MATH 331: THE LITTLE QUESTIONS: FALL 2014
HOMEWORK SOLUTION KEY

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ABSTRACT. A key part of any math course is doing the homework. This ranges from reading the material in the book so that you can do the problems to thinking about the problem statement, how you might go about solving it, and why some approaches work and others don't. Another important part, which is often forgotten, is how the problem fits into math. Is this a cookbook problem with made up numbers and functions to test whether or not you've mastered the basic material, or does it have important applications throughout math and industry? Below I'll try and provide some comments to place the problems and their solutions in context. Many of the comments below are from the TA, Jesse Freeman, or members of the class.

CONTENTS

Date: October 25, 2014.

1. HW #2: DUE SEPTEMBER 19, 2014

Note while we had two homeworks due (the comments on Uslan's speech, what you want to get out of class, and what you can do more efficiently) and those do count for 30 homework points, this shall be declared the first homework problem set.

1.1. Problems.

Due Friday, September 19 (note additional problems may be added): #0: Go to Project Euler

(https://projecteuler.net/)

and create an account for yourself, and solve the first problem. #1 to #4: Look at the problems above, and choose four that you find interesting and solve. It is important to learn how to generalize a problem; you want to get into this habit. You can choose these four problems from the links or from Project Euler, and do not need to hand anything in. First problem to be submitted for grading is #5. Looking at the problems on covering a $2 \times n$ board with $1 \times x$ dominoes, there are a lot of possibilities: #5: How many ways are there to cover a $3 \times n$ board using just 1×2 tiles? #6: What if now we have a $2 \times 2 \times n$ box and just $1 \times 1 \times 2$ tiles? #7: Choose at least one induction problem and at least one AM-GM problem to think about; you do not need to write about it. Email me if you'd like me to do it in class.

1.2. **Solutions.** #5: How many ways are there to cover a $3 \times n$ board using just 1×2 tiles?
Solution:

This problem will be easier after we cover recurrences; this was posed here to see how many of you wait till the last minute to start a HW problem and have trouble trying to do too much too quickly.

#6: What if now we have a $2 \times 2 \times n$ box and just $1 \times 1 \times 2$ tiles?
Solution:

This problem will be easier after we cover recurrences; this was posed here to see how many of you wait till the last minute to start a HW problem and have trouble trying to do too much too quickly.

2. HW #3: DUE SEPTEMBER 26, 2014

#1: How many ways are there to cover a $3 \times n$ board using just 1×2 tiles? #2: What if now we have a $2 \times 2 \times n$ box and just $1 \times 1 \times 2$ tiles? #3: Solve the double recurrence $f_n = f_{n-1} + 3g_{n-1}, g_n = -3f_{n-1} + 9g_{n-1}$. #4: Define a set to be selfish if it contains its cardinality (i.e., its number of elements) as an element; thus $\{1, 3, 5\}$ is selfish, while $\{1, 2, 3, 5\}$ is not. Find, with proof, the number of subsets of $\{1, 2, ..., n\}$ that are minimal selfish sets (that is, selfish sets none of whose subsets are selfish; thus $\{1, 3, 5\}$ is not minimal selfish as $\{1\}$ is a subset). This is a Putnam problem.....

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BELOW ARE SOME SUGGESTIONS / HINTS FOR THE PROBLEMS. THEY ARE COMPLETE IN SOME
PLACES, AND INCOMPLETE IN OTHERS. I STRONGLY URGE YOU TO DRAW PICTURES AND FILL IN ALL ARGUMENTS
THAT NEED HELP. I ALSO TRIED TO WRITE A BIT ABOUT THE THOUGHT
PROCESS AND HOW TO APPROACH THESE.
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#1: How many ways are there to cover a $3 \times n$ board using just 1×2 tiles?

Solution: We need to find a recurrence. Note that n has to be even, as otherwise we cannot cover as any number of 1×2 tiles covers an even number of squares. Thus, let us assume we have a $3 \times 2n$ board. Let A_n be the number of ways to cover a $3 \times 2n$ board with 1×2 tiles, and let B_n be the number of ways to cover a $3 \times 2n$ board where in the first column (in the far left) we only have the upper left corner entry (and not the middle or bottom left corner). We find a system of recurrences.

We have

$$A_n = 2A_{n-1} + B_n.$$

Why? Consider the bottom left square: either it is covered with a vertical or a horizontal tile. If it is a horizontal tile, then we have to cover the two rows of two blocks above it with two tiles, either vertically or horizontally; there are two ways to do that, and each gives us A_{n-1}. If it is a vertical tile then we have B_n ways to finish, as that is the configuration we get.

We now need a recurrence for B_n. Clearly we must have the first tile coming in from the upper left corner. If the next tile is vertical underneath its overhang, we now have a $3 \times 2(n-1)$ board and the number of ways to cover that is A_{n-1}. If instead our tile is horizontal then the one below is also horizontal, and we have a region that looks like our original but is two shorter, and thus the number of ways to cover it is B_{n-1}. Thus

$$B_n = A_{n-1} + B_{n-1},$$

which implies $A_{n-1} = B_n - B_{n-1}$, or shifting indices $A_n = B_{n+1} - B_n$.

