Math 331: The little Questions (Fall 2024)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 1 Video: https://youtu.be/n4nIJ2Si8ho

Lecture 1: 9-6-24: https://youtu.be/n4nIJ2Si8ho

Objectives

- Obviously learn problem solving.
- Emphasize techniques / asking the right questions.
- Learn to use computers to build intuition.
- Use these problems as a springboard to see good math.
- Uphold honor of Williams in competitions.
- Looking at equations and getting a sense: $\log -5$ Method: $\frac{p \pm pq}{p+q \pm 2pq}$.

Grading / Administrative

- Homework: 50%, Midterm 10%, Final 5%, Class Participation: 20%, Project Euler: 15%. Late, messy or unstapled HW will not be accepted.

- Pre-reqs: linear algebra (programming a plus).

Office hours

- TBD and when I'm in my office.

http://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/ : Numerous handouts, additional comments each day (mix of review and optional advanced material).

Opportunity to help with Pi Mu Epsilon Problem Section.

Opportunity to help with Math riddles page: http://mathriddles.williams.edu/.

Opportunity to help with Math Outreach.

Being Prepared

Never know when an opportunity presents itself....

S. J. Miller at the Sarnak 61st Dinner

Being Prepared

- Your Job:
  - Be prepared for class: do reading, think about material.
  - Come to me / come to each other with questions.
- My Job:
  - Provide resources, guiding questions.
  - Be available.

Other: Advice from Jeff Miller

- Party less than the person next to you.
- Take advantage of office hours / mentoring.
- Learn to manage your time: no one else wants to.

Happy to do practice interviews, adjust deadlines....

LaTeX and Mathematica Tutorials and Templates

Templates for using LaTeX for papers, talks, posters, and a Mathematica tutorial (with video):
http://web.williams.edu/Mathematics/sjmiller/public_html/math/handouts/latex.htm

Handout homepage:
http://web.williams.edu/Mathematics/sjmiller/public_html/handouts/handouts.htm

General advice:
https://web.williams.edu/Mathematics/sjmiller/public_html/advice.htm

Search Results on **fractions**

https://mathriddles.williams.edu/?s=fractions

Sum to 1

No comments

Use the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 (each exactly once) to form 3 fractions such that the 3 sum to 1. Each numerator must be 1 digit and each denominator 2 digits. Example: 1/23 + 4/56 + 7/89, (but of course this doesn't work).

Articles

Read more

$1, 2, \ldots, 9$ and $\dfrac{a}{10b+c} + \dfrac{d}{10e+f} + \dfrac{g}{10h+i} = 1$

Try Everything

$9!$

$6! = 720$ so $9! = \underbrace{9 \cdot 8 \cdot 7}_{\approx 500 \approx 1000/2} \cdot 720$

$9! \approx 360{,}000$

wlog 9 lives in first fraction, split into 3 cases

Case 1: $\dfrac{9}{10a+b} \longrightarrow 8!$

Case 2: $\dfrac{a}{10\cdot 9+b} \longrightarrow 8!$

Case 3: $\dfrac{a}{10b+9} \longrightarrow 8!$

So it is $3\cdot 8!$ vs $9!$

Not worth the savings as run only once!

$$\frac{a}{10b+c} + \frac{d}{10e+f} + \frac{g}{10h+i} = 1$$

Imagine smallest tens digit is a 3: could this work?

b, e, h are not 1 or 2

↳ each fraction is $< \frac{10}{30} = \frac{1}{3}$

cannot add to 1.

Imagine smallest tens digit is a 2: could that work?

$$\frac{9}{21} + \frac{8}{35} + \frac{7}{46} = \frac{1303}{1610} < 1$$

↳ cannot have 2 as smallest tens digit

Guess: 7!

9

___ + ___ + ___

known 8!

2 as tens digit or as ones digit in first

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Lecture 02: Induction: September 9, 2024: https://youtu.be/1KKfjTTGzys

Code from fractions:

```
bob=Permutations[{1,2,3,4,5,6,7,8,9}];
For[j=1,j<=9!,j++,
{
  permat=bob[[j]];
  For[i=1,i<=9,i++,dig[i]=permat[[i]]];

Numb=(dig[1]/(10*dig[2]+dig[3]))+(dig[4]/(10*dig[5]+dig[6])
)
       +(dig[7]/(10*dig[8]+dig[9]));
  If[Abs[1-Numb]<=.00001,{Print["We win",permat];j=10!;}];
  If[Mod[j,10000]==0,Print["We are at j = ",j," out of ",9!]];
}]
```

Answer: $\frac{9}{12}+\frac{5}{34}+\frac{7}{68}$

python: from **Angus Henderson**

```python
import numpy as np
from sympy.utilities.iterables import multiset_permutations

nums = [9, 8, 7, 6, 5, 4, 3, 2] # ordered in reverse as we know larger values more likely to be numerator

def fraction_value(nums):
    frac1 = nums[0] / (10 + nums[1]) # uses our knowledge that 1 must be a ten's denominator
    frac2 = nums[2] / (10 * nums[3] + nums[4])
    frac3 = nums[5] / (10 * nums[6] + nums[7])
    return frac1 + frac2 + frac3

for ns in multiset_permutations(nums):
    if fraction_value(ns) == 1:
        print(np.array(ns))
        break

# gives 9/12 + 5/34 + 7/68 = 1 as solution.
```

Code from Cameron White:

```
[[[[[[[[[print(a,b,c,d,e,f,g,h,i) for a in range(1,10)

    if len({a,b,c,d,e,f,g,h,i})==9

    if 1==(a/(10*b+c) + d/(10*e+f) + g/(10*h+i))]
      for b in range(1,10) if len({b,c,d,e,f,g,h,i})==8]
      for c in range(1,10) if len({c,d,e,f,g,h,i})==7]
      for d in range(1,10) if len({d,e,f,g,h,i})==6]
      for e in range(1,10) if len({e,f,g,h,i})==5]
      for f in range(1,10) if len({f,g,h,i})==4]
      for g in range(1,10) if len({g,h,i})==3]
      for h in range(1,10) if len({h,i})==2]
      for i in range(1,10)]
```

More on Fractions:

$\cancel{\frac{16}{64}} = \frac{1}{4}$

$\frac{12}{24} = \frac{1}{4}$

$\cancel{\frac{49}{98}} = \frac{1}{2}$

$\cancel{\frac{19}{95}} = \frac{1}{5}$

Euler's Prime
producing poly

Induction

One of the most important techniques we have for proving results.

Say we have some statement P(n). Perhaps P(n) is “the sum of the first n integers is n(n+1)/2”.

We can check this for various n; every time we check it is true but that is NOT the same as a proof!

Induction

Say we have some statement P(n). Perhaps P(n) is “the sum of the first n integers is n(n+1)/2”.

Imagine we can show the following two statements are true.

1. P(1) is true, and
2. Whenever P(n) is true then P(n+1) is true.

If these are true then have P(n) is true for all n!

(Note: Sometimes we start at n=0 not n=1)

Induction

Say we have some statement P(n). Perhaps P(n) is “the sum of the first n integers is n(n+1)/2”.

Imagine we can show the following two statements are true.

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Why does this imply that it holds for all n?

Induction (Box, Dirichlet, Pigeonhole Principle)

Imagine we can show the following two statements are true.

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Take n=1: thus the second becomes P(1) true implies P(2) true

P(1) is true

P(1) true implies P(2) true

THEREFORE since P(1) is true we now know P(2) is true.

Induction (Box, Dirichlet, Pigeonhole Principle)

Imagine we can show the following two statements are true.

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

We know P(1) and P(2) are true.
Take n=2: thus the second becomes P(2) true implies P(3) true
P(2) is true
P(2) true implies P(3) true
THEREFORE since P(2) is true we now know P(3) is true.

Induction (Box, Dirichlet, Pigeonhole Principle)

Imagine we can show the following two statements are true.

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

We know P(1), P(2) and P(3) are true.

Take n=3: thus the second becomes P(3) true implies P(4) true

P(3) is true

P(3) true implies P(4) true

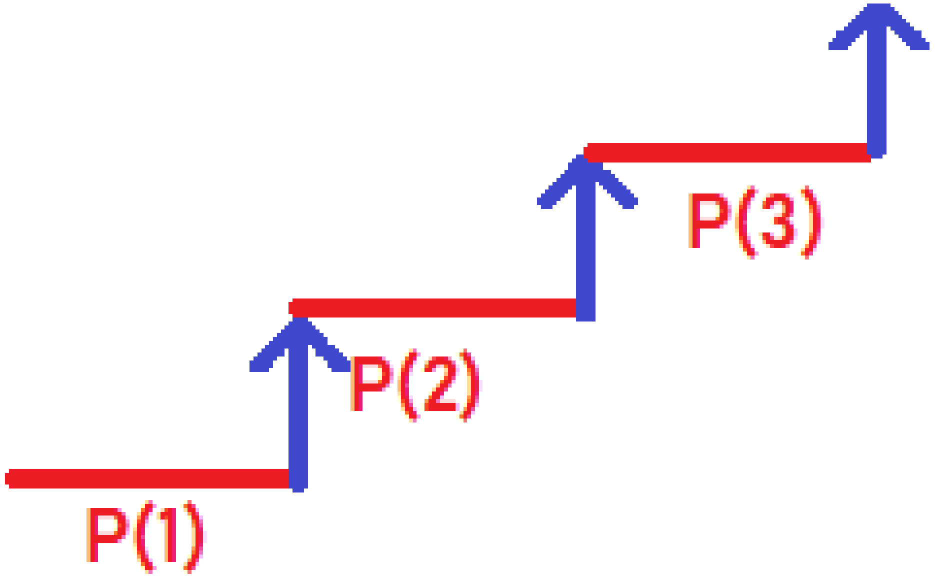
THEREFORE since P(3) is true we now know P(4) is true. AND SO ON!

Induction (Box, Dirichlet, Pigeonhole Principle)

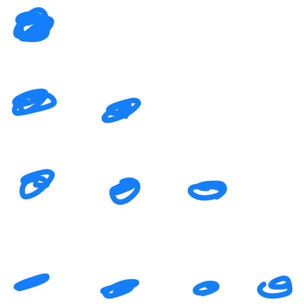
To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

This is often viewed as a staircase.



Example: P(n): $1 + 2 + \ldots + n = n(n+1)/2$



To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

We will prove this by induction. There are two steps.

First we prove P(1) is true, then we show IF P(n) is true THEN P(n+1) is true.

Example: $P(n)$: $1 + 2 + \ldots + n = n(n+1)/2$

To prove $P(n)$ is true for all n, must show

1. Base case: $P(1)$ is true, and
2. Inductive Step: Whenever $P(n)$ is true then $P(n+1)$ is true.

Step 1: Base Case: We must show $P(1)$ is true.

Thus we must show that when $n=1$, we have $1 = 1(1+1)/2$.

This however follows immediately!

We are done with the base case.

Example: $P(n)$: $1 + 2 + \ldots + n = n(n+1)/2$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: We now get to ASSUME that P(n) is true, and we must show that P(n+1) is true.

We are done with the base case. We could try to do n=2 or n=3 to build up intuition, but it is not necessary.

Example: $P(n)$: $1 + 2 + \ldots + n = n(n+1)/2$

To prove $P(n)$ is true for all n, must show

1. Base case: $P(1)$ is true, and
2. Inductive Step: Whenever $P(n)$ is true then $P(n+1)$ is true.

Step 2: Inductive Step: We now get to ASSUME that $P(n)$ is true, and we must show that $P(n+1)$ is true.

Extra work: If n=2 let's check: Does $1+2 = 2(2+1)/2$? YES!

Extra work: if n=3 let's check: Does $1+2+3 = 3(3+1)/2$? YES!

These extra checks are not a substitute for a proof, but the more values of n that work, the more confident we are that it is true.

Example: P(n): $1 + 2 + \ldots + n = n(n+1)/2$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: We now get to ASSUME that P(n) is true, and we must show that P(n+1) is true.

OK, we now get to assume P(n) is true, we want to prove P(n+1) is true.

What does this mean?

P(n) true means we assume $1 + 2 + \ldots + n = n(n+1)/2$.

We want to prove that P(n+1): $1 + 2 + \ldots + n + (n+1) = (n+1)(n+1+1)/2$ is true.

How should we proceed? When we look at P(n+1), do we see anything related to P(n)?

Example: $P(n)$: $1 + 2 + \ldots + n = n(n+1)/2$

To prove $P(n)$ is true for all n, must show

1. Base case: $P(1)$ is true, and
2. Inductive Step: Whenever $P(n)$ is true then $P(n+1)$ is true.

Step 2: Inductive Step: We now get to ASSUME that $P(n)$ is true, and we must show that $P(n+1)$ is true.

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How should we proceed? Notice that the sum for n+1 starts off exactly as the sum for n!

What are we assuming we know about $1 + 2 + \ldots + n$? We are assuming it equals ….

Example: P(n): 1 + 2 + … + n = n(n+1)/2

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
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We want to prove that P(n+1): $1 + 2 + \ldots + n + (n+1) = (n+1)(n+1+1)/2$ is true.

How should we proceed? Notice that the sum for n+1 starts off exactly as the sum for n!

What are we assuming we know about $1 + 2 + \ldots + n$? We are assuming it equals $n(n+1)/2$.

Thus let's substitute for $1 + 2 + \ldots + n$ in $1 + 2 + \ldots + n + (n+1)$.

Example: P(n): $1 + 2 + \ldots + n = n(n+1)/2$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
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OK, we now get to assume P(n) is true, we want to prove P(n+1) is true.

What does this mean?

P(n) true means we assume $1 + 2 + \ldots + n = n(n+1)/2$.

We want to prove that P(n+1): $1 + 2 + \ldots + n + (n+1) = (n+1)(n+1+1)/2$ is true.

Using the inductive assumption, we have

$1 + 2 + \ldots + n + (n+1) = (1 + 2 + \ldots + n) + (n+1) = n(n+1)/2 + (n+1)$.

Now we just need to show the far right equals our claim, $(n+1)(n+1+1)/2$. How do we add two fractions?

Example: P(n): 1 + 2 + … + n = n(n+1)/2

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: We now get to ASSUME that P(n) is true, and we must show that P(n+1) is true.

OK, we now get to assume P(n) is true, we want to prove P(n+1) is true.

What does this mean?

P(n) true means we assume 1 + 2 + … + n = n(n+1)/2.

