MATH 331: THE LITTLE QUESTIONS: FALL 2024
HOMEWORK SOLUTION KEY

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ABSTRACT. A key part of any math course is doing the homework. This ranges from reading the material in the book so that you can do the problems to thinking about the problem statement, how you might go about solving it, and why some approaches work and others don't. Another important part, which is often forgotten, is how the problem fits into math. Is this a cookbook problem with made up numbers and functions to test whether or not you've mastered the basic material, or does it have important applications throughout math and industry? Below I'll try and provide some comments to place the problems and their solutions in context. **NOTE: IT IS NOT ALWAYS THE CASE THAT PROBLEMS ARE WELL-STATED – SOMETIMES YOU NEED TO EMAIL ME AND SAY YOU THINK IT IS TOO VAGUE!**

CONTENTS

Date: October 21, 2024.

A000124 Central polygonal numbers (the Lazy Caterer's sequence): n(n+1)/2 + 1; or, maximal number of pieces formed when slicing a pancake with n cuts. 297
(Formerly M1041 N0391)

1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, 137, 154, 172, 191, 211, 232, 254, 277, 301, 326, 352, 379, 407, 436, 466, 497, 529, 562, 596, 631, 667, 704, 742, 781, 821, 862, 904, 947, 991, 1036, 1082, 1129, 1177, 1226, 1276, 1327, 1379 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS These are Hogben's central polygonal numbers with the (two-dimensional) symbol
2
.P
1 n
The first line cuts the pancake into 2 pieces. For n > 1, the n-th line crosses every earlier line (avoids parallelism) and also avoids every previous line intersection, thus increasing the number of pieces by n. For 16 lines, for example, the number of pieces is 2 + 2 + 3 + 4 + 5 + ... + 16 = 137. These are the triangular numbers plus 1 (cf. A000217).

Define a number of straight lines in the plane to be in general arrangement when (1) no two lines are parallel, (2) there is no point common to three lines. Then these are the maximal numbers of regions defined by n straight lines in general arrangement in the plane. - Peter C. Heinig (algorithms(AT)gmx.de), Oct 19 2006

Note that a(n) = a(n-1) + A000027(n-1). This has the following geometrical interpretation: Suppose there are already n-1 lines in general arrangement, thus defining the maximal number of regions in the plane obtainable by n-1 lines and now one more line is added in general arrangement. Then it will cut each of the n-1 lines and acquire intersection points which are in general arrangement. (See the comments on A000027 for general arrangement with points.) These points on the new line define the maximal number of regions in 1-space definable by n-1 points, hence this is A000027(n-1), where for A000027 an offset of 0 is assumed, that is, A000027(n-1) = (n+1)-1 = n. Each of these regions acts as a dividing wall, thereby creating as many new regions in addition to the a(n-1) regions already there, hence a(n) = a(n-1) + A000027(n-1). Cf. the comments on A000125 for an analogous interpretation. - Peter C. Heinig (algorithms(AT)gmx.de), Oct 19 2006

FIGURE 1. Proof from the OEIS.

1. HW #2: DUE FRIDAY SEPTEMBER 13, 2024

1.1. **Problems:** (1) Go to Project Euler (https://projecteuler.net/) and create an account for yourself, and solve the first problem. You do not need to submit this, just email me when done. (2) Read http://www.math.ucla.edu/~radko/circles/lib/data/Handout-142-159.pdf and do Problem #1: If n lines are drawn in a plane, and no two lines are parallel, how many regions do they separate the plane into? (3) Prove that $(1-1/4)(1-1/9)\cdots(1-1/n^2) = (n+1)/2n$.

1.2. **Solutions:** **(2): Read http://www.math.ucla.edu/~radko/circles/lib/data/Handout-142-159.pdf. Do Problem #1: If n lines are drawn in a plane, and no two lines are parallel, how many regions do they separate the plane into?**
Solution: The problem is not phrased well; it implies there is a unique answer, but if they all intersect in a common point the answer is different than if they do not. One student said a good way to rephrase is that no three lines may intersect in a common point. Doing a little work, if we have n lines (starting at 0) the number of regions is 1, 2, 4, 7, 11. Plugging this into the OEIS yields http://oeis.org/A000124 (there were other suggestions but reading them it is clear this is the one we want!).

As a nice additional problem, is the minimum all lines intersecting in a common point, giving $2n$ regions with n lines? If yes, can you show that you can get any number of intersections between $2n$ and M_n (call the maximum with n lines M_n)?