We can now find a recurrence for just B's by substituting for the A's in the first relation, which yields

$$B_{n+1} - B_n = 2B_n - 2B_{n-1} + B_n \quad \text{or} \quad B_{n+1} = 4B_n - 2B_{n-1}.$$

The initial conditions are easy: $B_1 = 1$ and $B_2 = 4$. We can now use this to solve for B_n (try $B_n = r^n$, get the characteristic equation), and then get A_n.

Of course, since we only care about A_n we could instead note that if $A_n = 2A_{n-1} + B_n$ then $A_{n-1} = 2A_{n-2} + B_{n-1}$. Subtracting the two yields $A_n - A_{n-1} = 2A_{n-1} - 2A_{n-2} + (B_n - B_{n-1})$; however, from the recurrence for B we know $B_n - B_{n-1} = A_{n-1}$ and thus

$$A_n - A_{n-1} = 2A_{n-1} - 2A_{n-2} + A_{n-1} \quad \text{or} \quad A_n = 4A_{n-1} - 2A_{n-2},$$

which is the same recurrence! The initial conditions are $A_1 = 3$ and $A_2 = 11$. We solve this using characteristic polynomials as before. The Mathematica code is

```
RSolve[{A[n] - 4 A[n - 1] + 2 A[n - 2] == 0, A[1] == 3, A[2] == 11}, A[n], n]
```

#2: What if now we have a $2 \times 2 \times n$ box and just $1 \times 1 \times 2$ tiles?

Solution: Let A_n be the number of ways to tile a $2 \times 2 \times n$ box. We again find a recurrence. There are 2 ways to tile the bottom level completely (both parallel to the x-axis, or both parallel to the y-axis), and thus our recurrence begins $A_n = 2A_{n-1} + \cdots$; we now figure out the remainder. There are two possibilities. The first is all tiles in the bottom level point up; there is one way to do this, and it leaves us with a a $2 \times 2 \times (n-2)$ box, which has A_{n-2} ways to tile. Thus $A_n = 2A_{n-1} + A_{n-2} + \cdots$, and the lone case remaining is that we have two vertical tiles in the bottom row and one horizontal (note the two vertical tiles must be next to each other). There are four ways to choose where to place the one horizontal tile. Thus let B_n be the number of ways to tile a a $2 \times 2 \times n$ box where there is a horizontal tile filled in the bottom row. We have

$$A_n = 2A_{n-1} + A_{n-2} + 4B_{n-1}.$$

We need a recurrence for B_n. If we add a horizontal tile in the last level, that gives us a completed level and now we have a $2 \times 2 \times (n-1)$ box, and there are A_{n-1} ways to tile. If we add two vertical tiles then we have the same region as we started but smaller, and the number of ways to fill that is B_{n-1}. Thus our second recurrence is

$$B_n \ = \ A_{n-1} + B_{n-1} \quad \text{or} \quad A_{n-1} \ = \ B_n - B_{n-1}.$$

The initial conditions can be found by straightforward computation.

We can now get a recurrence just involving B, solve by looking at the characteristic polynomial and doing linear combinations, and then deduce the one for A.

#3: Solve the double recurrence $f_n = f_{n-1} + 3g_{n-1}, g_n = -3f_{n-1} + 9g_{n-1}$.

Solution: We solve for one in terms of the other. Using the second relation we get $f_{n-1} = -\frac{1}{3}g_n + 3g_{n-1}$; as this holds for all indices we can increment n by 1 and find $f_n = -\frac{1}{3}g_{n+1} + 3g_n$. We now substitute these values into the first recurrence, and find

$$\left(-\frac{1}{3}g_{n+1} + 3g_n\right) \ = \ \left(-\frac{1}{3}g_n + 3g_{n-1}\right) + 3g_{n-1} \quad \text{or} \quad g_{n+1} \ = \ 10g_n - 18g_{n-1}.$$

We try $g_n = r^n$ and find a characteristic polynomial of

$$r^2 - 10r + 18 \ = \ 0.$$

If instead we tried to write the g's in terms of the f's we would again obtain this recurrence relation. We now solve in the usual way.

Explicitly, we assume that $f_n = r^n$ to see if we can satisfy the equation. This gives

$$r^{n+1} - 10r^n + 18r^{n-1} \ = \ 0,$$

which means that non-trivial solutions are the roots of the quadratic equation

$$r^2 - 10r + 18 \ = \ 0,$$

which are given by

$$r_1 \ = \ 5 + \sqrt{7}$$
$$r_2 \ = \ 5 - \sqrt{7}.$$

So, a general solution is of the form

$$\alpha_1 r_1^n + \alpha_2 r_2^n.$$

The problem doesn't give any initial values, so we cannot simplify further. To highlight the method, let's assume $f_0 = 0$ and $f_1 = 1$. This gives $\alpha_1 = -\alpha_2$. Consequently,

$$1 = -\alpha_2(5 + \sqrt{7}) + \alpha_2(5 - \sqrt{7})$$

and we have that

$$\alpha_2 = -\frac{1}{2\sqrt{7}}$$
$$\alpha_1 = \frac{1}{2\sqrt{7}},$$

which concludes the proof. A solution is of the form

$$f_n = \left(\frac{1}{2\sqrt{7}}\right)(5 + \sqrt{7})^n - \left(\frac{1}{2\sqrt{7}}\right)(5 - \sqrt{7})^n,$$

and we can use similar methods to find a recursion for the g_n.