We want to prove that P(n+1): 1 + 2 + … + n + (n+1) = (n+1)(n+1+1)/2 is true.

We have 1 + 2 + … + n + (n+1) = n(n+1)/2 + (n+1).

But $\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$, which is what we needed to show, completing the proof (as n+2 = n+1+1)!

Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

The proof is similar to what we just did!

Step 1: The Base Case: n=1: Is

Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
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The proof is similar to what we just did!

Step 1: The Base Case: n=1: Is $1^2 = 1(1+1)(2*1 + 1)/6$? YES!

We don't need to, but we can check other values of n.

If n=2 does

If n=3 does

Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

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The proof is similar to what we just did!

Step 1: The Base Case: n=1: Is $1^2 = 1(1+1)(2*1 + 1)/6$? YES!

We don't need to, but we can check other values of n.

If n=2 does $1^2 + 2^2 = 2(2+1)(2*2+1)/6$? YES!

If n=3 does $1^2 + 2^2 + 3^2 = 3(3+1)(2*3 + 1)/6$? YES!

Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

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P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

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We must show P(n+1) is true. What is that?

P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

What is in common with P(n) and P(n+1)?

Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

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P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

What is in common with P(n) and P(n+1)? We can now substitute….

Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

So is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (1^2 + 2^2 + \ldots + n^2) + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2$.

We have to combine the fractions – how do we do that?

Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

So is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = = n(n+1)(2n+1)/6 + (n+1)^{2.}$

We have $\frac{n(n+1)(2n+1)}{6} = \frac{6(n+1)2}{6} =$??? What is in common with the two fractions? Both have a ….

Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

So is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = = n(n+1)(2n+1)/6 + (n+1)^2$.

We have $\frac{n(n+1)(2n+1)}{6} = \frac{6(n+1)2}{6} = \frac{(n+1)(n(2n+1)+\ 6(n+1))}{6} = \frac{(n+1)(2n2+n+6n+6)}{6} = \frac{(n+1)(2n2+7n+6)}{6}$

Doing some algebra, we see $2n2 + 7n + 6$ equals (n+2)(2n+3) by FOIL, completing the proof.

Example: P(n): $1 + 3 + \ldots + (2n-1) = n^2$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

The proof is similar to what we just did!

Step 1: The Base Case: n=1: Is $1 = 1^2$? YES!

We don't need to, but we can check other values of n.

If n=2 does

If n=3 does

Rest of the proof is similar to what we've done before….

Example: P(n): $1 + 3 + \ldots + (2n-1) = n^2$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

The proof is similar to what we just did!

Step 1: The Base Case: n=1: Is $1 = 1^2$? YES!

We don’t need to, but we can check other values of n.

If n=2 does $1 + 3 = 2^2$? YES!

If n=3 does $1 + 3 + 5 = 3^2$? YES!

Rest of the proof is similar to what we’ve done before….

Example: P(n): $1 + 3 + \ldots + (2n-1) = n^2$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
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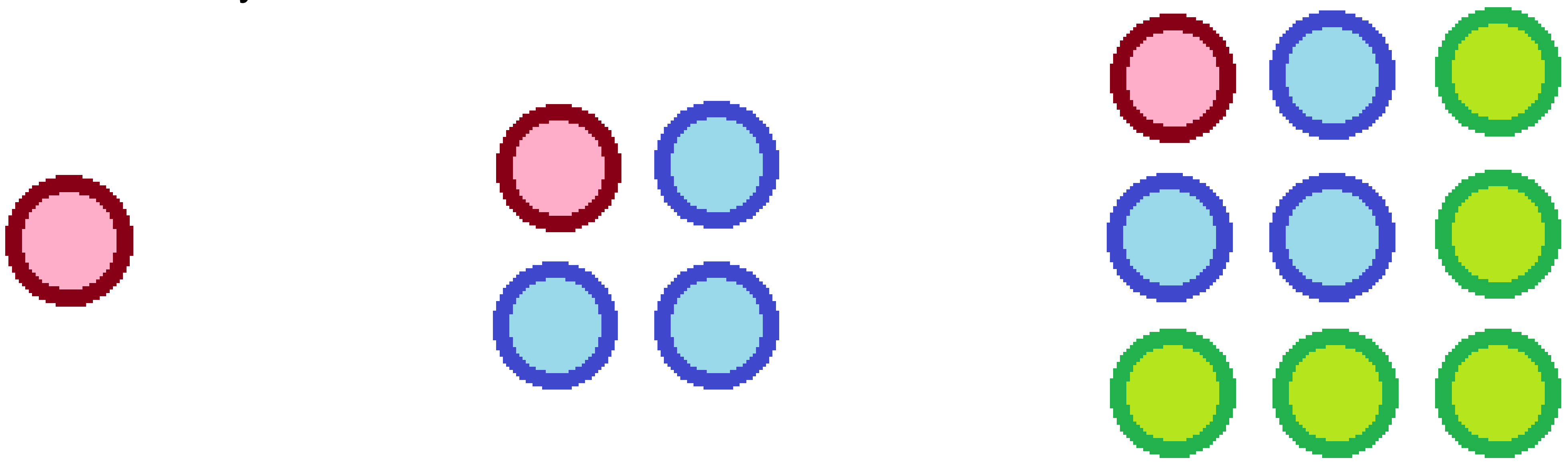
Can prove in other ways than Induction….



Example: P(n): 133 divides $11^{n+1} + 12^{2n-1}$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Let's try to show P(1) is true: does 133 divide $11^{1+1} + 12^{2*1-1}$?

Example: P(n): 133 divides $11^{n+1} + 12^{2n-1}$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Let's try to show P(1) is true: does 133 divide $11^{1+1} + 12^{2*1-1}$?

Yes, as $11^{1+1} + 12^{2*1-1} = 11^2 + 12 = 121 + 12 = 133$, which is clearly a multiple of 133.

Example: P(n): 133 divides $11^{n+1} + 12^{2n-1}$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Now assume P(n) is true, we must show P(n+1) is true.

Can assume 133 divides $11^{n+1} + 12^{2n-1}$, must show 133 divides $11^{n+1} + 12^{2n-1}$.

$$\begin{aligned} 11^{(n+1)+1} + 12^{2(n+1)-1} &= 11^{n+1+1} + 12^{2n-1+2} \\ &= 11 \cdot 11^{n+1} + 12^2 \cdot 12^{2n-1} \\ &= 11 \cdot 11^{n+1} + (133+11)12^{2n-1} \\ &= 11\left(11^{n+1} + 12^{2n-1}\right) + 133 \cdot 12^{2n-1}. \quad \text{(A.6)} \end{aligned}$$

By the inductive assumption 133 divides $11^{n+1} + 12^{2n-1}$; therefore, 133 divides $11^{(n+1)+1} + 12^{2(n+1)-1}$, completing the proof.

Getting a feel for the answer….

Showed 1 + 2 + … + n = n(n+1)/2 = $n^2/2$ + n/2. Is this reasonable?

How can we try to get an UPPER BOUND and a LOWER BOUND for the sum?

$0 \le 1 \le n \le \quad 1 + \cdots + n \quad \le n^2 \le n^n < \infty$

$(1 + 2 + \cdots + \frac{n}{2}) + \underbrace{(\frac{n}{2} + 1 + \cdots + n)}_{\text{lower bound}}$

$\geq \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$

Final thoughts on sums of powers….

Hardest part of the induction is knowing what to PROVE.

How can we find the formula?

Looking at the cases we've done it looks like it is always a polynomial of degree one higher than the power, constant term is zero, leading term (if sum of k^{th} powers) is $n^{k+1} / (k+1)$.

Note 2 points determine a line, 3 points a quadratic (parabola), 4 a cubic, and so on; we can evaluate the sum for a few points and then INTERPOLATE and figure out the polynomial!

Challenge: Prove $1^3 + 2^3 + \ldots + n^3 = n^2 (n+1)^2 / 4$.

False proofs by Induction

The following is my favorite false proof by induction. Where is the mistake?

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Let’s try to prove this by induction. We must show:

1. Base Case: In any group with 1 person, everyone has the same name.
2. Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

False proofs by Induction

The following is my favorite false proof by induction. Where is the mistake?

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Let's try to prove this by induction. We must show:

1. Base Case: In any group with 1 person, everyone has the same name.

PROOF OF BASE CASE: This follows immediately, as….

False proofs by Induction

The following is my favorite false proof by induction. Where is the mistake?

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Let’s try to prove this by induction. We must show:

1. Base Case: In any group with 1 person, everyone has the same name.

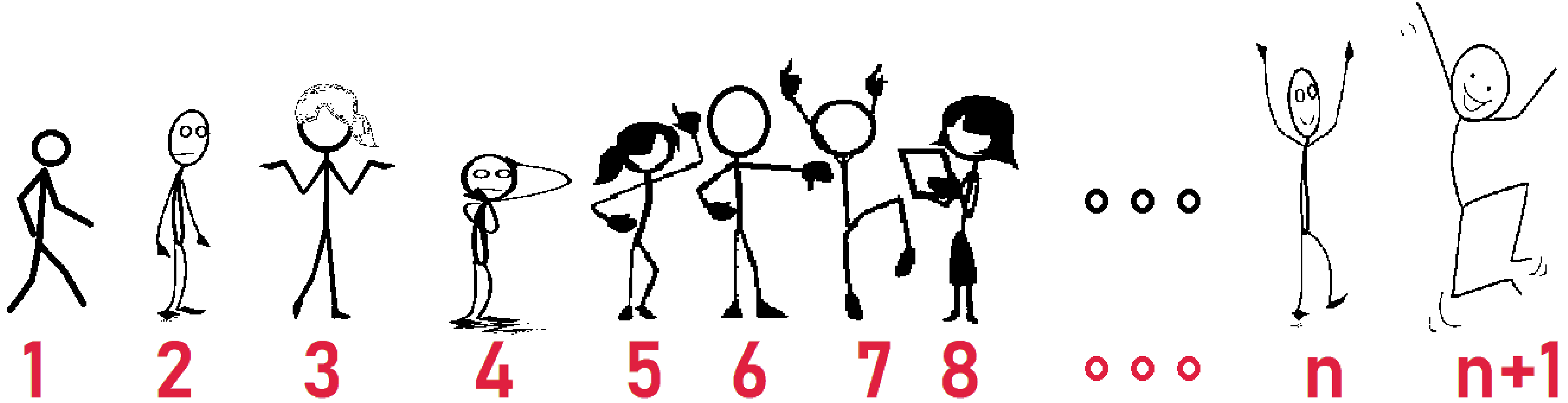
PROOF OF BASE CASE: This follows immediately, as there is only one person in the group, so clearly everyone in the group has the same name!

False proofs by Induction

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

“PROOF” OF INDUCTIVE STEP: We assume everyone in a group of size n has the same name, must show true for a group of size n+1. Consider a group of n+1 people. How can we use the inductive assumption (all groups of size n have all with the same name)? Can you find some groups of size n?



False proofs by Induction

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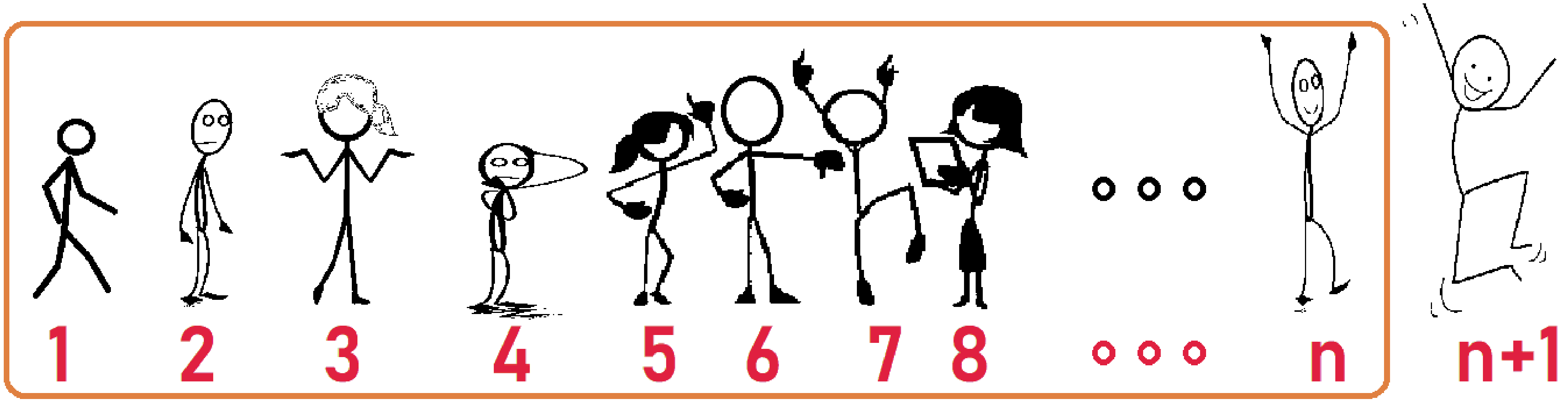

False proofs by Induction

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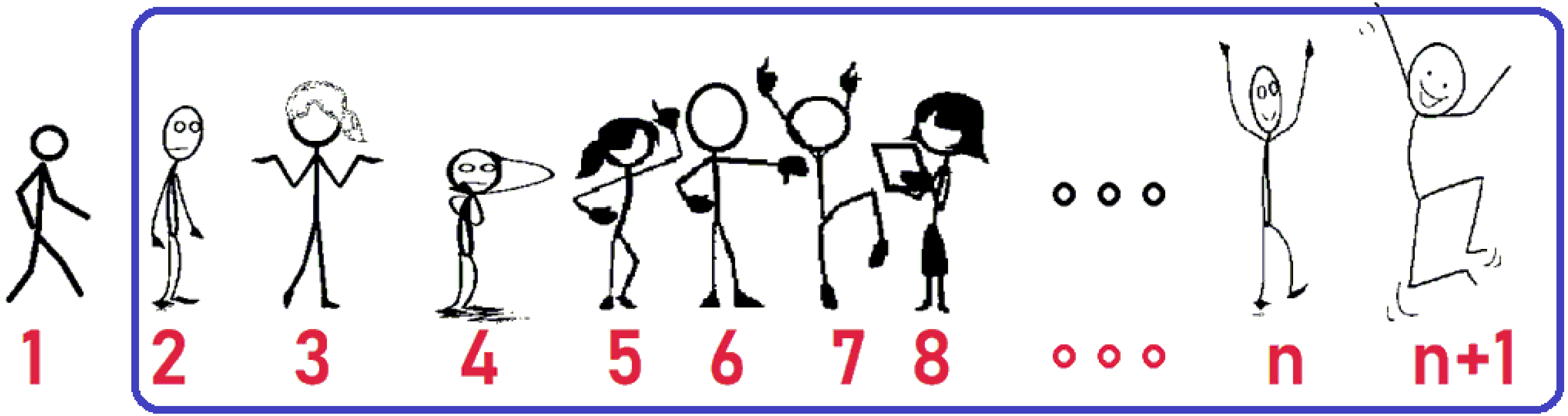

