1)(q-1)} = 1 + \alpha p = 1 + \beta q$.

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$a^{(p-1)(q-1)} = 1 + \alpha p = 1 + \beta q$.

Thus $\alpha p = \beta q$ so $q|\alpha$ and $p|\beta$, so $a^{(p-1)(q-1)} = 1 \mod pq$.
$\square$

Primality Tests from FIT

If $\gcd(a, n) = 1$ and $a^{n-1} \neq 1 \mod n$ then n cannot be prime.

If equalled 1 then n ***might be*** prime.

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- If can take high powers, very fast!
- Can suggest candidate primes, and then use better, slower test for certainty.
- Carmichael numbers: Composites that are never rejected: 561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341, ... (OEIS A002997).