#4: Define a set to be selfish if it contains its cardinality (i.e., its number of elements) as an element; thus $\{1, 3, 5\}$ is selfish, while $\{1, 2, 3, 5\}$ is not. Find, with proof, the number of subsets of $\{1, 2, ..., n\}$ that are minimal selfish sets (that is, selfish sets none of whose subsets are selfish; thus $\{1, 3, 5\}$ is not minimal selfish as $\{1\}$ is a subset). This is a Putnam problem.....

Solution: For problems like this, it's best to do a few cases and get a feel. Doing this we find the number of minimal selfish sets, for the first few n, to be the Fibonacci numbers!

Let S_n denote the number of subsets of $\{1, \dots, n\}$ that are minimal selfish. Consider one of the minimal selfish sets; it either contains n, or it doesn't. By definition the number of minimal selfish sets of $\{1, \dots, n\}$ not containing n is S_{n-1}. Imagine now we have a minimal selfish set containing n. Note it's cardinality is its size, and it has no selfish subset. Its cardinality cannot be n if $n > 1$ (as that would mean we have all numbers, and thus selfish subsets). If we subtract 1 from each element we now have a subset of $\{1, \dots, n-1\}$ (note we could not have had 1 and n both in our original set, and thus since we assumed n was in, 1 was not). We remove $n-1$ now, and notice we've decreased all the elements by 1 and removed one element from the original set which had n and was minimal selfish; we now have a minimal selfish subset of $\{1, \dots, n-2\}$ (its cardinality must be in here). Thus the number of minimal selfish sets containing n here is S_{n-2}, and we get the recurrence $S_n = S_{n-1} + S_{n-2}$. We just need the initial conditions, which are $S_1 = 1$ and $S_2 = 1$, to see that it's the Fibonaccis.

3. HW #4: Due October 3, 2014

#1: Find the final digit (i.e., the ones digit) of $2^{3^{4^5}}$. #2: Prove 2^{n-1} divides $n!$ if and only if n is a power of 2. #3: How many primes are there such that, if the prime is written in base 10, its digits are an alternating string of 0s and 1s with first digit and last digit 1? #4: Show that if n divides a Fibonacci number that it divides infinitely many Fibonacci numbers. #5: For all positive real numbers a, b, c show that $a^a b^b c^c >= a^b b^c c^a$. #6: Show that if a, b and c are positive numbers summing to 1 that $(a+1/a)^2 + (b+1/b)^2 + (c+1/c)^2$ is at least 100/3.

#1: Find the final digit (i.e., the ones digit) of $2^{3^{4^5}}$.
Solution: The powers of 2 go **2**, 4, 8, 16, 3**2**, 64, 128, 256, 51**2**, ... Notice that we just need to know the value of the exponent modulo 4 to figure out the ones digit, as the pattern repeats every four. Thus the problem reduces to what is 3^{4^5} modulo 4. As any multiple of 100 is a multiple of 4, we just need to find the last *two* digits of 3^{4^5} Looking at powers of 3, we notice that it goes 3, 9, 27, 81, and thus ever four powers returns us something that is 1 modulo 4 (a little more work gives 3^{20} has last two digits 01, and all the tens digits are even, so we just need to study the ones digits). While we could continue this analysis, 4^5 is small enough to compute directly – it equals $2^{10} = 1024$, and thus the ones digit of 3^{4^5} is just 1 (or, equivalently for us, it is 1 modulo 4). Of course, this is overkill – clearly 4^5 is a multiple of 4! Knowing this, we raise 2 to something which is 1 modulo 4, and thus we just get 2.

For the record, 3^{4^5} equals

$$\begin{array}{r}
373391848741020043532959754184866588225409776783734007750636931722079040617265251 \\
229993688938803977220468765065431475158108727054592160858581351336982809187314191 \\
748594262580938807019951956404285571818041046681288797402925517668012340617298396 \\
574731619152386723046235125934896058590588284654793540505936202376547807442730582 \\
144527058988756251452817793413352141920744623027518729185432862375737063985485319 \\
476416926263819972887006907013899256524297198527698749274196276811060702333710356481
\end{array}$$

(which is about 10^{488}. How big would $2^{3^{4^5}}$ be? Well, let's say we have $2^{10^{488}}$. As $2^{10} \approx 10^3$ (it's actually a bit more), we find

$$2^{10^{488}} \ = \ (2^{10})^{10^{487}} \ > \ (10^3)^{10^{487}} \ = \ \left(10^{10^{487}}\right)^3.$$

This is what I could do on my computer; online WolframAlpha does better an gives the answer directly (including the ones digit!).