False proofs by Induction

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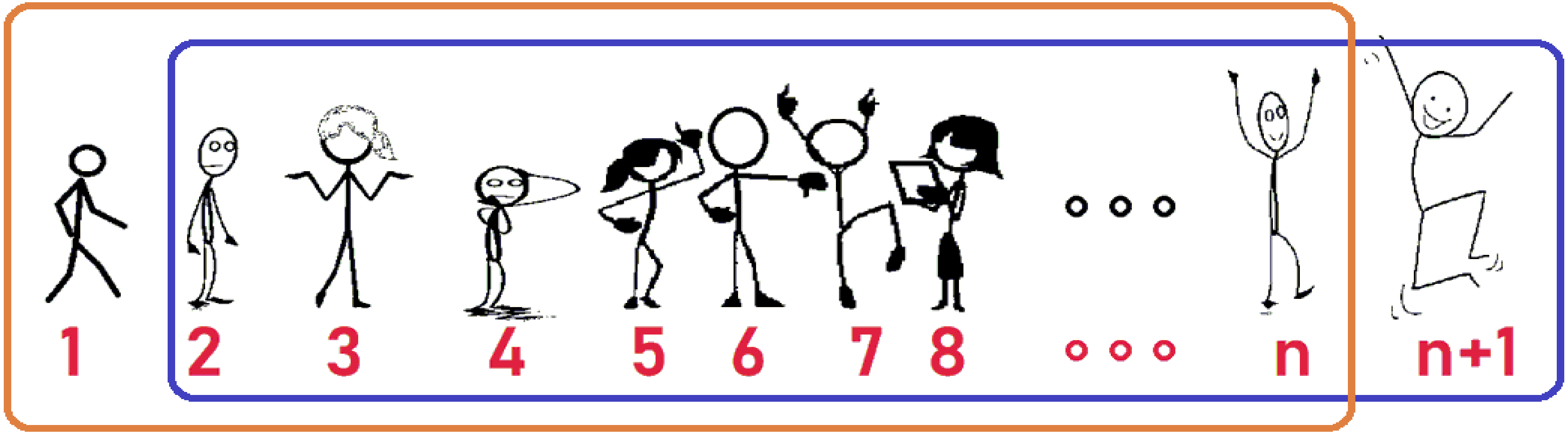
“PROOF” OF INDUCTIVE STEP: We assume everyone in a group of size n has the same name, must show true for a group of size n+1. Consider a group of n+1 people. How can we use the inductive assumption (all groups of size n have all with the same name)? Can you find some groups of size n? Note people 2, 3, …, n are in both groups!



False proofs by Induction

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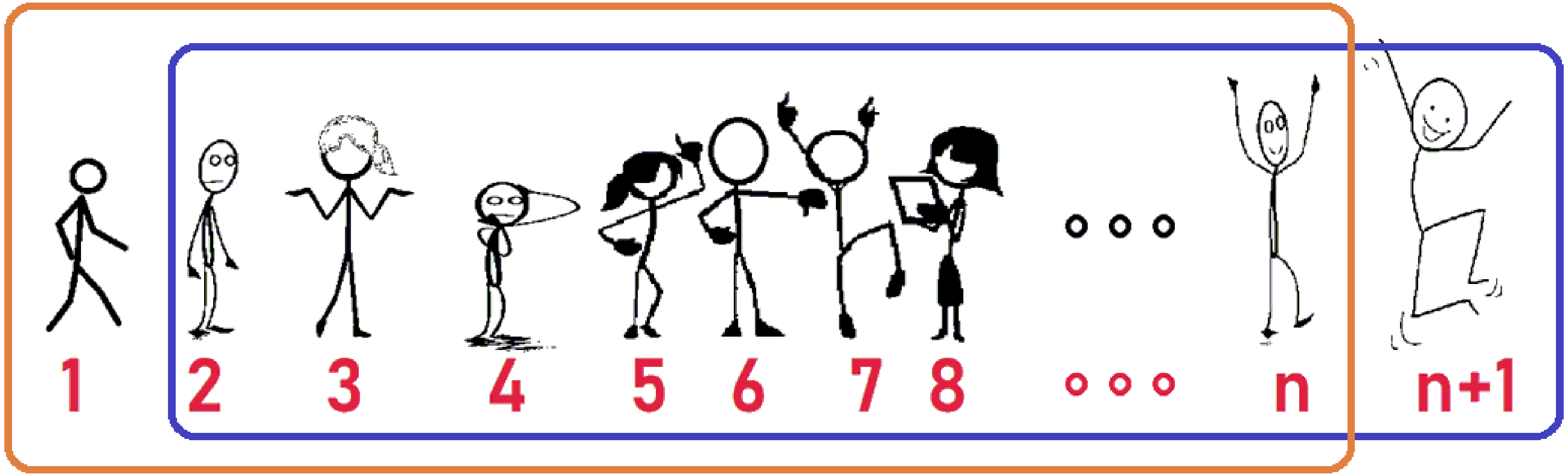
Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

“PROOF” OF INDUCTIVE STEP: We assume everyone in a group of size n has the same name, must show true for a group of size n+1. Consider a group of n+1 people. How can we use the inductive assumption (all groups of size n have all with the same name)? Can you find some groups of size n? Note people 2, 3, …, n are in both groups! Thus everyone in the first n has the same name, everyone in the last n has the same name, and since people 2, 3, …, n are in both that means those two names are the same and our proof is done! If your name is not Steve Miller, you should be skeptical. Mistake?

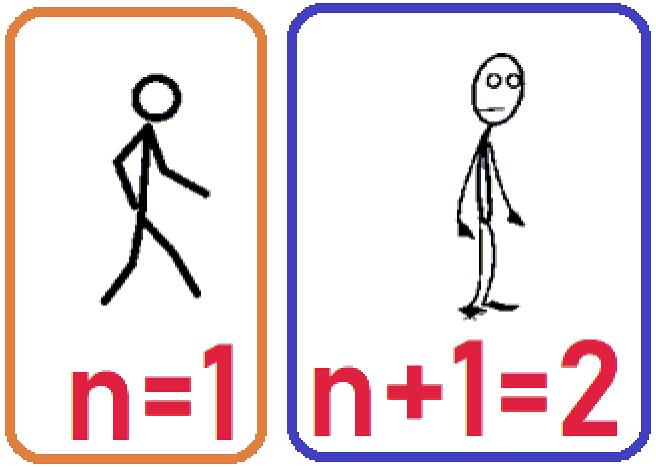


False proofs by Induction

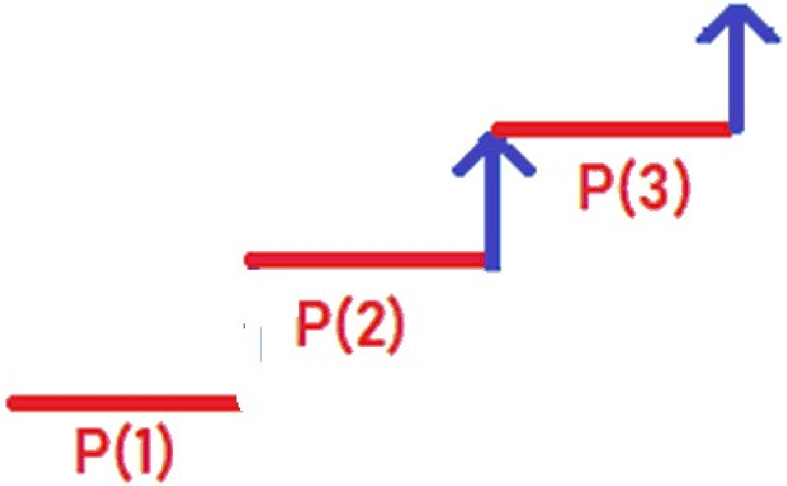
P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

“PROOF” OF INDUCTIVE STEP: The mistake is we drew this for a “large” n. Remember we must show for ANY n that if P(n) is true then P(n+1) is true. If n is 2 or more then there is a person in both groups, but if n=1 there is not!



When n=1 we see there is no overlap!



Product Rule and Induction

$A(x) = f(x)g(x)$

$$A'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x) + \lim_{h\to 0} f(x) \frac{g(x+h) - g(x)}{h}$$

$$= f'(x)g(x) + f(x)g'(x)$$

Know the deriv of x^n is nx^{n-1} if n positive integer

↳ Binomial Thm: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ (take $y=h$)

just need $(x+h)^n = x^n + nx^{n-1}h + (\text{terms of } h^2 \text{ and higher})$

Prove Binomial Thm

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ = # ways to choose k objects from n objects when order doesn't matter.

$(x+y)^{n+1} = (x+y)(x+y)^n = x(x+y)^n + y(x+y)^n$

"See" $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$

Gathering Data

$f(x) = x^n$ $\qquad$ $g(x) = x^m$ $\qquad$ $A(x) = f(x)g(x) = x^{n+m}$

$f'(x) = nx^{n-1}$ $\qquad$ $g'(x) = mx^{m-1}$ $\qquad$ $A'(x) = (n+m)x^{n+m-1}$

$A'(x)$ should be a fn of f, f', g, g'

↳ know if $f(x)$ or $g(x) = 1$

↳ symmetric in f's and g's

$$nx^{n-1}x^m + mx^n x^{m-1} = (n+m)x^{n+m-1}$$

Ex: $f(x) = \sin x$ $\qquad$ $g(x) = \cos x$ $\qquad$ $A(x) = \sin x \cos x = \frac{1}{2}\sin(2x)$

Claim: $(x^n)' = nx^{n-1}$

Assume we know the product rule.

Base Case: $n=1$ straightforward

Inductive Step: Assume $(x^n)' = nx^{n-1}$

Consider $(x^{n+1})' = (x \cdot x^n)'$

$= (x)' x^n + x (x^n)'$ by the product rule

$= 1 \cdot x^n + x \cdot nx^{n-1}$ by induction

$= (n+1) x^n$

Bypassed the Binomial Theorem!

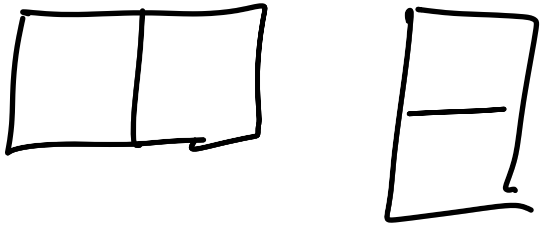
Fibonaccis from Tiling

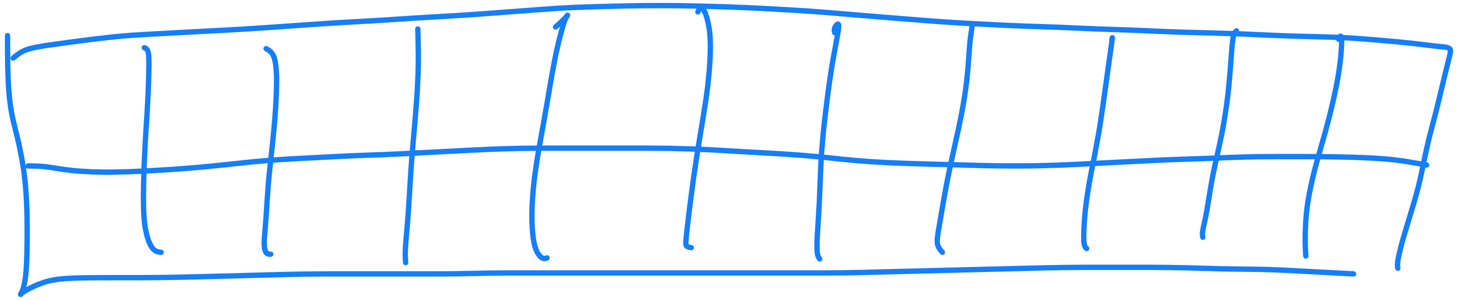
$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} : 0, 1, 1, 2, 3, 5, 8, \ldots$

Golden Mean: $\varphi = \frac{1+\sqrt{5}}{2} \qquad (1/\varphi)^{-1} = \varphi$

Binet's Formula

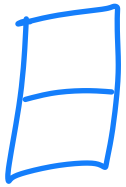
$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$





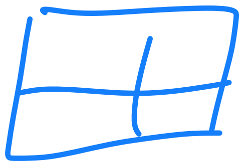
$2 \times n$ with 1×2 and 2×1 tiles?

$n = 1$



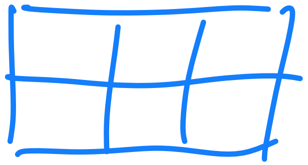
1

$n = 2$

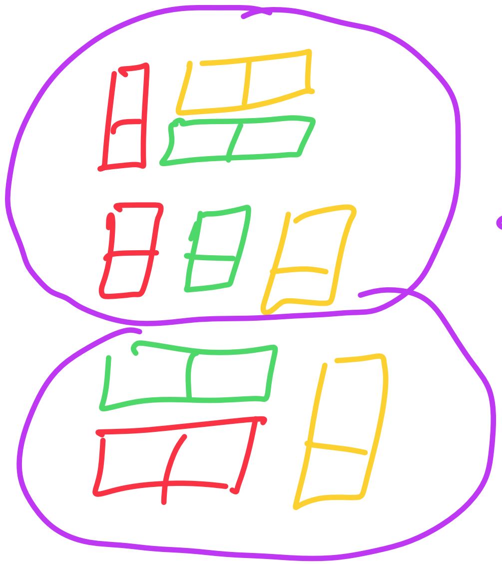


2

$n = 3$



3



1st vertical, reduces to $2 \times (3-1)$

1st horizontal, 2nd horizontal, reduces to $2 \times (3-2)$

Tile $2 \times n$ with 1×2 and 2×1 tiles

$a_n = \#$ ways

$a_{n+1} = a_n + a_{n-1}$ with $a_0 = 0$, $a_1 = 1$

Fibonacci Numbers!