(3): (3) Prove that $(1-1/4)(1-1/9)\cdots(1-1/n^2) = (n+1)/2n$.
Solution: This follows by induction. Let $P(n)$ be the statement $(1-1/4)(1-1/9)\cdots(1-1/n^2) = (n+1)/2n$. The base case is immediate. We now assume $P(n)$ holds and must show $P(n+1)$ is true. We have

$$\begin{aligned}\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right) &= \frac{n+1}{2n}\frac{(n+1)^2-1}{(n+1)^2}\\ &= \frac{n^2+2n}{2n(n+1)} = \frac{n+2}{2n+2} = \frac{(n+1)+1}{2(n+1)},\end{aligned}$$

as claimed, completing the proof.

Another proof (from John Fan):

$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \prod_{k=2}^{n}\frac{(k-1)(k+1)}{k\cdot k} = \prod_{k_2=2}^{n}\frac{1}{k_1}\prod_{k_2=2}^{n}\frac{1}{k_2}\prod_{k_3=2}^{n}(k_3-1)\prod_{k_4=2}^{n}(k_4+1) = \frac{1\cdot(n+1)}{2\cdot n}.$$

HW #3: Due September 27, 2024: (1) Make sure you have done the first 10 problems on Project Euler. (2) How many ways are there to cover a $3 \times n$ board using just 1×2 tiles? (3) What if now we have a $2 \times 2 \times n$ box and just 1×2 tiles? (For fun, no need to write up, just email me: can you find the exact answer for n = 1000? 10,000? 100,000? 1,000,000? What is the highest you can go in one hour of computing on your device?) (4) Let a, b be positive integers each at least 2. Prove that the number of ways to tile an a x b x n box is given by a recurrence relation of finite depth and constant coefficients; can you bound the size of the coefficients or the depth of the relation?

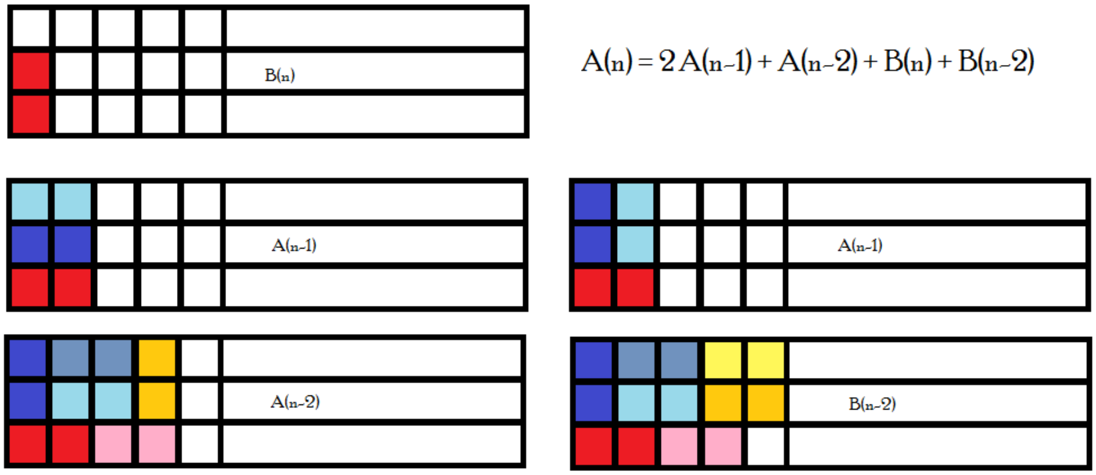

FIGURE 2. The different configurations needed to study to find the recurrence for A_n.

2. HW #3: DUE SEPTEMBER 27, 2022

HW #3: Due September 27, 2024: (1) Make sure you have done the first 10 problems on Project Euler. (2) How many ways are there to cover a $3 \times n$ board using just 1×2 tiles? (3) What if now we have a $2 \times 2 \times n$ box and just 1×2 tiles? (For fun, no need to write up, just email me: can you find the exact answer for n = 1000? 10,000? 100,000? 1,000,000? What is the highest you can go in one hour of computing on your device?) (4) Let a, b be positive integers each at least 2. Prove that the number of ways to tile an a x b x n box is given by a recurrence relation of finite depth and constant coefficients; can you bound the size of the coefficients or the depth of the relation?

2.1. Solutions. #2: How many ways are there to cover a $3 \times n$ board using just 1×2 tiles?

Solution: We need to find a recurrence. Note that n has to be even, as otherwise we cannot cover as any number of 1×2 tiles covers an even number of squares. Thus, let us assume we have a $3 \times 2n$ board. Let A_n be the number of ways to cover a $3 \times 2n$ board with 1×2 tiles, and let B_n be the number of ways to cover a $3 \times 2n$ board where in the first column (in the far left) we only have the upper left corner entry (and not the middle or bottom left corner). We find a system of recurrences.