#2: Prove 2^{n-1} divides $n!$ if and only if n is a power of 2.
Solution: Assume $n = 2^L$ is a perfect power of 2. We count how many times 2 goes into the terms of $n!$. Of the 2^L numbers 1, 2, 3, ..., 2^L, half of them (or 2^{L-1}) are multiples of 2 once, one-fourth of them (or 2^{L-2}) are multiples of 2 twice, one-eight of them (or 2^{L-3}) are multiples of 2 thrice, and so on until one of them (or 2^0) is a multiple of 2 a total of L times. By writing it like this, we count certain numbers multiple times; thus 8 and 24 are both counted exactly three times as each is a multiple of 2, 4 and 8 but neither are multiples of 16 or anything higher. Thus the total number of 2's in $n!$ is

$$2^{L-1} + 2^{L-2} + 2^{L-3} + \cdots + 2^0 \ = \ 2^L - 1 \ = \ n - 1;$$

this follows from using the geometric series formula to sum the geometric series (or just add one and see how everything bumps up, so one more than the sum is 2^L or the sum is $2^L - 1$), and then noting that $n = 2^L$. Thus if $n = 2^L$ we do have 2^{n-1} divides $n!$.

What about the other case? We have to show that we have too many powers of 2 in 2^{n-1}. Here's a plan of attack. If $n = 2^L$ it just works. Now look what happens as you increase n. Every time you hit an odd number, n increases by 1 and you get no additional powers of 2. Thus it fails when $n = 2^L + 1$. Keep track as you move up and you see there's always a deficit. Things get better when you hit big powers of 2, but you don't overcome the deficit until you hit 2^{L+1}. Obviously you should make this more rigorous. You can also note that if you go from $2^L + 1$ to $2^L + k$ the powers of 2 that you gain are exactly what you would get going from 1 to k, and you can then argue by induction.

#3: How many primes are there such that, if the prime is written in base 10, its digits are an alternating string of 0s and 1s with first digit and last digit 1?
Solution: The only number that works is 101. If we have an even number of 1's it is a multiple of 101, and this can be seen by using numbers of the form 10001000100010001; when you multiply by 101 it fills things in and gives 1010101010101010101.

We are left with an odd number of 1's. At first I tried breaking into cases. If we have a multiple of 3 as the number of 1's then it's divisible by 3 (one of the old divisibility rules), but we're starting to break into too many cases. I then tried factoring to see if there's something that goes into all the odd number of 1 cases, but no luck. Here's code to create a number with a given number of digits and test for primality:

```
f[n_] := Print[Sum[10^(2 k), {k, 0, Prime[n] - 1}], " ",
PrimeQ[Sum[10^(2 k), {k, 0, Prime[n] - 1}]]]
```

What next? We could try factoring the numbers to see *why* they are composite.

```
f[n_] := Print[Sum[10^(2 k), {k, 0, Prime[n] - 1}], " ",
FactorInteger[Sum[10^(2 k), {k, 0, Prime[n] - 1}]]]
```

Rather than looking at the prime factors, it's better to look at all the divisors – maybe it's how we *group* the primes that will be enlightening.

```
f[n_] := Print[Sum[10^(2 k), {k, 0, Prime[n] - 1}], " ",
Divisors[Sum[10^(2 k), {k, 0, Prime[n] - 1}]]]
```

Playing with some odd n we see that if the number of 1's is n then our number appears to be divisible by $111\dots1$, where the number of 1's in this number is n. A little more work shows that when we divide we get 9090909091, where the number of 9's in general appears to be $(n-1)/2$. This now gives us something very concrete to work with and try to prove, and we can try and prove it by induction, or maybe use the geometric series formula for the sums. Another good suggestion from a student was to do long division....

#4: Show that if n divides a Fibonacci number that it divides infinitely many Fibonacci numbers.

Solution: Note that the Fibonacci numbers are periodic modulo m for any m. The reason is the pigeonhole principle. Modulo m there are only m possible residues, and thus only m^2 possible pairs of two numbers modulo m. Once we look at m^2+2 consecutive Fibonacci numbers we have m^2+1 pairs, and thus at least two pairs are the same.

For our problem, let's look at the Fibonacci numbers modulo n. By assumption we know n divides one of them; we now prove it divides infinitely many as the pattern repeats. To see this, imagine we have repeating pairs at indices (i_1, i_1+1) and (i_2, i_2+1), and let's assume F_k is our given multiple of n. If k is one of these indices, or between them, it's clear. What if k isn't? Well, we had to hit k as we walked from indices $(0,1)$ to (i_1, i_1+1); thus if we run backwards from (i_1, i_1+1) we must hit k; however, this will give us the same residues as we would get walking backwards from (i_2, i_2+1), and so we must have something between our two pairs that's a multiple of n.

#5: For all positive real numbers a, b, c show that $a^a b^b c^c >= a^b b^c c^a$.

Solution: If we wanted, we could rescale and assume $abc = 1$. Why? If we multiply each by r we get each side increases by $r^{r(a+b+c)}$, and thus the relation still holds or doesn't hold. It doesn't help us, but for awhile I thought about making their product 1, or setting b equal to 1.... What is more useful is there is a cyclic symmetry, and without loss of generality we may assume $a \le b \le c$. Some ordering exists, the left hand side is independent of the ordering, and seeing the cyclicity (the right hand side is also $b^c c^a a^b$ or $c^a a^b b^c$) there is no harm in assuming an ordering.