Question: $2 \times 2 \times n$ and "$1 \times 1 \times 2$" tiles

Challenge

Math 331: The little Questions (Fall 2024)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 03: Recurrences, Inequalities: September 11, 2024:
https://youtu.be/fDaNheZa33k

Binet's Formula, Growth Rates and Tilings

$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}$

Know $F_{n+1} \le 2F_n$ as $F_{n-1} \le F_n$

$F_{n+1} \le 2 \cdot 2F_{n-1} \le 2 \cdot 2 \cdot 2 \cdot F_{n-2} \cdots$

$F_{n+1} \le 2^{n+1}$

Also $F_{n+1} \ge 2F_{n-1}$

↳ algebra: "basically" get $F_n \ge (\sqrt{2})^n$

$(\sqrt{2})^n$ "$\le$" $F_n \le 2^n$

Maybe $F_n \sim r^n$ for some r

Guess Try $F_n = r^n$

Get $F_{n+1} = F_n + F_{n-1}$ becomes

$r^{n+1} = r^n + r^{n-1}$ divide by r^{n-1} since $r \neq 0$

$\Rightarrow r^2 = r + 1$ or $r^2 - r - 1 = 0$

Characteristic polynomial for the recurrence

roots are $r = \frac{1 \pm \sqrt{5}}{2}$ call r_1, r_2

So r_1^n and r_2^n solve the recurrence

So too does $a_1 r_1^n + a_2 r_2^n$ for any a_1, a_2.

Our conditions $F_0 = 0, F_1 = 1 \Rightarrow F_n = \frac{1}{\sqrt{5}} \underset{\text{"}r_+\text{"}}{r_1^n} - \frac{1}{\sqrt{5}} \underset{\text{"}r_-\text{"}}{r_2^n}$

New Problems

Consider $a_{n+1} = \alpha a_n + \beta a_{n-1}$

with $0 \le \alpha, \beta$ and

(1) $\alpha + \beta = 1$ (or 2) (or c)

(2) $\alpha^2 + \beta^2 = 1$ (or 2) (or c)

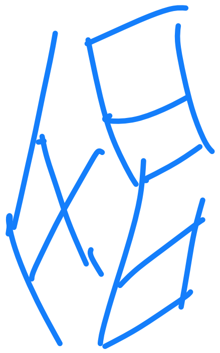
(3) $\alpha^{1/2} + \beta^{1/2} = 1$ (or 2) (or c)

For each, find α, β leading to fastest growth rate of the $\{a_n\}$

Tiling

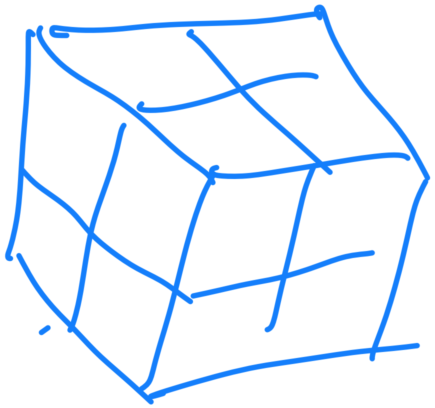
$2 \times n$ tiled by 1×2 is Fibonacci

Question: $2 \times 2 \times n$ by $1 \times 1 \times 2$: What is that?



$2 \times 2 \times 1$: # cases is 2 :

$$C_1 = 2$$



$C_2 = 9$

C_2 is

$C_1 + C_1$

$+ 1 + 4\cancel{C_1}$

$4(C_1 - 1)$

$= 9$

all 4

1

or just 2

4 ways

say

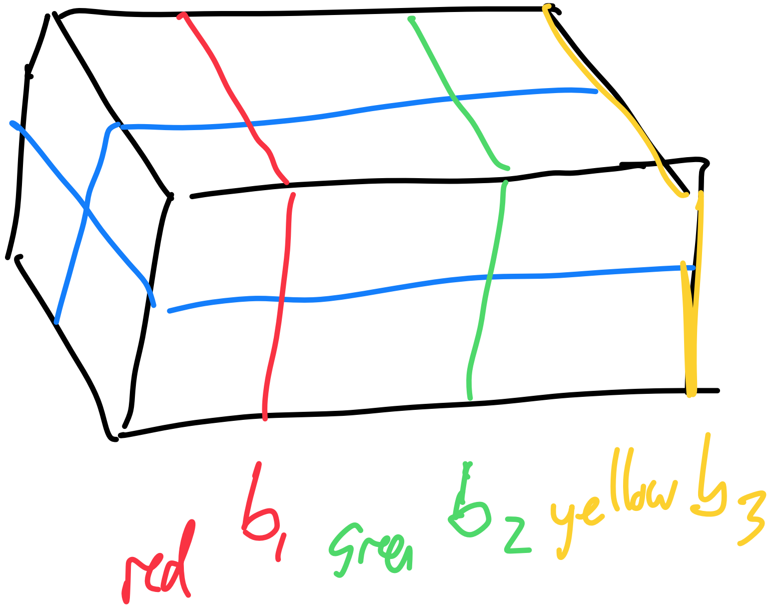
$\cancel{C_1}$

$(C_1 - 1)$

$\updownarrow \ \updownarrow \ * \ C_1$

$\nwarrow\!\!\!\searrow \ \nwarrow\!\!\!\searrow \ * \ C_1$

C_3 ?



$$C_3 = 2C_2 + \cdots$$

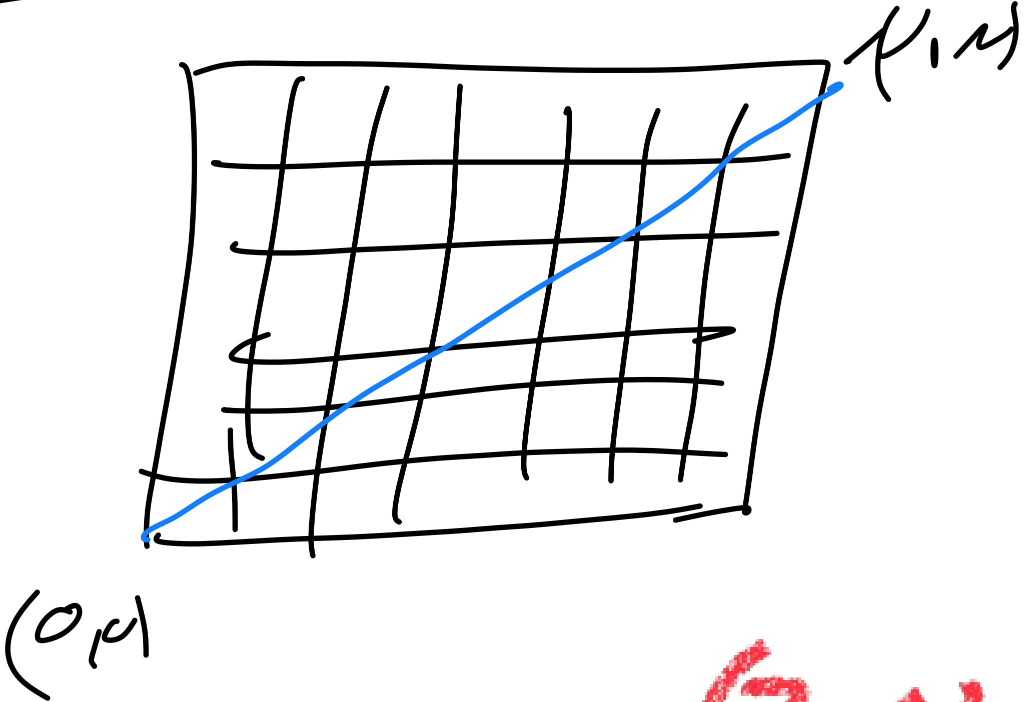
one option is completely fill first two thirds and then move on, that is $C_2 * C_1$

but double counts

$$(C_2 - C_1 * C_1) * C_1$$

remove double counting

Catalan #s:



only move E and N

need n of each

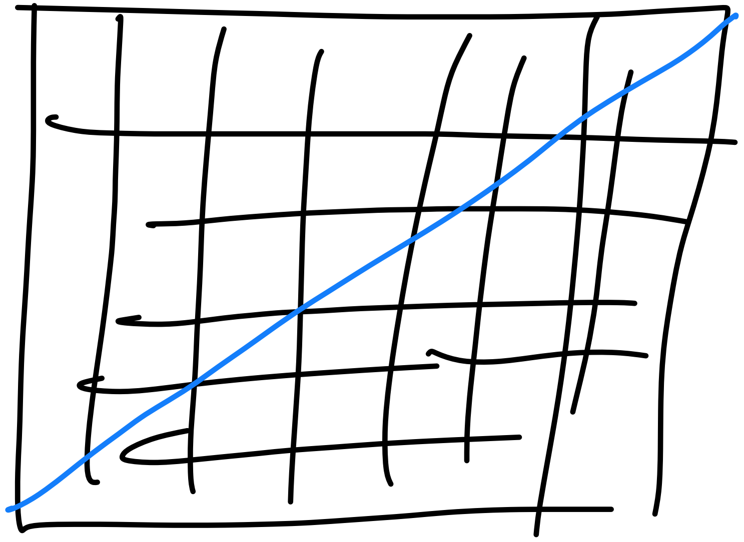
Stay on or below the main diagonal.

paths is $\binom{2n}{n}$

how many valid paths staying below the diagonal?

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Auxiliary Quantities



C_N = # paths from $(0,0)$ to (N,N) always on or below the main diagonal

b_N = # paths from $(0,0)$ to (N,N) such that first hit the main diagonal at (N,N) and below rest of the time

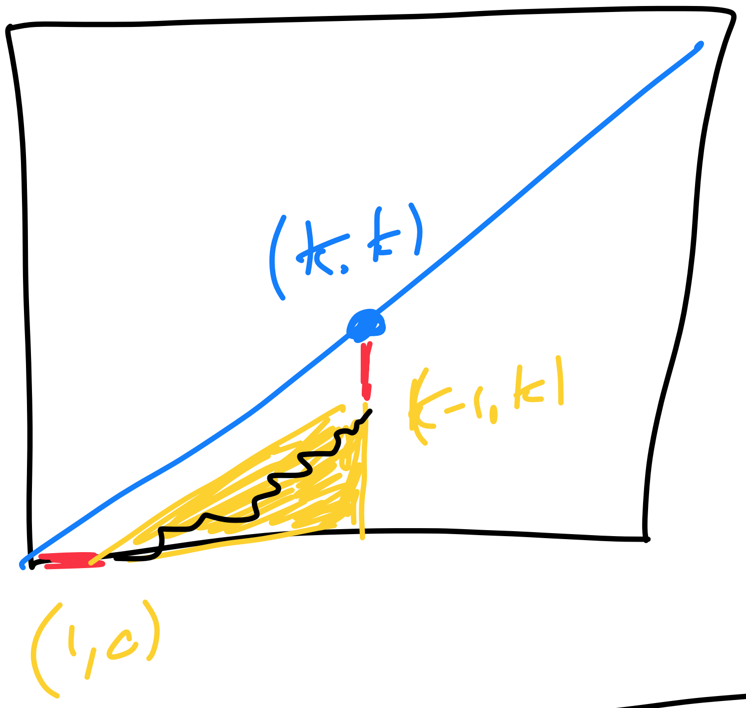
↳ first hit main diagonal at (N,N)

$$C_N = b_1 C_{N-1} + b_2 C_{N-2} + \dots + b_N C_0$$

$b_1 C_{N-1}$: 1st hit at $(1,1)$

$b_2 C_{N-2}$: 1st hit at $(2,2)$

$C_0 = 1$



$$b_k = C_{k-1}$$

$$\boxed{C_N = C_0 C_{N-1} + C_1 C_{N-2} + \cdots + C_{N-1} C_0}$$

Since 1st step east, last north: have $N-1$ east steps, $N-1$ north steps original Catalan Problem, stay on or below, but one fewer East and one fewer north steps.

Math 331: The little Questions (Fall 2024)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 04: Coding Efficiency, Recurrences: September 13, 2024:
https://youtu.be/F6ZQFjIzfEQ

Triangle Numbers

```
T[n_] := n (n + 1) / 2

naive[max_] :=
 For[n = 1, n ≤ max, n++,
  If[IntegerQ[Sqrt[T[n]]] == True,
   Print["n = ", n, " leads to a square: ", T[n]]]
 ]

Timing[naive[10 000 000]]

n = 1 leads to a square: 1
n = 8 leads to a square: 36
n = 49 leads to a square: 1225
n = 288 leads to a square: 41 616
n = 1681 leads to a square: 1 413 721
n = 9800 leads to a square: 48 024 900
n = 57 121 leads to a square: 1 631 432 881
n = 332 928 leads to a square: 55 420 693 056
n = 1 940 449 leads to a square: 1 882 672 131 025
{130.875, Null}
```

$$T_n = 1 + 2 + \cdots + n$$
$$= \frac{n(n+1)}{2}$$

Proof:

$$T_n = 1 + 2 + \cdots + n$$
$$\underline{T_n = n + (n-1) + \cdots + 1}$$
$$2T_n = (n+1) + \cdots + (n+1)$$
$$T_n = n(n+1)/2$$

(Gauss)

```
fast[max_] :=
 For[n = 1, n ≤ Sqrt[max], n = n + 1,
  {
   m = (2 n - 1)^2;
   If[IntegerQ[Sqrt[(m + 1) / 2]] == True, Print["n = ", m, " leads to a square: ", T[m]]];
   If[IntegerQ[Sqrt[(m - 1) / 2]] == True, Print["n = ", m - 1, " leads to a square: ", T[m - 1]]];
  }]
```

```
Timing[fast[10000000]]
n = 1 leads to a square: 1
n = 0 leads to a square: 0
n = 8 leads to a square: 36
n = 49 leads to a square: 1225
n = 288 leads to a square: 41616
n = 1681 leads to a square: 1413721
n = 9800 leads to a square: 48024900
n = 57121 leads to a square: 1631432881
n = 332928 leads to a square: 55420693056
n = 1940449 leads to a square: 1882672131025
n = 11309768 leads to a square: 63955431761796
{0.046875, Null}
```

OEIS

8, 49, 288, 1681, . . .

