MATH 331: THE LITTLE QUESTIONS: COMMENTS ON PROBLEMS

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ABSTRACT. These notes are expanded comments about some of the problems discussed in class.

CONTENTS

1. COVERING A SPHERE

Exercise 1.1. *Let S be a unit sphere, and let $H_1, \dots, H_n$ be hemisphere subsets of S. If $S \subset H_1 \cup \cdots \cup H_n$ with $n \geq 4$, then there is a subset of the H_i's of size four that covers S; in other words, if S is covered by $n \geq 4$ hemispheres then there is a sub-cover using just four hemispheres.*

Some general comments:

- Whenever you have a problem with numbers, you want to see what the numbers are, and which matter. Sometimes the numbers are clear (such as n spheres and a subcover of size 4), while other times the numbers are hidden (the hidden number here is that we're in 3-dimensional space, although as someone said in class perhaps it's better to view it as we have a 2-dimensional surface).
- It is often useful to consider simple cases and generalize. For this problem, instead of considering a sphere in 3-dimensions we can consider a circle in 2-dimensions, or two points in 1-dimension. Often these extra data points help us see a pattern.

What is the relationship between the numbers? Why 4? Maybe it's 4 is 3+1. To see if that's right, it's worth looking at other dimensions....

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Key words and phrases. Problem Solving, Induction.

The comments here joint work of discussions with students in and outside of class; it is a pleasure to thank everyone who spoke up for contributing.

1.1. **1-Dimensional Case:** The 1-dimensional case is clear. In that case we have the boundary of the 1-dimensional unit sphere is just the points $\{-1, 1\}$. In this case half-spheres are just singleton points, and we see any cover must have a subcover of just two items. Why? The only possibilities for the half-spheres are $\{-1\}$ and $\{1\}$, and we clearly don't need two copies of the same. Thus in the 1-dimensional case we just need 2 half-spheres. Note this does support our belief that the number needed is just one more than the dimension.

1.2. **2-Dimensional Case:** Things are now more interesting as we turn to the 2-dimensional case. There were a lot of great suggestions in class, but the arguments were not mathematically precise, and they weren't elegant (until the final one about a minimal sector). Frequently in math riddles and problems the final answer or argument is elegant. It leads to an a-ha moment (at least if it's a good problem!), so if we haven't hit it, we haven't done the problem as well as possible yet. A good sign that we haven't found the right perspective yet is when our argument seems tortuous, full of words and phrases like 'clearly', 'obviously', 'we must have'.

The sector argument was very nice: Look at all pairs of semi-circles. Either their union is the entire circle (in which case we're done) or they exclude a sector. So, let's take the pair whose excluded sector has minimal area (or equivalently minimal arc length). If another semicircle covers the entire missing sector we're done and have three whose union covers the circle. Thus, assume no other semicircle covers the entire missing region. As every point is covered, we must have some semicircle covering some of this. We then look and compare how much we miss if we use this new semicircle with each of the two semicircles being considered. *After some work we can show that one of these configurations leads to a smaller missed region, violating minimality.*

While I like this minimality idea and think it can be made to work, a bit more detail is needed. Thus, I'd like to suggest another approach to this problem. We proceed by induction.

- Base case: Assume we have the boundary of the disk covered by three semi-circles. Then there is a subset of three semi-circles that covers the boundary of the disk. OK, this isn't that hard to prove!

- Inductive step: Assume that whenever we have the boundary of the disk covered by $n \geq 3$ semicircles there is a subset of three that cover; we must show if $n+1$ semicircles cover then a subset of 3 must cover. To do this, it suffices to show that we can *always* remove one semicircle from our list and be left with a set of n semicircles that still cover, as we will then be done by induction.

  We show that if we ever have four semi-circles, we can always remove at least one without affecting the region they cover. Let their centers be at C_1, C_2, C_3 and C_4; without loss of generality we may assume C_1 is at $(1, 0)$ and the centers are labeled counter-clockwise (as that's how mathematicians walk).

  Actually, rather than viewing the centers as a point in the plane, it's better to view them as an angle; thus the centers are at the angles $\theta_1 = 0, \theta_2, \theta_3, \theta_4$, and we consider θ_3 the angle before θ_4, and θ_1 the angle after θ_4. Here's the key (and elegant!) observation.

> *If we ever have an angle between two angles whose distance is less than 180 then it is not needed, and we can safely remove it without affecting what we cover.*

There are thus three possibilities for θ_3.

$\diamond$ If $\theta_3 < 180$ then the semicircles θ_1 and θ_3 overlap (as each extends 90 degrees to each side of the center). Thus the second semicircle is entirely contained in the first and third and is not needed. We may therefore remove it, and now we've covered the disk with n semicircles, and are done by induction.

$\diamond$ If $\theta_3 = 180$ then the first and third semicircle cover the entire circle, and we're done (as two semicircles is fewer than three!).

$\diamond$ If $\theta_3 > 180$ then the distance from θ_3 to θ_1 is less then 180, and we may argue as in subcase 1, finding that the fourth circle is not needed.

I like this phrasing for the problem. It's easy to write-up, it's clear, there's no convoluted logic or hand-waving on where things are. We have four points in order, and unless the first and third semicircles cover everything, they're either too close on one side or the other!

1.3. **3-Dimensional Case:** I now leave this as an exercise for the class.

2. Ordering a deck

Consider the following problem: take the numbers 1, 2, ..., n and randomly sort them. If the number k is on top, reverse the order of the first k numbers. Continue as long as the top number isn't a 1. Prove that eventually this process terminates with a 1 at the top. (So, if we start with the ordering 4 2 1 3 5, the next ordering is 3 1 2 4 5, then 2 1 3 4 5 and finally 1 2 3 4 5).

In class we searched for several different monoinvariants. I think the following solution works. It clearly suffices to show that eventually either 1 or n is at the top, or n is at the bottom (if it's 1 we win, while if n is at the top then on the next turn it's on the bottom, and an n on the bottom reduces us to the case of $n-1$ and we are done by induction).

I claim that either 1 is eventually on top, or n is eventually on the bottom (note n on top implies n is on the bottom at the next step). Let us assume this does not happen for some n, and without loss of generality let N be the smallest integer such that 1 is never on top and N is never on the bottom. As there are only finitely many ways to order $N-1$ numbers, the ordering must be cycle (and in all the elements of the cycle, N is never on the top or bottom).

We have N is never on the top or the bottom; thus whatever card is on the bottom must always stay on the bottom; let's call that card k. Since it is only the top card that causes change to happen, note that we could safely switch N and k. If we do this, we would still have an infinite cycle but now it would only involve the numbers 1, 2, ..., $N-1$. This contradicts the minimality of N and thus our assumption is wrong.

Let us choose n to be the smallest integer such that 1 never ends

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