In some sense, if you look at this problem the right way it's "obvious". Why? Imagine our numbers are integers. We're talking about having some number of powers of a, b and c. We can choose $a+b+c$ numbers. Clearly you want to have as many powers of c as possible, so give it the exponent c. Then let's take as many b's as we can, namely b of them, and finally let's take the rest to be a.

More formally, we have the following chain (which holds for positive real numbers $a \le b \le c$):

$$\begin{aligned} a^a b^b c^c &= a^a b^b c^{c-(b-a)+(b-a)} \\ &\ge a^{a+(b-a)} b^b c^{c-(b-a)} \\ &= a^b b^b c^{(c-b)+b-(b-a)} \\ &\ge a^b b^{b+(c-b)} c^{b-(b-a)} \;=\; a^b b^c c^a. \end{aligned}$$

Note that all the exponents are positive, and the inequalities are true as we replace larger numbers in the product with smaller ones.

For another good inequality to know, see Jensen's inequality:

http://www.artofproblemsolving.com/Wiki/index.php/Jensen's_Inequality

#6: Show that if a, b and c are positive numbers summing to 1 that $(a+1/a)^2 + (b+1/b)^2 + (c+1/c)^2$ is at least 100/3.

Solution: I tried lots of approaches. First I multiplied things out, and got

$$a^2 + a^{-2} + b^2 + b^{-2} + c^2 + c^{-2} + 6.$$

I tried to attack this with the arithmetic mean - geometric mean inequality, sometimes writing 6 as 1 + 1 + 1 + 1 + 1 + 1, or as $6(a+b+c)$. These all gave bounds, but not the desired ones. The problem is that this doesn't seem to bring in the sum $a+b+c$ equals 1, so it's probably not the right way.

Early on I also checked to see if the claimed lower bound is reasonable. In problems like this we see that if one of the terms goes to 0 then the expression goes to infinity, so we're looking for a minimum value. That often happens when all variables are equal (if it's symmetric), and taking $a = b = c = 1/3$ gives $3(10/3)^2 = 100/3$, as claimed.

I tried doing Lagrange multipliers, as that's a *great* way to work in constraints. Unfortunately while having all the terms equal is a solution, there could be others. Explicitly, we'd look at $f(a,b,c) = a^2 + a^{-2} + b^2 + b^{-2} + c^2 + c^{-2}$ (we can make our life a little easier and remove the 6), $g(a,b,c) = a+b+c-1$, and then extrema of $f(a,b,c)$ subject to $g(a,b,c) = 0$ satisfy

$$\nabla f(a,b,c) \ = \ 2(a - a^{-3}, b - b^{-3}, c - c^{-3} \ = \ \lambda(1,1,1) \ = \ \nabla g(a,b,c), \quad a+b+c \ = \ 1.$$

This implies

$$a - a^{-3} \ = \ b - b^{-3} \ = \ c - c^{-c}, \quad a+b+c \ = \ 1;$$

again, clearly $a = b = c = 1/3$ is a solution, but is there another? We are looking to see how many x can satisfy $x - x^{-3} = r$ for some fixed r. This is the same as $rx^3 - x + 1 = 0$. This is a cubic equation, it has three roots (either one real and two complex conjugate, or three real; it depends on r). We could try using the cubic formula.... Remember we have to look at both $r > 0$ and $r < 0$ (if $r = 0$ there are two solutions, ± 1, but remember we are only looking for solutions in the positive numbers). A better approach is to show the function is strictly increasing (or strictly decreasing) for positive x; if $f(x) = x - 1/x^3$ is strictly increasing for positive x then we cannot have two such x giving the same value, and thus the minimum will occur when all are 1/3. A **`great`** way to show that a function is strictly increasing (or decreasing) is to analyze the derivative. Here we find

$$f'(x) \ = \ 1 + \frac{3}{x^4},$$

which is clearly positive for $x > 0$. Thus f is strictly increasing, and there cannot be two inputs with the same output. This completes the proof using Lagrange Multipliers.

Here is a more standard inequality approach. We start using the Cauchy-Schwarz Inequality; see for example

http://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality

which says

$$\left|\sum_{i=1}^{n} x_i y_i\right|^2 \ \le \ \sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2.$$

A very good time to use an inequality like this is to replace a^2 with sums of $a \cdot 1$ (or a^{-2} with $a^{-1} \cdot 1$), as we then get a sum of 1's, which is easily handled.