0 1 3 6 2 7 13 20 12 21 11 22 10 23 OEIS

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A001108 a(n)-th triangular number is a square: a(n+1) = 6*a(n) - a(n-1) + 2, with a(0) = 0, a(1) = 1. 64
(Formerly M4536 N1924)

0, 1, 8, 49, 288, 1681, 9800, 57121, 332928, 1940449, 11309768, 65918161, 384199200, 2239277041, 13051463048, 76069501249, 443365544448, 2584123765441, 15061377048200, 87784138523761, 511643454094368, 2982076586042449, 17380816062160328, 101302819786919521 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS b(0)=0, c(0)=1, b(i+1)=b(i)+c(i), c(i+1)=b(i+1)+b(i); then a(i) (the number in the sequence) is 2b(i)^2 if i is even, c(i)^2 if i is odd and b(n)=A000129(n) and c(n)=A001333(n). - Darin Stephenson (stephenson(AT)cs.hope.edu) and Alan Koch

For n > 1 gives solutions to A007913(2x) = A007913(x+1). - Benoit Cloitre, Apr 07 2002

If (X,X+1,Z) is a Pythagorean triple, then Z-X-1 and Z+X are in the sequence.

For n >= 2, a(n) gives exactly the positive integers m such that 1,2,...,m has a perfect median. The sequence of associated perfect medians is A001109. Let a_1,...,a_m be an (ordered) sequence of real numbers, then a term a_k is a perfect median if Sum_{j=1..k-1} a_j = Sum_{j=k+1..m} a_j. See Puzzle 1 in MSRI Emissary, Fall 2005. - Asher Auel, Jan 12 2006

This is the r=8 member of the r-family of sequences S_r(n) defined in A092184 where more information can be found.

Also, 1^3 + 2^3 + 3^3 + ... + a(n)^3 = k(n)^4 where k(n) is A001109. - Anton Vrba (antonvrba(AT)yahoo.com), Nov 18 2006

Project Euler .net https://projecteuler.net/problem=1

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Multiples of 3 or 5

Problem 1

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get $3, 5, 6$ and 9. The sum of these multiples is 23.

Find the sum of all the multiples of 3 or 5 below 1000.

1) march from 1 to 1000, adding number if multiple of 3 or 5.

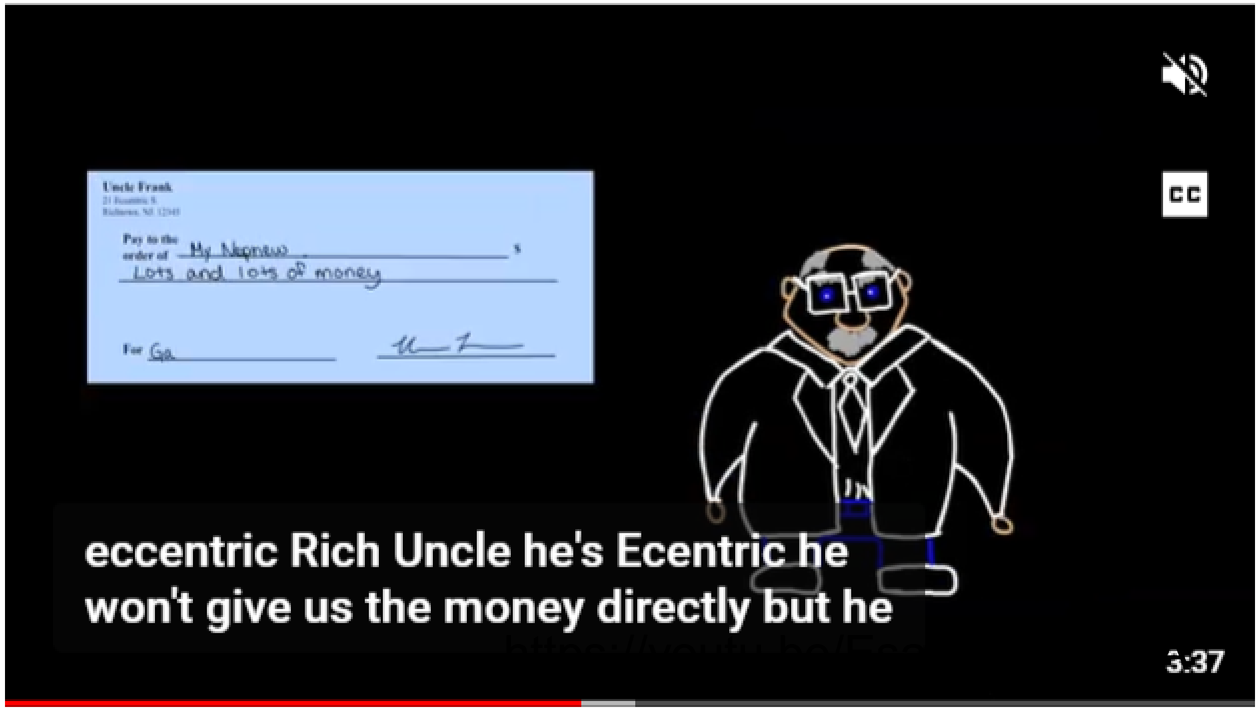
2) add multiples of 3 and 5 then subtract doubly counted multiples of 15

3) note $3+6+9+\cdots+3m = 3(1+2+\cdots+m) = 3\frac{m(m+1)}{2}$

4) blocks of 30

$(0+3+5+6+9+10+12+15+18+20+21+24+25+27)$

next block = previous sum + 30 * (# elements)

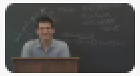


Double Plus Ungood WEB

1K views • 11 years ago

Steven Miller

Double Plus Ungood: From the Fibonacci numbers to roulette.



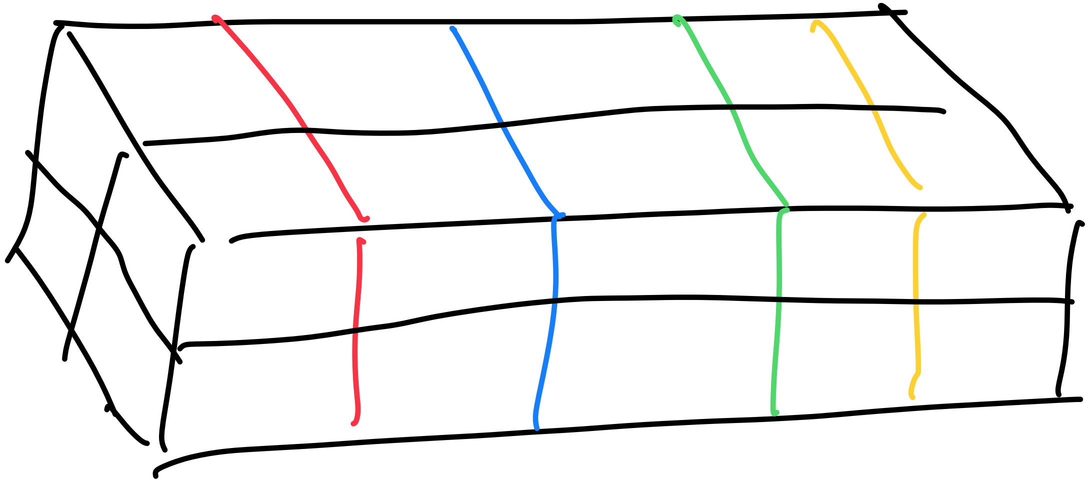
3 chapters Introduction | Double Plus Strategy | Outro

https://youtu.be/Esa2TYwDmwA

Recurrences to Inequalities

Tiles: $1 \times 1 \times 2$

C_n = # ways to tile the $2 \times 2 \times n$

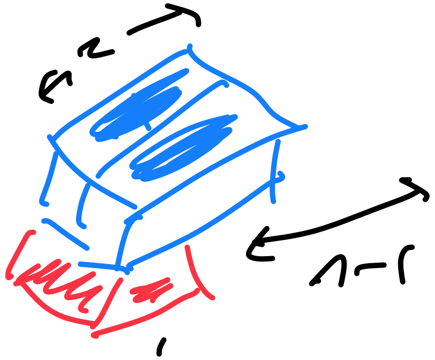


$$C_n = 2 * C_{n-1} + 1 * C_{n-2} + 4 * d_{n-1}$$

tile "red" horiz or vertically

tile "red" and blue by all 4 goin "inward"

ways just 2 going "in" and one on bottom



$d_{n-1} = C_{n-2} + d_{n-2}$

red block under

two blocks on bottom sticking out

$$c_n = 2c_{n-1} + c_{n-2} + 4d_{n-1}$$

$$d_{n-1} = c_{n-2} + d_{n-2}$$

$c_0 = 0$

$c_1 = 2$

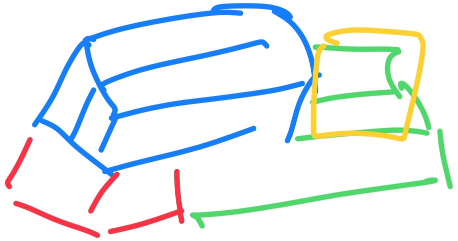
$c_2 = 9$

$d_1 = 1$

$d_2 = 3$

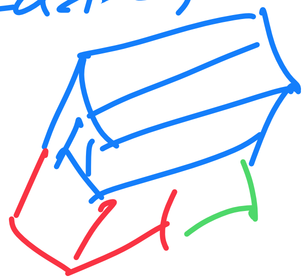
$n=3$: $d_2 = c_1 + d_1$ YES!

$n=2$: $d_1 = c_0 + d_0 \Rightarrow 1 = 0 + d_0$



$$c_{n-2} = d_{n-1} - d_{n-2}$$

$$d_{n+1} - d_n = 2(d_n - d_{n-1}) + (d_{n-1} - d_{n-2}) + 4d_{n-1}$$



c_1

$m = n-2$ or $n = m+2$

$$\boxed{c_m = d_{m+1} - d_m}$$

$$d_3 = c_2 + d_2 = 9 + 3 = 12$$

Math 331: The little Questions (Fall 2024)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 05: Recurrences, Inequalities: September 16, 2024:
https://youtu.be/3LzFI3cfJYI
(video didn’t record, only audio – need to advance slides with audio playing)

Read Appendix A: The Average Gap Distribution for Generalized Zeckendorf Decompositions (with Olivia Beckwith, Amanda Bower, Louis Gaudet, Rachel Insoft, Shiyu Li and Philip Tosteson), the Fibonacci Quarterly (**51** (2013), 13--27**).** pdf

Theorem A.1 (Generalized Binet's Formula). *Consider the linear recurrence*

$$G_{n+1} \;=\; c_1 G_n + c_2 G_{n-1} + \cdots + c_L G_{n+1-L} \tag{A.1}$$

with the c_i's non-negative integers and $c_1, c_L > 0$. Let $\lambda_1, \ldots, \lambda_L$ be the roots of the characteristic polynomial

$$f(x) \;:=\; x^L - \left(c_1 x^{L-1} + c_2 x^{L-2} + \cdots + c_{L-1}x + c_L\right) \;=\; 0, \tag{A.2}$$

ordered so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_L|$. Then $\lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_L|$, $\lambda_1 > 1$ is the unique positive root, and there exist constants such that

$$G_n \;=\; a_1 \lambda_1^n + O\left(n^{L-2} \lambda_2^n\right). \tag{A.3}$$

More precisely, if $\lambda_1, \omega_2, \ldots, \omega_r$ denote the distinct roots of the characteristic polynomial with multiplicities 1, $m_2, \ldots, m_r$, then there are constants $a_1 > 0, a_{i,j}$ such that

$$G_n \;=\; a_1 \lambda_1^n + \sum_{i=2}^{r} \sum_{j=1}^{m_r} a_{i,j} n^{j-1} \omega_i^n. \tag{A.4}$$

Lecture given to young scholars:

- What do you MEAN? https://youtu.be/jBKZaCxpgSE (word file here, pdf here) (3/19/2020): Comfort with Algebra sufficient: 40 minutes

Lectures from Math 349 (Operations of Order):

- Lecture 34: 11/29/23: What do you mean? https://youtu.be/6fal8vRN-Ew
- Lecture 36: 12/04/23: What do you mean, II? https://youtu.be/azABrUnQklg

Handouts:

- Notes from Miller: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/handouts/ArithMeanGeoMean.pdf
- Paper (Ben-Ari and Conrad): Maclaurin's Inequality and a Generalized Bernoulli Inequality
- Video from 2018 iteration of our class: AM-GM inequality, Triangle Game: https://youtu.be/k1XF420-8QY

Important Competition Inequalities:

- https://artofproblemsolving.com/articles/files/MildorfInequalities.pdf
- https://web.williams.edu/Mathematics/sjmiller/public_html/161/articles/Riasat_BasicsOlympiadInequalities.pdf
- https://esp.mit.edu/download/8a5f8efe-59f5-407d-9252-607ace7aa190/M11250_Intro%20to%20ol%20ineq%20hssp.pdf
- https://artofmaths.wordpress.com/wp-content/uploads/2014/06/inequalities-a-mathematical-olympiad-approach.pdf
- https://artofproblemsolving.com/wiki/index.php/Inequality

Most Important Inequality

If $x \in \mathbb{R}$ then $x^2 \geq 0$

What properties should $\text{Mean}(x,y)$ satisfy? $0 < x \le y$

- $x \le \text{Mean}(x,y) \le y$
  - $\hookrightarrow$ if $y = x \implies \text{Mean}(x,x) = x$

If $\text{Mean}(x,y)$ is halfway b/w x and y then

$\text{Mean}(x,y)$ must be $\frac{1}{2}x + \frac{1}{2}y = \frac{x+y}{2}$

$\hookrightarrow$ Arithmetic $\text{Mean}(x,y)$

Define $\text{Mean}_p(x,y) = px + (1-p)y$

for $0 \le p \le 1$

What about multiplying?

$$\sqrt{xy} = x^{1/2} y^{1/2} = \text{Geom Mean}(x,y)$$

More Generally: $\text{Geom Mean}_p(x,y) = x^p y^{1-p}, \ 0 \le p \le 1$

BIG Question:

$$\text{Arith Mean}(x,y) \quad \text{vs} \quad \text{Geom Mean}(x,y)$$

$$\mathrm{AMean}(x,y) = \frac{x+y}{2} \qquad \mathrm{GMean}(x,y) = \sqrt{xy}$$

x	y	AM	GM
4	9	6.5	6
1	1	1	1
1	N^2	$\approx \frac{1}{2}N^2$	N
	N big		

Wlog, can I adjust so that $x=1$?