We have (note the re-grouping is to simplify some algebra later)

$$A_n \;=\; 2A_{n-1} + A_{n-2} + B_n + B_{n-2} \;=\; 2A_{n-1} + B_n + (A_{n-2} + B_{n-2})\,.$$

Why? Consider the bottom left square: either it is covered with a vertical or a horizontal tile. See Figure 2.

- If it is covered by a vertical tile then we are left with just one square in the upper left corner in the first column. By definition the number of ways to cover what remains is B_n.
- If it is covered by a horizontal tile we have several options. We could have two horizontal tiles above it, which would completely cover the first two columns and leave us with a $3 \times (2n-2)$ board to cover; there are A_{n-1} ways to do this. We could have two vertical tiles, again completely covering the first two columns and leaving us with a $3 \times (2n-2)$ board to cover, which again can be done A_{n-1} ways. Finally we could have a vertical tile for the last column, and then two horizontal tiles. In that case we would need a horizontal in the bottom row (so now the bottom four squares are covered). We either now have a vertical tile completing the covering of column four (which leaves us with a $3 \times (2n-4)$ board, which can be covered in A_{n-2} ways), or we have two horizontal tiles and thus only the bottom element in column 5 is left in the first five columns (and by definition there are B_{n-2} ways to tile what remains).

We now need a recurrence for B_n; see Figure 3. Clearly we must have the first tile coming in from the upper left corner. If the next tile is vertical underneath its overhang, we now have a $3 \times 2(n-1)$ board and the number of ways to cover that is A_{n-1}. If instead our tile is horizontal then the one below is also horizontal, and we have a region that looks like our original but is two shorter, and thus the number of ways to cover it is B_{n-1}. Thus

$$B_n \;=\; A_{n-1} + B_{n-1},$$

which implies $A_{n-1} = B_n - B_{n-1}$, or shifting indices $A_n = B_{n+1} - B_n$.

We can now find a recurrence for just B's by substituting for the A's in the first relation (note we can replace the grouping we did there with B_{n-1}), which yields

$$B_{n+1} - B_n \;=\; 2B_n - 2B_{n-1} + B_n + B_{n-1} \quad\text{or}\quad B_{n+1} \;=\; 4B_n - B_{n-1}.$$

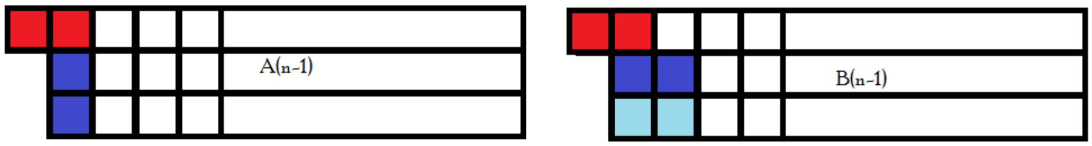

FIGURE 3. The different configurations needed to study to find the recurrence for B_n.

The initial conditions are easy: $B_1 = 1$ and $B_2 = 4$. We can now use this to solve for B_n (try $B_n = r^n$, get the characteristic equation), and then get A_n. The first few values for B_n are 1, 4, 15, 56 and 209, and the general formula is

$$B_n \;=\; -\frac{\left(2\sqrt{3}+3\right)\left(\left(2-\sqrt{3}\right)^n-\left(\sqrt{3}+2\right)^n\right)}{6\left(\sqrt{3}+2\right)}.$$