Specifically, if we remove the 6 from the cross terms, we need to study

$$S \ := \ a^2 + b^2 + c^2 + a^{-2} + b^{-2} + c^{-2} \ = \ \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} x_i^{-2}$$

(in hopefully obvious notation). Applying the Cauchy-Schwarz inequality to each gives

$$\begin{aligned} S \ &= \ \left(\sum_{i=1}^{3} x_i^2\right)\left(\sum_{i=1}^{3} 1\right) \cdot \frac{1}{3} + \left(\sum_{i=1}^{3} x_i^{-2}\right)\left(\sum_{i=1}^{3} 1\right) \cdot \frac{1}{3} \\ &\ge \ \frac{1}{3}\left(\sum_{i=1}^{3} x_1 \cdot 1\right)^2 + \frac{1}{3}\left(\sum_{i=1}^{3} x_{-1} \cdot 1\right)^2 \\ &= \ \frac{1}{3} \cdot 1^2 + \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2. \end{aligned}$$

We now need to determine the largest value of $1/a + 1/b + 1/c$ subject to $a+b+c = 1$ and $a, b, c > 0$. This can easily be done with Lagrange multipliers. We get all terms must be equal by symmetry, and are thus 1/3; thus the maximum value here is 9.

Putting everything together, and remembering the 6, gives

$$(a+1/a)^2+(b+1/b)^2+(c+1/c)^2 \ \geq\ 6+\frac{1}{3}\cdot 1+\frac{1}{3}\cdot 9^2 \ =\ \frac{100}{3}.$$

4. HW #5: DUE OCTOBER 10, 2014

The 1992 Green Chicken Exam:

http://web.williams.edu/Mathematics/sjmiller/public_html/greenchicken/exams/gcexam1992.pdf

#1: First, we will show this can be done in three turns. Then, we will show that it cannot be done in two turns.

Break the coins into three groups given by

$$g_{11} = \{1, 2, 3, 4\}$$
$$g_{12} = \{5, 6, 7, 8\}$$
$$g_{13} = \{9, 10, 11, 12\}.$$

We will use the notation $g_{ij} > g_{ik}$ to articulate that g_{ij} weighs more than g_{jk}. Weigh g_{11}, g_{12}. We use similar notation for n-tuples $\{k_1, \dots, k_n\}$.

Case 1 $g_{11} = g_{12}$. In this case, we know that the special coin is in g_{13}. So, weigh $\{9, 10, 11\}$ against $\{1, 2, 3\}$.

Case 1a $\{9, 10, 11\} = \{1, 2, 3\}$. Then, we know that the special coin is 12 and we are done.

Case 1b $\{9, 10, 11\} \neq \{1, 2, 3\}$. Without loss of generality, assume $\{9, 10, 11\} > \{1, 2, 3\}$. Then, we know that the special coin is heavy. So, weigh 9 against 10. If these coins have the same weight, 11 is special. If they have different weights, the heavy one is special.

Case 2 $g_{11} \neq g_{12}$. Without loss of generality, assume that $g_{11} > g_{12}$. Then, we will weigh $\{1, 5, 6\}$ against $\{2, 7, 8\}$.

Case 2a $\{1, 5, 6\} = \{2, 7, 8\}$. Then, the special coin can be in neither set. So, it is either 3 or 4. This means that the special coin lies in g_{11}, which is heavier than g_{12}. So, the special coin is heavy. Weigh 3 against 4; the heavier one is the special coin.

Case 2b $\{1, 5, 6\} \neq \{2, 7, 8\}$. Without loss of generality, assume $\{1, 5, 6\} > \{2, 7, 8\}$. This implies that 2,5,and 6 cannot be special coins. All were in both a heavy and a light set. Now, weigh 7 against 8. If their weights are equal, 1 is the special coin. If their weights are not equal, then the special coin is in $\{2, 7, 8\}$, which means it is light. So, choose the lighter of 7 and 8.

Now, we will argue that we cannot find the special coin in two moves. We will argue that after one move, one cannot in general have fewer than 4 candidates or know whether the special coin is lighter or heavier. It is clear that one cannot use one weighing to determine whether the coin is special when there are 4 candidates and no knowledge of whether the coin is heavier or lighter than others.

Break the 12 coins into three groups, G_1, G_2, G_3. Suppose that we weigh G_1, G_2 and that those two groups are unequal in size. If the larger group is heavier, we cannot conclude anything. The coin could be heavy or light in any groups, provided it is not an extremely heavy coin in G_1.

So, assume $|G_1| = |G_2|$. No individual outcome gives us guaranteed knowledge of whether the coin is heavy or light. If the weights are equal, we get nowhere in this regard, and we could be left with as many as $|G_3|$ coins. So, the optimal case is when $|G_1| = |G_2| = |G_3|$. But then, we are left with not knowing whether the coin is heavy or light hand having at least 4 candidates to choose from.

#2: Let $W = w_1 w_2 w_3 w_4$. We know that w_4 satisfies $3w_4 \equiv 3 \mod 10$. As 3 is relatively prime to 10, it has order 10 in $\mathbb{Z}_{10}^{\times}$. Therefore, the only solution is $w_4 = 1$ because we require w_4 to be a 1-digit number.

The third digit satisfies $9 + 3w_3 \equiv 9$. By the same reasoning, we must have $w_3 = 0$.

The second digit satisfies $7 + 3w_2 \equiv 9$, which forces $w_2 = 4$.

Finally, the first digit satisfies $8 + 3w_1 = 1$, which forces $w_1 = 1$.

So, our number is 1401.

#3: We make 3 remarks:

Remark 1. If $f \geq g$, $\int_0^1 f \geq \int_0^1 g$

This is true by the definition of the integral as a positive linear functional. Namely, $\int_0^1 (f-g) \geq 0$ because $f \geq g$.