$$\frac{x+y}{2} \quad \text{vs} \quad \sqrt{xy}$$

$$(x,y) \longrightarrow (a,b) = \left(x/x,\ y/x\right) = (1,t) \qquad t = y/x$$

$$AM(1,t) = \frac{1+t}{2} \qquad AM(x,y) = \frac{x+y}{2} = x\left(\frac{1+y/x}{2}\right) =$$

$$= x\ AM(1,t)$$

$$\boxed{\begin{array}{l}\text{Another property:}\\ \quad \mathrm{Mean}(rx, ry) = r\,\mathrm{Mean}(x,y) \text{ if } r>0\end{array}}$$

$(x,y) \longrightarrow (a,b) = (1,t)$ with $t = y/x$

$AM(1,t) = \frac{1+t}{2}$ $\quad AM(x,y) = x\, AM(1,t)$

$GM(1,t) = \sqrt{t}$ $\quad GM(x,y) = \sqrt{xy} = \sqrt{x \cdot x \cdot \frac{y}{x}} = x\sqrt{\frac{y}{x}}$

$= x\, GM(1,t)$

$AM(x,y)$ vs $GM(x,y)$ same as $AM(1,t)$ vs $GM(1,t)$

WLOG, $x = 1$ and $y = t \geq 1$

$\frac{1+t}{2}$ (AM) vs $\sqrt{t}$ (GM) $t \geq 1$

$$f(t) = \frac{1+t}{2} - \sqrt{t} \quad \text{for } 1 \leq t < \infty$$

critical points: $f'(t) = 0$

endpoints: $t = 1$ at $t = 1$, $f(1) = 0$

$$f'(t) = \frac{1}{2} - \frac{1}{2} t^{-1/2}$$

↳ only critical point is at $t = 1$

↳ note $f'(t) > 0$ for $t > 1$ so fn is increasing

Implies $\frac{1+t}{2} - \sqrt{t} \geq 0$ and is $= 0$ only at $t = 1$

AM $\geq$ GM without Calculus

Compare $\frac{1+t}{2}$ vs $\sqrt{t}$ let $u = \sqrt{t} \geq 1$

Compare $\frac{1+u^2}{2}$ vs u

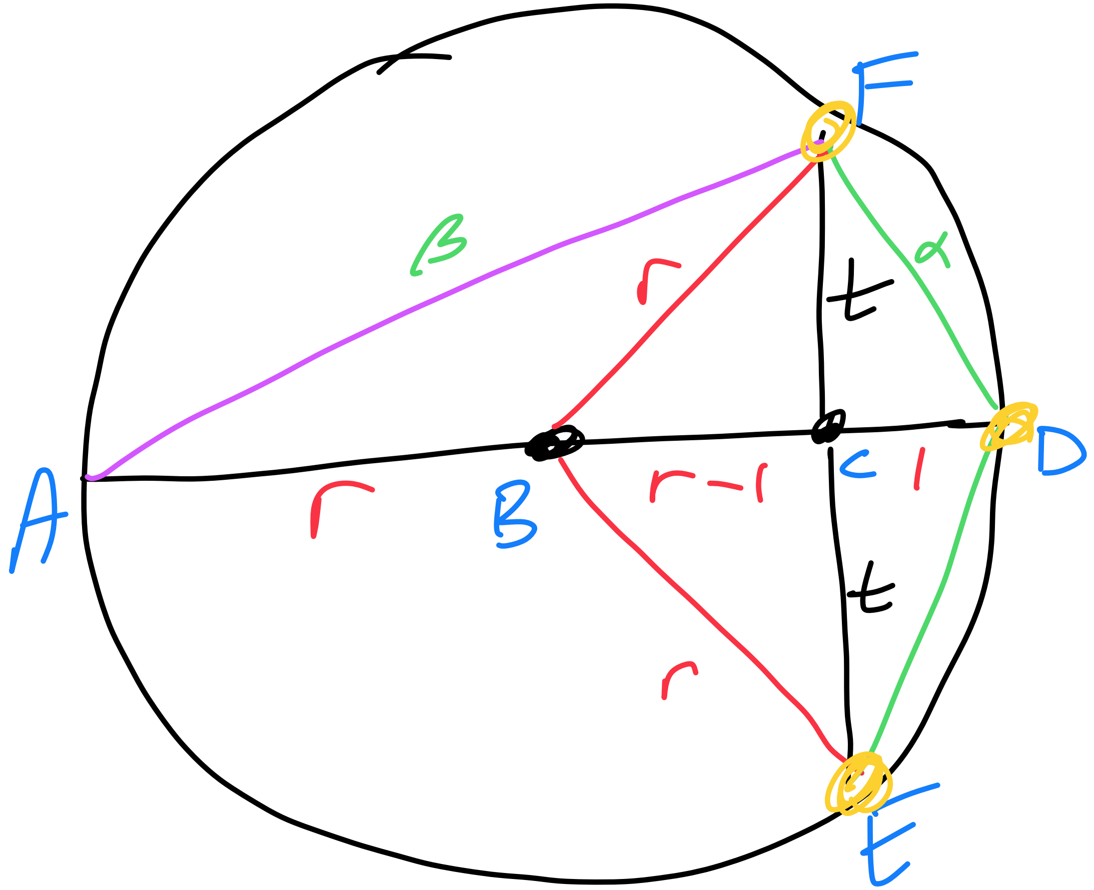
or $u^2 - 2u + 1$ vs 0

$(u-1)^2$ vs 0

as $u \geq 1$, LHS is non-neg and is pos if $u > 1$

AM vs GM

right triangle: $a^2+b^2=c^2$



diameter =

$\alpha^2 = t^2+1$

$r^2 = t^2+(r-1)^2$

$\beta^2 = t^2+(2r-1)^2$

$\alpha^2+\beta^2 = 4r^2 = (2r)^2$

Note: should do $\sqrt{t}$ not t

Yields (after doing that) $r = \frac{1+t}{2}$

clearly $r \geq \sqrt{t}$

$|FC|\,|CE| = |DC|\,|CA|$

(hint)

$0 < x \le y \le z$ $\qquad x \le \mathrm{Mean}(x,y,z) \le z$

Standard: AM: $\frac{1}{3}(x+y+z)$

SYMMETRIC

$$\mathrm{Mean}_{a,b}(x,y,z) = \left(\frac{xy+xz+yz}{b}\right)^a$$

GM: $(x\,y\,z)^{1/3}$

$$\text{Mean}_{a,b}(x,y,z) = \left(\frac{xy+xz+yz}{b}\right)^a$$

If $x=y=z$ get $x \Rightarrow \left(\frac{3x^2}{b}\right)^a = x$

Thus $b=3$ $a=1/2$

Suggests $$\left(\frac{xy+xz+yz}{3}\right)^{2}$$

is a mean, and is like AM and GM

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Lecture 06: Inequalities II: September 18, 2024: https://youtu.be/0fTiXEQua7Q

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- https://artofmaths.wordpress.com/wp-content/uploads/2014/06/inequalities-a-mathematical-olympiad-approach.pdf
- https://artofproblemsolving.com/wiki/index.php/Inequality

Geometric Proof of AM-GM

$AM(x,y) = \frac{x+y}{2}$ $\quad GM(x,y) = \sqrt{xy}$ $\quad 0 < x \le y$

wlog $x = 1$, $t = y/x$

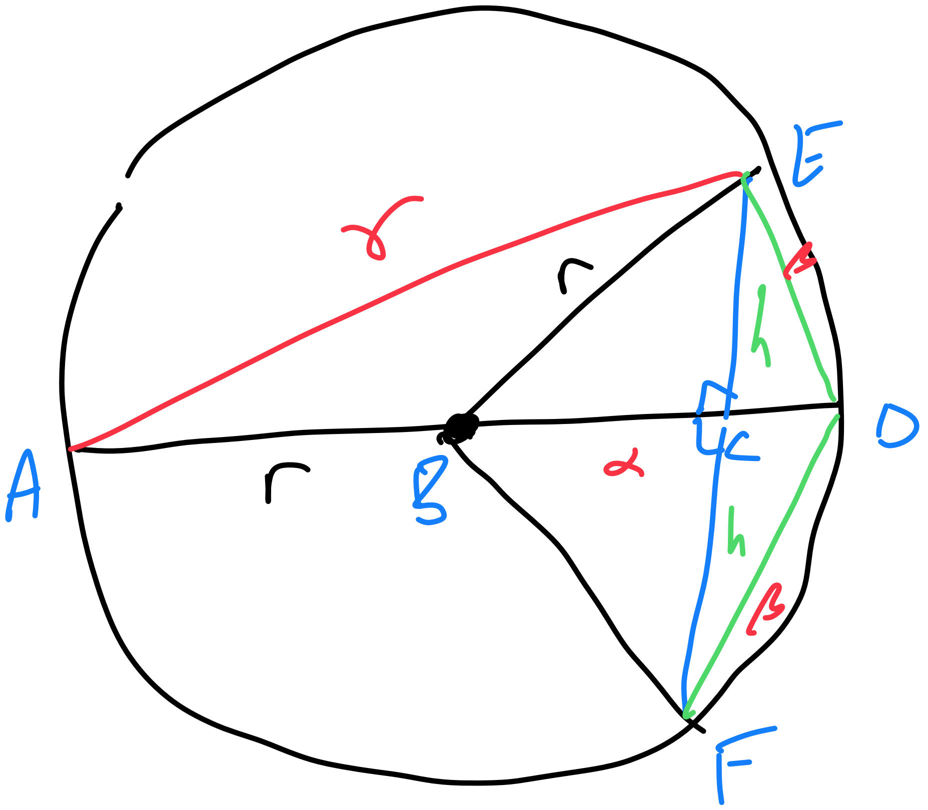
$AM(rx, ry) = r\,AM(x,y)$

$GM(rx, ry) = r\,GM(x,y)$

NORMALIZATION

(1) $x = 1$ (2) $AM = 1$ (3) $GM = 1$

Circle of radius $r = \frac{x+y}{2}$ (could do $\frac{1+t}{2}$)



Geometry: $h \cdot h = |AC| \cdot |CD|$

$h = |CE|$

$|CD| = r - \alpha$

$|BC| = \alpha$

$$h^2 = r^2 - \alpha^2$$

$$h^2 = \gamma^2 - (r+\alpha)^2$$

$$h^2 = \beta^2 - (r-\alpha)^2$$

$$4r^2 = \beta^2 + \gamma^2$$

$$\Longrightarrow (r+\alpha)(r-\alpha) = h^2$$

Note r is the AM

If h is the geom mean then done as $r \geq h$

Know: $\dfrac{x_1+x_2}{2} \geq \sqrt{x_1 x_2}$ $\quad 0 < x_1, x_2$

Cont 1, 2, 4, 8, 16, ...

$$\frac{x_1+x_2+x_3+x_4}{4} = \frac{\overbrace{\left(\frac{x_1+x_2}{2}\right)}^{u_1} + \overbrace{\left(\frac{x_3+x_4}{2}\right)}^{u_2}}{2}$$

want to compare with $GM(x_1, x_2, x_3, x_4)$ which is $(x_1 x_2 x_3 x_4)^{1/4}$

$$\geq GM(u_1, u_2) = \sqrt{u_1 u_2}$$

$$= \sqrt{\frac{x_1+x_2}{2} \cdot \frac{x_3+x_4}{2}}$$

$$\geq \sqrt{\sqrt{x_1 x_2}\sqrt{x_3 x_4}} \quad \text{by AM-GM Ineq}$$

$$= (x_1 x_2 x_3 x_4)^{1/4}$$

$\blacksquare$

$$\frac{x_1 + \cdots + x_4 + x_5 + \cdots + x_8}{8} = \frac{\left(\frac{x_1 + \cdots + x_4}{4}\right) + \left(\frac{x_5 + \cdots + x_8}{4}\right)}{2}$$

$$\geq \sqrt{\frac{x_1 + \cdots + x_4}{4} \cdot \frac{x_5 + \cdots + x_8}{4}}$$ by AM-GM with $n=2$

$$\geq \sqrt{(x_1 \cdots x_4)^{\frac{1}{4}} (x_5 \cdots x_8)^{1/4}}$$ by AM-GM with $n=4$

$$= (x_1 \cdots x_8)^{1/8}$$ ∎

INDUCT: Get true for $n = 1, 2, 4, 8, 16 \ldots$

Imagine have $x_1, x_2, \dots, x_n$ and $2^{k-1} < n < 2^k$

$$\frac{x_1 + x_2 + x_3 + \dots + x_n + x_{n+1} + x_{n+2} + \dots + x_{2^k}}{2^k}$$

$$\geq \left(x_1 x_2 \cdots x_n \cdot x_{n+1} x_{n+2} \cdots x_{2^k}\right)^{1/2^k}$$

Try: $x_{n+1} = \dots = x_{2^k}$ and all are

(1) 0 or 1 $\leftarrow$ bad idea

(2) $(x_1 \cdots x_n)^{\delta}$ $\quad$ δ is free

(3) $\beta(x_1 + \dots + x_n)$ $\quad$ β is free

$$\frac{x_1+\cdots+x_n+x_{n+1}+\cdots+x_{2^k}}{2^k}$$

$x_{n+1}=\cdots=x_{2^k}=(x_1+\cdots+x_n)\beta$

$S':=x_1+\cdots+x_n$

Typo! fixed in next lecture

$$\frac{S'+(2^k-n)\beta S'}{2^k}=\frac{(2^k-n)\beta}{2^k}S'$$

take $\beta=\frac{2^k}{2^k-n}\frac{1}{n}$

and get $AM(x_1,\dots,x_n)$

Want $\frac{S'}{n}=\frac{x_1+\cdots+x_n}{n}$

Start with $\frac{x_1+\cdots+x_n+x_{n+1}+\cdots+x_{2^k}}{2^k}$

If $\beta=\frac{2^k}{2^k-n}\frac{1}{n}$ Then $AM(x_1,\dots,x_n,x_{n+1},\dots,x_{2^k})=$