This can be found by using the Method of Divine Inspiration, or using RSolve in Mathematica:

```
RSolve[{B[n + 1] == 4 B[n] - B[n - 1], B[1] == 1, B[2] == 4}, B[n], n]
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Of course, since we only care about A_n we could instead note that if $A_n = 2A_{n-1} + B_n + B_{n-1}$ then $A_{n-1} = 2A_{n-2} + B_{n-1} + B_{n-2}$. Subtracting the two yields

$$A_n - A_{n-1} = 2A_{n-1} - 2A_{n-2} + (B_n - B_{n-1}) + (B_{n-1} - B_{n-2});$$

however, from the recurrence for B we know $B_n - B_{n-1} = A_{n-1}$ and thus

$$A_n - A_{n-1} \;=\; 2A_{n-1} - 2A_{n-2} + A_{n-1} + A_{n-2} \quad \text{or} \quad A_n \;=\; 4A_{n-1} - A_{n-2},$$

which is the same recurrence!

#3: What if now we have a $2 \times 2 \times n$ box and just $1 \times 1 \times 2$ tiles?

Solution: Let A_n be the number of ways to tile a $2 \times 2 \times n$ box. We again find a recurrence. There are 2 ways to tile the bottom level completely (both parallel to the x-axis, or both parallel to the y-axis), and thus our recurrence begins $A_n = 2A_{n-1} + \cdots$; we now figure out the remainder. There are two possibilities. The first is all tiles in the bottom level point up; there is one way to do this, and it leaves us with a a $2 \times 2 \times (n-2)$ box, which has A_{n-2} ways to tile. Thus $A_n = 2A_{n-1} + A_{n-2} + \cdots$, and the lone case remaining is that we have two vertical tiles in the bottom row and one horizontal (note the two vertical tiles must be next to each other). There are four ways to choose where to place the one horizontal tile. Thus let B_n be the number of ways to tile a a $2 \times 2 \times n$ box where there is a horizontal tile removed from the bottom row. We have

$$A_n \;=\; 2A_{n-1} + A_{n-2} + 4B_{n-1}.$$

We need a recurrence for B_n. If we add a horizontal tile in the last level, that gives us a completed level and now we have a $2 \times 2 \times (n-1)$ box, and there are A_{n-1} ways to tile. If we add two vertical tiles then we have the same region as we started but smaller, and the number of ways to fill that is B_{n-1}. Thus our second recurrence is

$$B_n \;=\; A_{n-1} + B_{n-1} \quad \text{or} \quad A_{n-1} \;=\; B_n - B_{n-1}.$$

The initial conditions can be found by straightforward computation.

We can now get a recurrence just involving B, solve by looking at the characteristic polynomial and doing linear combinations, and then deduce the one for A. Using $2B_n = 2A_{n-1} + 2B_{n-1}$ we find

$$A_n \;=\; A_{n-2} + 2B_n + 2B_{n-1} \quad \text{or} \quad B_{n+1} \;=\; 3B_n + 3B_{n-1} - B_{n-2},$$

with initial conditions $B_1 = 1$, $B_2 = 3$ and $B_3 = 11$ (this last takes a bit of counting). Typing