Remark 2. For $0 \leq \alpha \leq \beta \leq \pi$, $\cos(\alpha) \geq \cos(\beta)$

This holds because $\cos$ is monotonically decreasing on $[0, \pi]$.

Remark 3. For $x \in [0,1], \pi x^3 \leq \pi x^2$

This is obvious. For $x \leq 1, x^3 \leq x^2$.

Using remarks 3,2,1 in that order, the desred inequality follows immediately, namely

$$\int_0^1 \cos(\pi x^3)\, dx \geq \int_0^1 \cos(\pi x^2)\, dx \tag{4.1}$$

Can also solve with trig identities. We have

$$\cos(A) - \cos(B) \ = \ -2\sin((A+B)/2)\sin((A-B)/2).$$

For us $A = \pi x^3$ and $B = \pi x^2$. The first sine term is positive, the second negative. Thus we're integrating a non-negative function....

#4: Consider the n-digit numbers, of which there are $9 \times 10^{n-1}$. As a fraction, we have that $\frac{8}{9}\left(\frac{9}{10}\right)^{n-1}$ of these contain no 7s. In particular, the contribution from each n digit number is at most $\frac{1}{10^{n-1}}$. So, we have

$$\sum_{\substack{n \geq 1 \\ n \text{ 7-free}}} \frac{1}{n} \leq \frac{9*8}{9} \sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^{n-1}$$
$$< \infty$$

where the final equality comes from the geometric series formula.

#5:
Part I: "I am a poor knave".

Claim: Nobody on this island can say "I am a knave".
Proof: If a knave says so, it is true, violating knave-ness. If a knight says, so, it is false, violating the knight's code. □
Let N denote knave, K denote knight, P denote poor and R denote rich. These are all mutually exclusive in the context of this problem. We are given the statement $P \cap N$. By Lemma **??**, our lady knows this statement is false. So, she is left with $R \cup K$. If K is true, then the original statement was a lie. But, that is not possible because knights only tell the truth. So, K must be false, N must be true and R must be true because either R or K is true.
Part II: "I am not a poor knight"

We will first show that this statement cannot be false. Suppose the author is either rich or a knave. If the author is a knave, then S is true, a contradiction. If the author is not a knave, but poor, the statement is false, another contradiction.

The author must then be a knight. So, S must be true. In order for this to be the case, the author must be rich.

#6: Let $S(R)$ denote the side length of a regular polygon or figure R. We begin with a proposition:

Proposition 4. Let R be a square of side length k. If $T_1, \dots, T_n$ are n squares that tile R such that $S(T_i) \neq S(T_j)$ for $i \neq j$, then

$$\min\{S(T_1), \dots, S(T_n)\} < \frac{S(R)}{2} \tag{4.2}$$

Proof. Divide R into four quadrants and place tiles in the corners. If one tile has side length at least $S(R)/2$, the other three cannot. □

We also note the following fact:

Lemma 5. Let R be a square and $T_1, \dots, T_n$ be a finite tiling of the square by squares of different size. Then, the tile of smallest size cannot lie in the corner or on an edge of R

Proof. Suppose the smallest tile, T^* were in the corner. To fix ideas, suppose it is in the lower righthand corner of R. Then, the tile lying above T^* overhangs the top edge of T^* to the left. This implies that the square immediately to the left of T^* can be no taller than T^*. And, because no two squares have the same size, it must have smaller side length than T^*. But, this contradicts the minimality of T^*.

If T^* were on an edge but not a corner, the tiles to the left and right would be larger, forcing us to put a smaller tile on top of T^*. □

Without loss of generality, suppose C has volume 1. Assume that C has a tiling by finitely many smaller cubes. Then, the bottom face of C is tiled by the bottom faces of cubes lying in C. There is a cube of minimum side length on this bottom face. Call it C_1. By proposition 4.2, $S(C_1) \leq 1/2$.

Now, consider the top face of C_1. Note that all tiles surrounding C_1 are taller than C_1 because our tiles are cubes. If C is tiled, then the top face of C_1 is tiled by squares contained entirely in the top face of C_1. This tiling cannot have any spillover onto the faces of other cubes, because those other cubes have a higher elevation than C_1. So, we have some cube of minimal volume lying on the top face of C_1. Call this cube C_2. We know

$$S(C_2) \leq \frac{S(C_1)}{2}. \tag{4.3}$$

Proceed to construct C_n as the tile of minimal side length on top of C_{n-1}. If C_{n-1} were in a corner, and had just jumped over two cubes bounding C_{n-2}, we might be able to construct C_n to be a large cube that has its corner lying in C_{n-1} and mostly overhangs the two bounding cubes. However, lemma 5 tells us that this cannot happen because C_{n-1} cannot lie in a corner or on an edge. In particular, we have

Lemma 6. As defined above, any cube that covers part of the top face of C_{n-1} is contained entirely within the top face of C_{n-1}.