$AM(x_1,\dots,x_n)$

Know $AM(x_1, \dots, x_n, x_{n+1}, \dots, x_{2^k}) \geq GM(x_1, \dots, x_n, x_{n+1}, \dots x_{2^k})$

for or β / $\beta^{1/n}$ as we know for powers of 2

$$AM(x_1, \dots, x_n) \geq (x_1 x_2 \cdots x_n x_{n+1} \cdots x_{2^k})^{1/2^k}$$

$$x_{n+1} = \cdots = x_{2^k} = \beta(x_1 + \cdots + x_n) = \beta \cdot AM(x_1, \dots, x_n) \cdot n$$

$$AM(x_1, \dots, x_n)^{2^k} \geq \Big(\underbrace{x_1 \cdots x_n}_{\text{looks like a geom mean}} \ (\underbrace{\beta \cdot n}_{\text{typo fixed next lecture}} \cdot AM(x_1, \dots, x_n))^{2^k - n} \Big)$$

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Lecture 07: Inequalities III: September 20, 2024: https://youtu.be/p_yx7p8UOoc

$$\frac{x_1+\dots+x_n+x_{n+1}+\dots+x_{2^k}}{2^k}$$

$x_{n+1}=\dots=x_{2^k}=(x_1+\dots+x_n)\beta$

$S:=x_1+\dots+x_n$

$$\frac{S+(2^k-1)\beta S}{2^k}=\frac{(2^k-1)\beta+1}{2^k}S$$

take $\beta=\frac{2^k}{2^k-1}\frac{1}{n}$

and get $AM(x_1,\dots,x_n)$

$(2^k-1)\beta+1=\frac{2^k}{n}$

$(2^k-1)\beta=\frac{2^k-1}{n}$

$\beta=1/n$

Want $\frac{S}{n}=\frac{x_1+\dots+x_n}{n}$

Start with $\frac{x_1+\dots+x_n+x_{n+1}+\dots+x_{2^k}}{2^k}$

If $\beta=\cancel{\frac{2^k}{2^k-1}}\frac{1}{n}$ Then $AM(x_1,\dots,x_n,x_{n+1},\dots,x_{2^k})=$

$\beta=1/n$

$AM(x_1,\dots,x_n)$

Given $x_1, \dots, x_n$

Free to choose $x_{n+1}, \dots, x_{2^k}$

Can use AM-GM ineq for 2^k inputs

$$\hookrightarrow \underbrace{AM(x_1, \dots, x_n, x_{n+1}, \dots, x_{2^k})}_{\text{involves dividing by } 2^k} \geq \underbrace{GM(x_1, \dots, x_n, x_{n+1}, \dots, x_{2^k})}_{\text{involves power } 1/2^k}$$

Know $AM(x_1,\dots,x_n,x_{n+1},\dots,x_{2^k}) \geq GM(x_1,\dots,x_n,x_{n+1},\dots,x_{2^k})$

for arb β

$\beta^{-1/n}$ as we know for powers of 2

$$AM(x_1,\dots,x_n) \geq \left(x_1 x_2 \cdots x_n \, x_{n+1} \cdots x_{2^k}\right)^{1/2^k}$$

$$x_{n+1} = \cdots = x_{2^k} = \beta(x_1+\cdots+x_n) = \beta \cdot AM(x_1,\dots,x_n)\cdot n$$

$$AM(x_1,\dots,x_n)^{2^k} \geq \Big(\underbrace{x_1 \cdots x_n}_{\text{looks like a geom mean}} \big(\underbrace{\beta \cdot n}_{\text{equals } 1} \cdot AM(x_1,\dots,x_n)\big)^{2^k-n}\Big)$$

$$AM(x_1,\dots,x_n)^n \geq x_1 \cdots x_n \iff AM(x_1,\dots,x_n) \geq GM(x_1,\dots,x_n)$$

$A(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n}$ $\quad G(x_1, \ldots, x_n) = (x_1 \cdots x_n)^{1/n}$

wlog assume $0 \leq x_i$

rescale inputs we rescale output:

$A(r\vec{x}) = r A(\vec{x}) \quad G(r\vec{x}) = r G(\vec{x})$

wlog either $A(\vec{x})$ or $G(\vec{x})$ is 1 (otherwise AM-GM ineq is trivial as GM is zero)

Calc III: Constrained Optimization:

$$\left.\begin{array}{l} \text{Constraint } g(\vec{x}) = C \\ \nabla f = \lambda \nabla g \end{array}\right\}$$

$n+1$ eqs

$n+1$ unknowns: $x_1, \ldots, x_n, \lambda$

$$\underbrace{x_1 + x_2} + \cdots + x_n = S$$

2-dim case: $x + y = S$

Farmer Brown Problem

Maximize xy

$f(x) = x(S - x)$

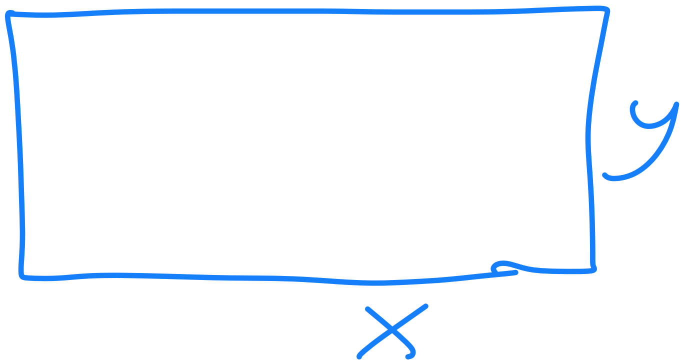
Thoreau it: Simplify, Simplify

$f(x) = xS - x^2$

$f'(x) = S - 2x \Rightarrow$ C.P. is $x = S/2$

Boundary points are $x = 0$ and S

$f(0) = 0$, $f(S) = 0$, $x = S/2 = y$ is Max

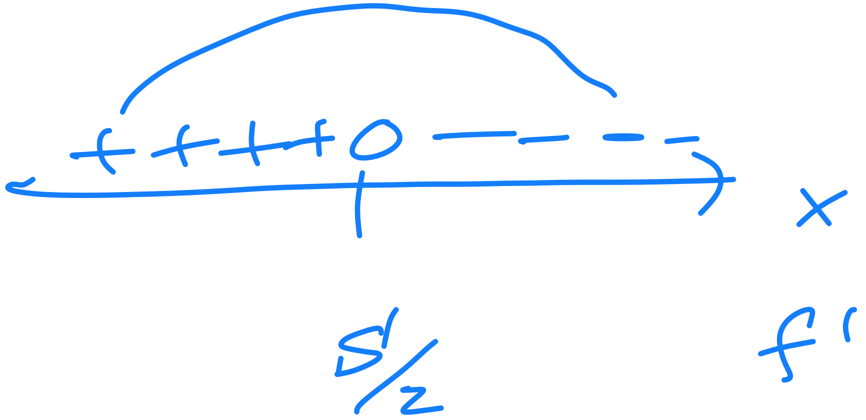


Max area given semi-perimeter

$f(x) = x(S - x)$

$f'(x) = S - 2x$

$f''(x) = -2$ max



$x + y = S$ Then max area is $x = y$ area is $x^2 = \frac{S^2}{4}$

$\frac{x+y}{2} = S$ or $\left(\frac{x+y}{2}\right)^2 = \frac{S^2}{4} \geq xy$

or $AM(x, y) \geq GM(x, y)$

Consider $x_1 + x_2 + x_3 + \cdots + x_n = S$

When is this maximized for $x_i \geq 0$, wlog $S = 1$

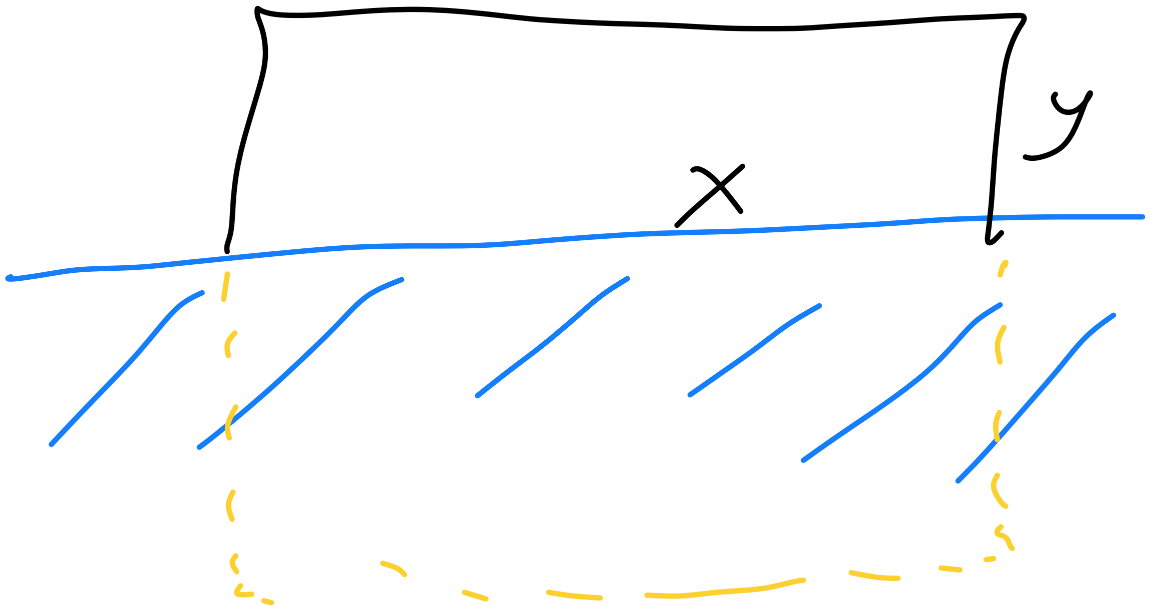
wlog, if two unequal say x_1, x_2

Consider $\frac{x_1+x_2}{2} + \frac{x_1+x_2}{2} + x_3 + \cdots + x_n = 1$

largest product has each $x_i = 1/n$

Real Analysis to make rigorous

Farmer Bob



Aquaman pen for fish

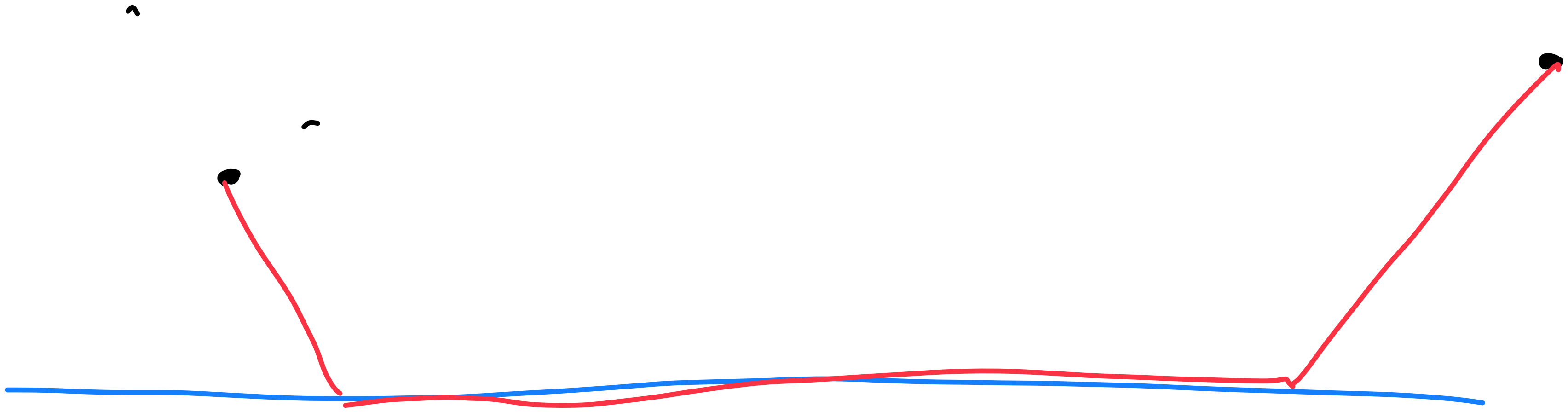
$$x + y + y = S$$

Maximize xy

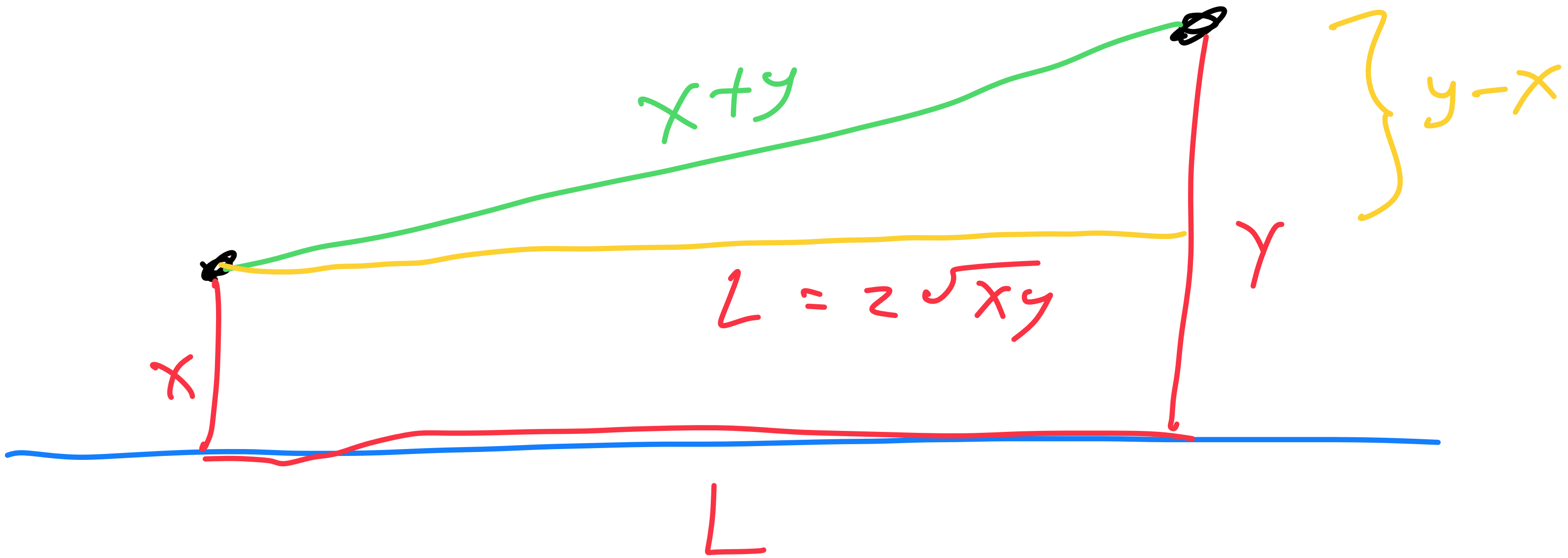
$$f(x) = x(S - x)/2$$

Combined optimal with $2S$ of fencing is a square so $2y = x$

Tim Pennings



fast on lead



$$(x+y)^2 = L^2 + (y-x)^2$$

$$x^2+2xy+y^2 = L^2+y^2-2xy+x^2$$

$$4xy = L^2$$

$$L = 2\sqrt{xy}$$

$$x+y \geq 2\sqrt{xy}$$

$$AM(x,y) = \frac{x+y}{2} \geq \sqrt{xy} = GM(x,y)$$

Do dogs know calculus? https://www.youtube.com/watch?v=h96ZNc3Z17Q
https://www.csun.edu/~dgray/BE528/Pennigs2003Dogs_Calculus.pdf

Do dogs know bifurcations?
https://www.tandfonline.com/doi/abs/10.1080/07468342.2007.11922260

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Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 8: September 23, 2024: https://youtu.be/u54ss-alhNc
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/GeneralizingPythagorasExpanded.pdf

2.5. **Integration III. GRE Practice #5:** *Solution by Steven Miller, Williams College*

From a GRE Practice Exam: Let a, b be positive; determine

$$\int_0^\infty \frac{\exp(ax) - \exp(bx)}{(1 + \exp(ax)) \cdot (1 + \exp(bx))}.$$

~~(a) 0~~ ~~(b) 1~~ ~~(c) $a - b$~~ ~~(d) $(a - b)\log(2)$~~ (e) $\frac{a-b}{ab}\log(2)$.

a=b get 0 : must be a function of a-b

$a = ra \quad b = rb \quad x = \frac{u}{r} \quad dx = \frac{du}{r}$

suggests (e)

eliminate a as not zero!