```
RSolve[{B[n+1] == 3 B[n] + 3 B[n-1] - B[n-2], B[1] == 1, B[2] == 3, B[3] == 11}, B[n], n]
```

yields

$$B_n \;=\; \frac{\left(4\sqrt{3}+7\right)\left(2-\sqrt{3}\right)^n+\left(\sqrt{3}+2\right)^{n+1}}{5\sqrt{3}+9},$$

and hence the first few values of A_n are 0, 2, 8, 30 and 112, or

$$A_n \;=\; \frac{\left(\sqrt{3}-1\right)\left(\sqrt{3}+2\right)^n-\left(3\sqrt{3}+5\right)\left(2-\sqrt{3}\right)^n}{\sqrt{3}+3}.$$

Problem #4: Let a, b be positive integers each at least 2. Prove that the number of ways to tile an a x b x n box is given by a recurrence relation of finite depth and constant coefficients; can you bound the size of the coefficients or the depth of the relation?

Solution: If we look at an $a \times b \times 2$ box there are only finitely many possibilities for which cells are filled and which are unfilled when using $1 \times 1 \times 2$ tiles; clearly there are at most 2^{ab} possibilities. We thus have at most finitely many states to consider. Unfortunately the recurrence could be hard to determine as starting with nothing filled we could have double counting as we add the tiles, but we can always remove that at the cost of a more complicated relation. Thus the number of auxiliary variables to introduce is at most 2^{ab} (not all of which will be possible), and the coefficients are bounded (a trivial bound should be the tower of $2^{ab}!$, where $T_1(x) = x, T_2(x) = x^x$ and $T_{k+1}(x) = x^{T_k(x)}$.

Homework #4: Due Oct 4, 2024: #1: Prove for $a_i > 0$ that $(1+a_1)\cdots(1+a_n) \geq 2^n\sqrt{a_1 \cdots a_n}$. #2: Prove for $a, b > 0$ that $a/b + b/a \geq 2$, both by using an inequality approach *and* without using an inequality! #3: Solve the double recurrence $f_n = f_{n-1} + 3g_{n-1}, g_n = -3f_{n-1} + 9g_{n-1}$. #4: Define a set to be selfish if it contains its cardinality (i.e., its number of elements) as an element; thus $\{1, 3, 5\}$ is selfish, while $\{1, 2, 3, 5\}$ is not. Find, with proof, the number of subsets of $\{1, 2, ..., n\}$ that are minimal selfish sets (that is, selfish sets none of whose subsets are selfish; thus $\{1, 3, 5\}$ is not minimal selfish as $\{1\}$ is a subset). This is a Putnam problem..... Also, make sure you have done the first 15 Project Euler Problems.

3. HW #4: Due Oct 4, 2024

#1: Prove for $a_i > 0$ that $(1+a_1)\cdots(1+a_n) \ge 2^n\sqrt{a_1\cdots a_n}$. #2: Prove for $a, b > 0$ that $a/b + b/a \ge 2$, both by using an inequality approach *and* without using an inequality! #3: Solve the double recurrence $f_n = f_{n-1} + 3g_{n-1}, g_n = -3f_{n-1} + 9g_{n-1}$. #4: Define a set to be selfish if it contains its cardinality (i.e., its number of elements) as an element; thus $\{1,3,5\}$ is selfish, while $\{1,2,3,5\}$ is not. Find, with proof, the number of subsets of $\{1,2,...,n\}$ that are minimal selfish sets (that is, selfish sets none of whose subsets are selfish; thus $\{1,3,5\}$ is not minimal selfish as $\{1\}$ is a subset). This is a Putnam problem..... Also, make sure you have done the first 15 Project Euler Problems.

#1: Prove for $a_i > 0$ that $(1+a_1)\cdots(1+a_n) \ge 2^n\sqrt{a_1\cdots a_n}$.
Solution: This follows by the AM-GM inequality applied to each factor: $(1+a_i)/2 \ge \sqrt{1\cdot a_i}$. The claim now follows by multiplication. Note there is an asymmetry here in that, for the first time, our quantities are not all of the same dimension. We can't just rescale the a_i's without changing things. The solution is to introduce $b_1,\dots,b_n$ and see this is the same as $(b_1+a_1)\cdots(b_n+a_n) \ge 2^n\sqrt{a_1b_1\cdots a_nb_n}$, and now if each variable is in meters, both sides are in meters^n. See http://www.aam.org.in/site/st_material/14.pdf for more.

#2: Prove for $a, b > 0$ that $a/b + b/a \ge 2$, both by using an inequality approach *and* without using an inequality!
Solution: We can do this via the AM-GM: we would get

$$\frac{a/b + b/a}{2} \;\ge\; \sqrt{\frac{a}{b}\frac{b}{a}} \;=\; 1.$$

We can also do this with one variable calculus: let $x = a/b$. Then we must show, for $x > 0$, that $x + 1/x \ge 2$. It's nice to have a compact set (closed and bounded) so we can use the wonderful result from real analysis that a continuous function on a closed and bounded set attains its maximum and minimum. Without loss of generality we may assume $x \in [1,2]$; the claim is clearly true for $x \ge 2$, and for $x < 1$ we just consider $1/x$ instead. We now have the function $f(x) = x + 1/x$ on $[1,2]$ and we want to find its minimum.

Since $f'(x) = 1 - 1/x^2$, we see the critical points (where it equals zero) in our set is just $x = 1$, which also happens to be an endpoint!. We find $f(1) = 2, f(2) = 2.5$, and thus the minimum is 2.

Notice if we tried to do this by scaling, we can say without loss of generality $ab = 1$; to see this, replace a by $a' = a\sqrt{t}$ and b by $b' = b\sqrt{t}$, which doesn't change the sum of the fractions but now $a'b' = t$. So, using $ab = 1$ leads to showing $a^2 + 1/a^2 \ge 2$. While we could differentiate or apply results to this expression, we can of course just replace a^2 with x and use the previous argument.

#3: Solve the double recurrence $f_n = f_{n-1} + 3g_{n-1}, g_n = -3f_{n-1} + 9g_{n-1}$.
Solution: We solve for one in terms of the other. Using the second relation we get $f_{n-1} = -\frac{1}{3}g_n + 3g_{n-1}$; as this holds for all indices we can increment n by 1 and find $f_n = -\frac{1}{3}g_{n+1} + 3g_n$. We now substitute these values into the first recurrence, and find