Proof. Lemma 5 tells us C_{n-1} must live in the interior of C_{n-2}. So, C_{n-1} is surrounded by 4 taller cubes. So, any cube in its top face must have its base completely contained in that top face. □

Lemma 6 reduces the problem to that of tiling the top face of C_n as if it were a square. This allows us to apply lemma 4 to show that there is a stack $C_1, \dots, C_n, \dots$ where the base of C_{n+1} is contained entirely within the top face of C_n and moreover

$$S(C_{n+1}) \leq \frac{S(C_n)}{2}$$

However, no finite substack of this stack $C_1, \dots, C_n, \dots$ can ever reach the ceiling because

$$\sum_{n=1}^{\infty} S(C_n) \leq (1/2 - \varepsilon) \sum_{n=1}^{\infty} \frac{1}{2^n} < 1. \tag{4.4}$$

So, such a tiling necessarily involves infintely many tiles.

5. HW #6: Due October 24, 2014

1993 Green Chicken exam:

http://web.williams.edu/Mathematics/sjmiller/public_html/greenchicken/exams/gcexam1993.pdf

#1: Reflect the isoceles triangle over itself to form a quadrilateral. We optimize the area of the triangle if and only if we optimize the area of the quadrilateral. However, the quadrilateral with optimal area is a square. So, the angle between the to sides of length 10 is a right angle. Thus the base has length $10\sqrt{2}$.

#2: By calculation, we have

$$\begin{aligned}
a_1 &= 1\\
a_2 &= 1\\
a_3 &= 10^6 + 10^3\\
a_4 &= 10^9 + 2\,10^6\\
a_5 &= \frac{10^{12} + 2\,10^9 + 10^6}{10^6 + 10^3}\\
&= \frac{(10^6 + 10^3)^2}{(10^6 + 10^3)}\\
&= 10^6 + 10^3\\
a_6 &= \frac{10^9 + 2\,10^6}{10^9 + 2\,10^6}\\
&= 1\\
a_7 &= \frac{10^6 + 10^3}{10^6 + 10^3}\\
&= 1
\end{aligned}$$

so by induction, this cycle repeats. Therefore, we only care about the congruence of 1993 mod 5. This is 3. So, $a_{1993} = 10^6 + 10^3$. *This is a nice example of getting a few data points and looking for the pattern!*

#3: We claim that the number of students in the three classes will always occupy the congruence class of $(0, 1, 2)$ (in some order) mod 3. Observe that it is true initially. Now, note that when we rearrange students by changing classes, we subtract one from each congruence mod 3. This is just a permutation of the congruence classes mod 3. *This is a nice example of an invariant / mono-invariant problem!*

#4: We will assume knowledge of the relation

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \tag{5.1}$$

Using this, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f^n(x+h) - f^n(x)}{h} &= \lim_{h\to 0} \frac{(f(x+h) - f(x))\left(\sum_{i=0}^{n-1} f^{n-i}(x+h) f^i(x)\right)}{h}\\
&= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h}\right)\left(\sum_{i=0}^{n-1} f^{n-1-i}(x+h) f^i(x)\right)\\
&= f'(x) n f^{n-1}(x)
\end{aligned}$$

where the final equality holds by continuity and differentiability.

#5: Let B denote the number of black marbles, W the number of white marbles, and T the total number of marbles. By the statement of the problem, we have

$$\left(\frac{2BW}{T(T-1)}\right) = \frac{1}{2} \tag{5.2}$$

from this we immediately have

$$\begin{aligned} 0 &= (B+W)\left((B+W)-1\right) - 4BW \\ &= (B+W)^2 - 4BW - (B+W) \\ &= (B-W)^2 - (B+W) \end{aligned}$$

which proves that $B+W$ is a perfect square.

Without loss of generality, assume $B > W$ (If B = W, then by the previous part we have 0 total marbles, in which case the problem does not make sense) To show that B, W are triangle numbers, note that

$$\begin{aligned} \frac{(B-W)\left((B-W)-1\right)}{2} &= \frac{(B-W)^2 - (B-W)}{2} \\ &= \frac{(B+W) - (B-W)}{2} \\ &= W \end{aligned}$$

A similar argument shows that

$$B = \frac{|(W-B)||((W-B)-1)|}{2} \tag{5.3}$$

which is also a triangle number.

#6: We are given the following equation:

$$F(x)F(x+1) + F(x+1) + 1 = 0 \tag{5.4}$$

First, note that it can never hold that $F(x) = 0$. If this were true, then $0 = 1$ by our equation in the problem.

We will use the intermediate value theorem to show that if F is continuous, F must have a zero. This will be a contradiction and will show F cannot be continuous.

Suppose $F(0) > 0$. Then, by (5.4), $F(-1)$ must be negative. This gives our zero.

Now, suppose $-1 \leq F(0) \leq 0$. If $F(0) = -1$, then (5.4) shows that $F(-1) = 0$. If $F(0) > -1$, then we have $F(-1)F(0) = -\varepsilon$ for $\varepsilon > 0$. Thus, $F(-1) > 0$. Finally, suppose $F(0) < -1$. Then, $|1 + F(x+1)| < |F(x+1)$ and so by (5.4), we have $|F(x)| < 1$, which reduces the proof to a previous case.