$$\left(-\frac{1}{3}g_{n+1} + 3g_n\right) \;=\; \left(-\frac{1}{3}g_n + 3g_{n-1}\right) + 3g_{n-1} \quad \text{or} \quad g_{n+1} \;=\; 10g_n - 18g_{n-1}.$$

We try $g_n = r^n$ and find a characteristic polynomial of

$$r^2 - 10r + 18 \;=\; 0.$$

If instead we tried to write the g's in terms of the f's we would again obtain this recurrence relation. We now solve in the usual way.

Explicitly, we assume that $f_n = r^n$ to see if we can satisfy the equation. This gives

$$r^{n+1} - 10r^n + 18r^{n-1} \;=\; 0,$$

which means that non-trivial solutions are the roots of the quadratic equation

$$r^2 - 10r + 18 \;=\; 0,$$

which are given by

$$r_1 \;=\; 5 + \sqrt{7}$$
$$r_2 \;=\; 5 - \sqrt{7}.$$

So, a general solution is of the form

$$\alpha_1 r_1^n + \alpha_2 r_2^n.$$

The problem doesn't give any initial values, so we cannot simplify further. To highlight the method, let's assume $f_0 = 0$ and $f_1 = 1$. This gives $\alpha_1 = -\alpha_2$. Consequently,

$$1 \ = \ -\alpha_2(5+\sqrt{7}) + \alpha_2(5-\sqrt{7})$$

and we have that

$$\alpha_2 = -\frac{1}{2\sqrt{7}}$$
$$\alpha_1 = \frac{1}{2\sqrt{7}},$$

which concludes the proof. A solution is of the form

$$f_n = \left(\frac{1}{2\sqrt{7}}\right)(5+\sqrt{7})^n - \left(\frac{1}{2\sqrt{7}}\right)(5-\sqrt{7})^n,$$

and we can use similar methods to find a recursion for the g_n.

#4: Define a set to be selfish if it contains its cardinality (i.e., its number of elements) as an element; thus $\{1,3,5\}$ is selfish, while $\{1,2,3,5\}$ is not. Find, with proof, the number of subsets of $\{1,2,...,n\}$ that are minimal selfish sets (that is, selfish sets none of whose subsets are selfish; thus $\{1,3,5\}$ is not minimal selfish as $\{1\}$ is a subset). This is a Putnam problem.....

Solution: For problems like this, it's best to do a few cases and get a feel. Doing this we find the number of minimal selfish sets, for the first few n, to be the Fibonacci numbers!

Let S_n denote the number of subsets of $\{1,\dots,n\}$ that are minimal selfish. Consider one of the minimal selfish sets; it either contains n, or it doesn't. By definition the number of minimal selfish sets of $\{1,\dots,n\}$ not containing n is S_{n-1}. Imagine now we have a minimal selfish set containing n. Note it's cardinality is its size, and it has no selfish subset. Its cardinality cannot be n if $n > 1$ (as that would mean we have all numbers, and thus selfish subsets). If we subtract 1 from each element we now have a subset of $\{1,\dots,n-1\}$ (note we could not have had 1 and n both in our original set, and thus since we assumed n was in, 1 was not). We remove $n-1$ now, and notice we've decreased all the elements by 1 and removed one element from the original set which had n and was minimal selfish; we now have a minimal selfish subset of $\{1,\dots,n-2\}$ (its cardinality must be in here). Thus the number of minimal selfish sets containing n here is S_{n-2}, and we get the recurrence $S_n = S_{n-1} + S_{n-2}$. We just need the initial conditions, which are $S_1 = 1$ and $S_2 = 1$, to see that it's the Fibonaccis.

Homework #5: Due Friday, October 11, 2024: (0) Show that no matter what 5 points are chosen on the surface of a unit sphere, there is at least one closed hemisphere containing at least 4 of the points. (1) Prove the law of cosines: if a, b and c are the sides of a triangle and θ is the angle between a and b, then $c^2 = a^2 + b^2 - 2ab\cos(\theta)$. (2-21) Complete the first 20 Project Euler Problems, and include in your HW a screenshot showing that you have completed all of these. Note this problem is worth 200 points (20 questions), and is thus giving you credit for all the work you have been doing. We will spend Friday discussing the coding and these problems, so let me know in advance ones you find particularly interesting. Homework (optional): Geometry problems typically invoke extreme reactions: some love, and some hate. If you like geometry problems look at the resources above, and choose 1-2 problems to do and submit. You may use these are HW exemptions for problems in future weeks (i.e., if you get full credit on either of these, you can skip a future problem and receive full credit).

4. HW #5: Due Friday, October 11, 2024

(0) Show that no matter what 5 points are chosen on the surface of a unit sphere, there is at least one closed hemisphere containing at least 4 of the points. (1) Prove the law of cosines: if a, b and c are the sides of a triangle and θ is the angle between a and b, then $c^2 = a^2 + b^2 - 2ab\cos(\theta)$. (2-21) Complete the first 20 Project Euler Problems, and include in your HW a screenshot showing that you have completed all of these. Note this problem is worth 200 points (20 questions), and is thus giving you credit for all the work you have been doing. We will spend Friday discussing the coding and these problems, so let me know in advance ones you find particularly interesting. Homework (optional): Geometry problems typically invoke extreme reactions: some love, and some hate. If you like geometry problems look at the resources above, and choose 1-2 problems to do and submit. You may use these as HW exemptions for problems in future weeks (i.e., if you get full credit on either of these, you can skip a future problem and receive full credit).

(0) Show that no matter what 5 points are chosen on the surface of a unit sphere, there is at least one closed hemisphere containing at least 4 of the points.

Solution: This should hopefully feel like a pigeonhole principle, but what are the boxes and pigeons? The pigeons are almost surely related to the five points, but what are the boxes? Sometimes it helps to try to look at a simpler case first. What would the two-dimensional version on a circle be? Perhaps it is if we have 4 points on a unit circle, at least 3 are on the same semicircle. There are unfortunately infinitely many semi-circles. One excellent choice is to take a point at random and look at all the semi-circles generated from it. Without loss of generality we may assume our point is at $(1, 0)$, and our family of semi-circles containing that range from the semi-circle in the left half of the circle (with points at $(-1, 0)$ and $(0, -1)$) to the semi-circle on the right half of the circle (with points at $(1, 0)$ and $(0, -1)$). Notice the two extreme semi-circles separate the circle into two pieces (that overlap at our point $(0, 1)$ and at $(0, -1)$), and each of our three points is in either the left, the right or both. Thus by the Pigeonhole Principle either the left or the right must get at least two of the three additional points, and thus either the left or the right must have at least 4 of the 5 points. Notice if two points are at $(0, 1)$ and two are at $(0, -1)$ then we still have a semi-circle containing at least 3 points but only if we use the endpoints; if we don't use the endpoints there is no way to do it!

Building on this we now return to the sphere. Without loss of generality we may assume one point is at the north pole, $(0, 0, 1)$. No matter what point we take next, those two points lie on a great circle and split the sphere into two halves, let's call then the 'top' and 'bottom'. We now have 3 = 5-2 points left to add, and by the Pigeonhole Principle at least two of the three must go to either the top or the bottom, proving the claim.

(1) Prove the law of cosines: if a, b and c are the sides of a triangle and θ is the angle between a and b, then $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.

Solution: Not surprisingly, the idea is to reduce to applications of the Pythagorean Theorem. See Figure 4.

Drop the perpendicular onto the side c to get (see Fig. 5)

$$c = a\cos\beta + b\cos\alpha.$$

(This is still true if α or β is obtuse, in which case the perpendicular falls outside the triangle.) Multiply through by c to get

$$c^2 = ac\cos\beta + bc\cos\alpha.$$

By considering the other perpendiculars obtain

$$a^2 = ac\cos\beta + ab\cos\gamma,$$
$$b^2 = bc\cos\alpha + ab\cos\gamma.$$

Adding the latter two equations gives

$$a^2 + b^2 = ac\cos\beta + bc\cos\alpha + 2ab\cos\gamma.$$

Subtracting the first equation from the last one we have

$$a^2 + b^2 - c^2 = -ac\cos\beta - bc\cos\alpha + ac\cos\beta + bc\cos\alpha + 2ab\cos\gamma$$

which simplifies to

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

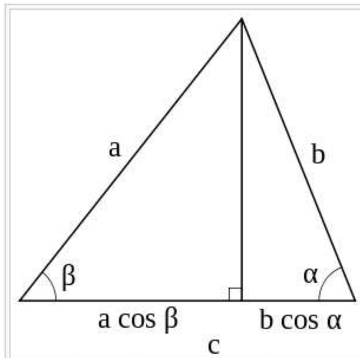

Fig. 5 – An acute triangle with perpendicular

Figure 4. Proof from https://en.wikipedia.org/wiki/Law_of_cosines.

HW #6: Due Friday, October 18: #1: Let $a_1, a_2, \dots, a_n$ be positive integers. Show a subset sums to a multiple of n. #2: Given any n, show there is a number x_n whose digits are only 0's and 7's such that n divides x_n. #3: Consider the previous problem. Find such a number for $n = 2017$; what is the smallest such number? #4: Show that if n divides a Fibonacci number that it divides infinitely many Fibonacci numbers. #5: For all positive real numbers a, b, c show that $a^a b^b c^c >= a^b b^c c^a$.

5. HW #6: Due Friday, October 18, 2024

HW #6: Due Friday, October 18: #1: Let $a_1, a_2, \dots, a_n$ be positive integers. Show a subset sums to a multiple of n. #2: Given any n, show there is a number x_n whose digits are only 0's and 7's such that n divides x_n. #3: Consider the previous problem. Find such a number for $n = 2017$; what is the smallest such number? #4: Show that if n divides a Fibonacci number that it divides infinitely many Fibonacci numbers. #5: For all positive real numbers a, b, c show that $a^a b^b c^c >= a^b b^c c^a$.

#1: Let $a_1, a_2, \dots, a_n$ be positive integers. Show a subset sums to a multiple of n.
Solution: As we are trying to prove a sum is a multiple of n, it is natural to look at sums modulo n. For pigeon hole problems we need to identify the boxes and the pigeons. The boxes should be the n residue classes modulo n. What about the pigeons? There are 2^n possible sums, but that's a lot more than n. Let's try looking at something smaller such as $a_1, a_1+a_2, a_1+a_2+a_3, \dots, a_1+\cdots+a_n$. Either one of these n sums is zero modulo n and we win, or else two by the Pigeon Hole Principle have the same remainder, say $i > j-1$. If we subtract, we get $a_i + \cdots + a_j = 0 \bmod n$, completing the claim.

#2: Given any n, show there is a number x_n whose digits are only 0's and 7's such that n divides x_n.
Solution: This problem is very similar to the previous. We start looking at the numbers 7, 77, 777 and so on; if we look at $n+1$ then two must have the same remainder modulo n. Subtracting the smaller from the larger leaves $777\cdots77000\cdots0$, which is congruent to zero modulo n and thus is our desired solution.

#3: Consider the previous problem. Find such a number for $n = 2017$; what is the smallest such number?
We could of course apply the method from above; as 10 is a multiple of just 2 and 5 we see there can be no trailing 7's in our answer. Instead, as we are asked to find the smallest such number, we just write a simple code to do that.

```
findsmallestsevenzero[max_, target_] := Module[{},
   smallest = Infinity;
   For[n = 1, n <= max, n++,
     {
      (* next line converts n to binary, multiples all digits by 7 *)
      digits = 7 IntegerDigits[n, 2];
      numdigits = Length[digits];
      number = Sum[digits[[d]] 10^(numdigits - d), {d, 1, numdigits}];
      If[Mod[number, target] == 0,
        {
         If[number < smallest, smallest = number];
         n = max + 10; (* exit for loop if found soln *)
         }];
      }]; (* end of n for loop *)
   If[smallest < Infinity, Print[smallest]];
   ];
```

This generates the answer 70077077707007; i.e., this is the first non-zero number whose digits are just 0's and 7's which is congruent to 0 modulo 2017. It might be interesting to see how the length of the smallest number varies as a function of the target. If we did 2016 we would find 77777777700000 (same number of digits), while 2018 is 700700077777770 and 2019 is the significantly shorter 700700007.

#4: Show that if n divides a Fibonacci number that it divides infinitely many Fibonacci numbers.
Solution: Note that the Fibonacci numbers are periodic modulo m for any m. The reason is the pigeonhole principle. Modulo m there are only m possible residues, and thus only m^2 possible pairs of two numbers modulo m. Once we look at m^2+2 consecutive Fibonacci numbers we have m^2+1 pairs, and thus at least two pairs are the same.

For our problem, let's look at the Fibonacci numbers modulo n. By assumption we know n divides one of them; we now prove it divides infinitely many as the pattern repeats. To see this, imagine we have repeating pairs at indices (i_1, i_1+1) and (i_2, i_2+1), and let's assume F_k is our given multiple of n. If k is one of these indices, or between them, it's clear. What if k isn't? Well, we had to hit k as we walked from indices $(0, 1)$ to (i_1, i_1+1); thus if we run backwards from (i_1, i_1+1) we must hit k; however, this will give us the same residues as we would get walking backwards from (i_2, i_2+1), and so we must have something between our two pairs that's a multiple of n.

#5: For all positive real numbers a, b, c show that $a^a b^b c^c \ge a^b b^c c^a$.

Solution: If we wanted, we could rescale and assume $abc = 1$. Why? If we multiply each by r we get each side increases by $r^{r(a+b+c)}$, and thus the relation still holds or doesn't hold. It doesn't help us, but for awhile I thought about making their product 1, or setting b equal to 1.... What is more useful is there is a cyclic symmetry, and without loss of generality we may assume $a \le b \le c$. Some ordering exists, the left hand side is independent of the ordering, and seeing the cyclicity (the right hand side is also $b^c c^a a^b$ or $c^a a^b b^c$) there is no harm in assuming an ordering.

In some sense, if you look at this problem the right way it's "obvious". Why? Imagine our numbers are integers. We're talking about having some number of powers of a, b and c. We can choose $a + b + c$ numbers. Clearly you want to have as many powers of c as possible, so give it the exponent c. Then let's take as many b's as we can, namely b of them, and finally let's take the rest to be a.

More formally, we have the following chain (which holds for positive real numbers $a \le b \le c$):

$$\begin{aligned} a^a b^b c^c &= a^a b^b c^{c-(b-a)+(b-a)} \\ &\ge a^{a+(b-a)} b^b c^{c-(b-a)} \\ &= a^b b^b c^{(c-b)+b-(b-a)} \\ &\ge a^b b^{b+(c-b)} c^{b-(b-a)} = a^b b^c c^a. \end{aligned}$$

Note that all the exponents are positive, and the inequalities are true as we replace larger numbers in the product with smaller ones.

For another good inequality to know, see Jensen's inequality:

http://www.artofproblemsolving.com/Wiki/index.php/Jensen's_Inequality

Homework #7: Due October 25, 2024: #1, #2, #3, #4 (counts as four problems): Show that any decomposition of N as a sum of Fibonacci numbers cannot have fewer summands than the Zeckendorf decomposition. Is there a monovariant that can help?

ALSO: For Monday, think about which is larger: e^π or π^e. You are NOT allowed to use a computer to calculate anything; try to prove elementarily which wins.