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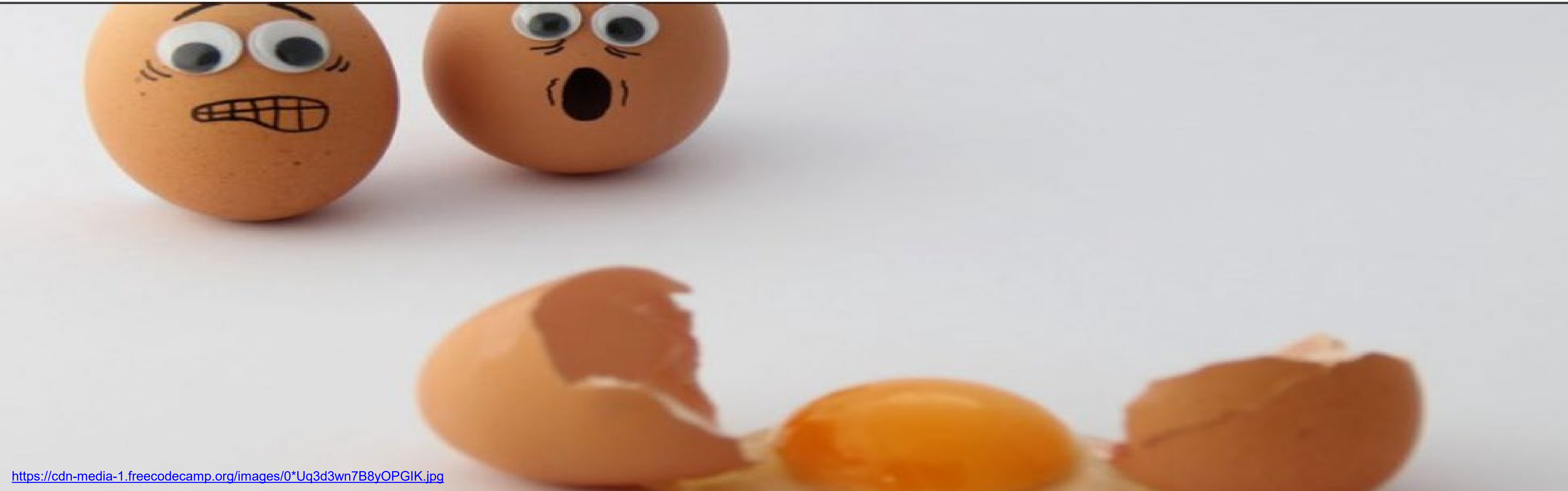
Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 9: September 25, 2024: https://youtu.be/pGixuGkTuQM
Egg Drop Mathematics: it IS all it's cracked up to be!

Egg Drop Mathematics:
It IS all its cracked up to be!

Steven J Miller, Williams College (sjm1@williams.edu)
https://web.williams.edu/Mathematics/sjmiller/public_html/



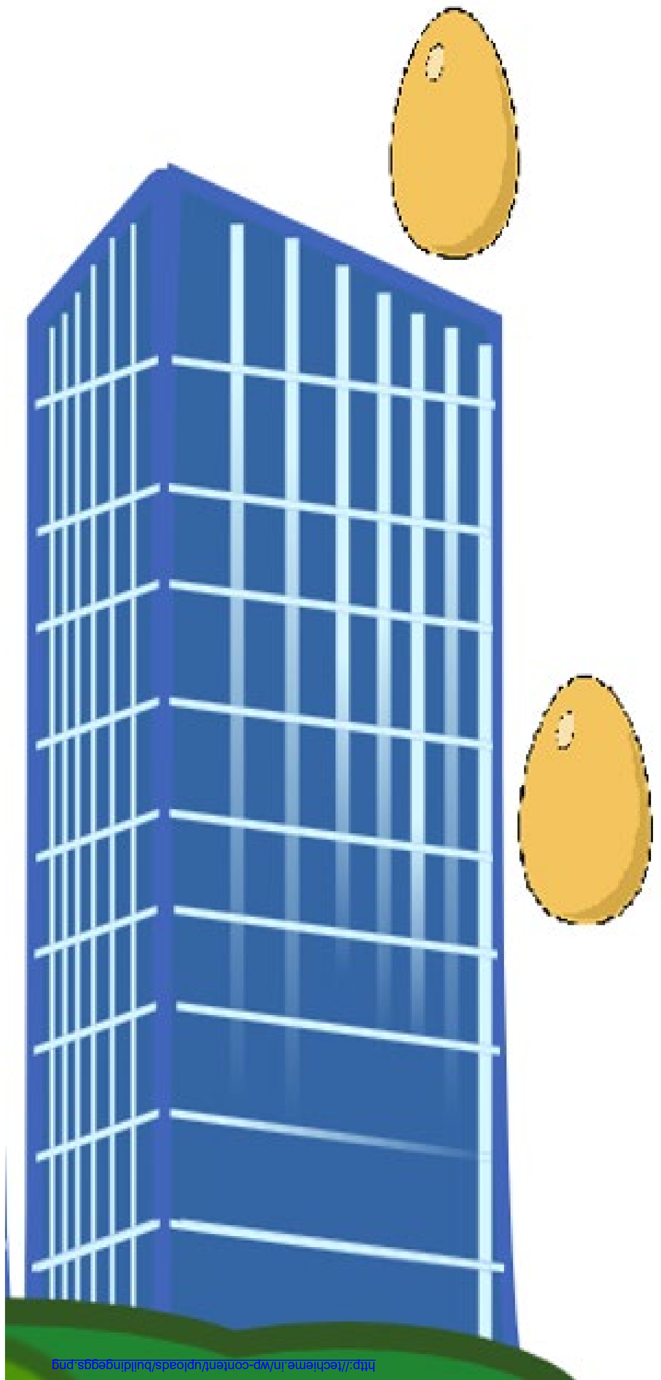
https://cdn-media-1.freecodecamp.org/images/0*Uq3d3wn7B8yOPGIK.jpg

Building with N floors, have 2 golden eggs.

Special eggs: some floor **n** such that if you drop from below **n** no damage; can drop as many times as wish.

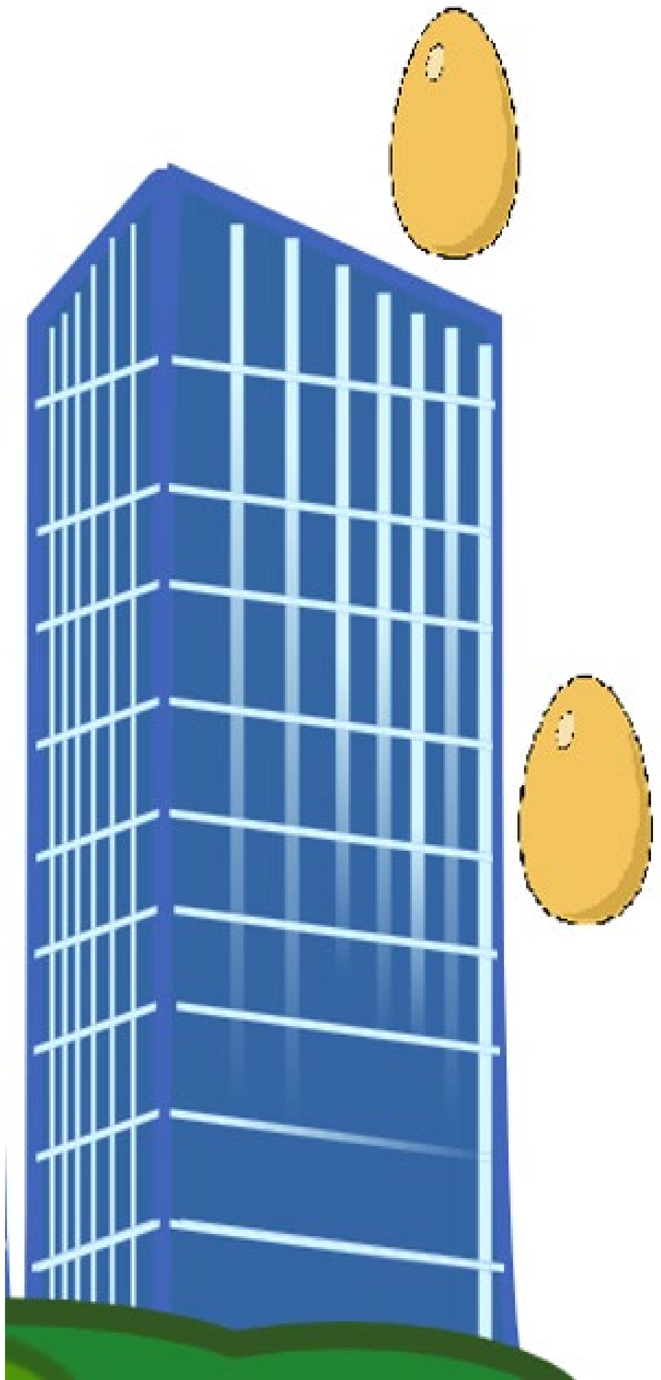
If drop even once from floor **n** or higher immediately break.

Find in as few drops as you can what **n** is (the lowest floor where if you drop from there it breaks). Doesn't matter if have any of the golden eggs at the end - just want to know **n**.



Interpretation:

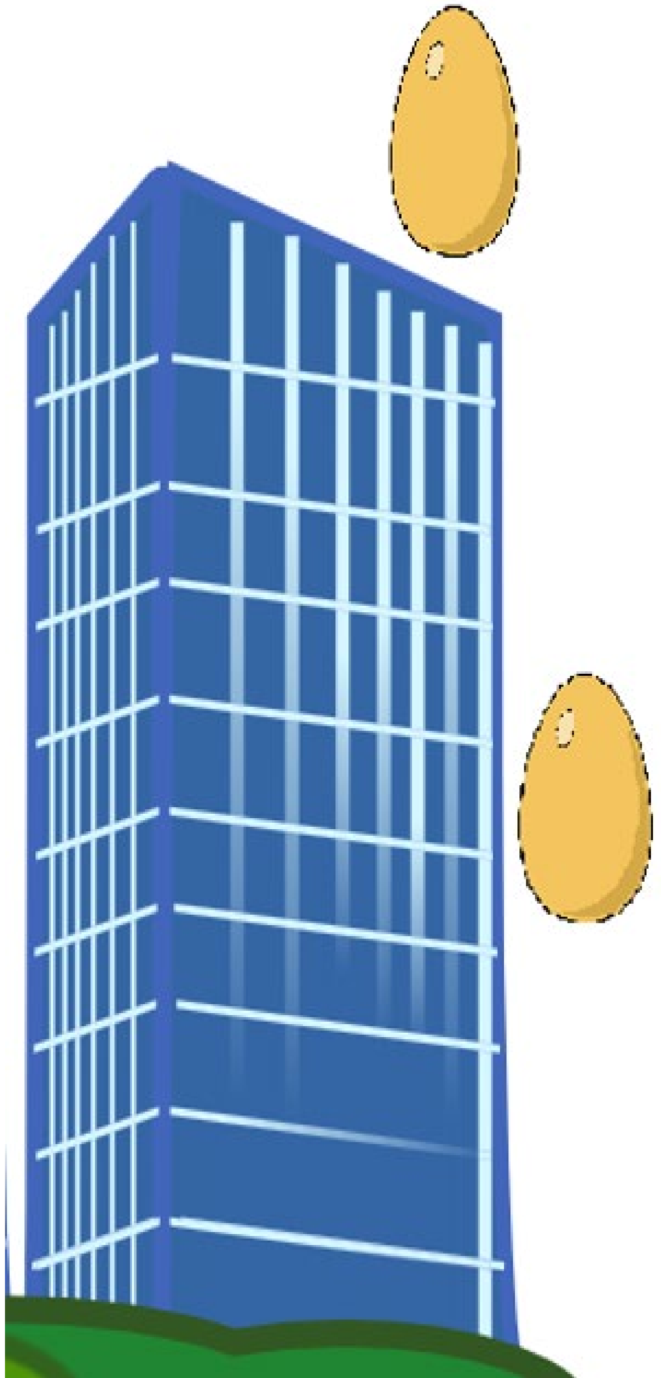
How do you interpret finding n in as few drops as possible?



Interpretation:

How do you interpret finding n in as few drops as possible?

- Minimize worse case.
- Minimize average case.

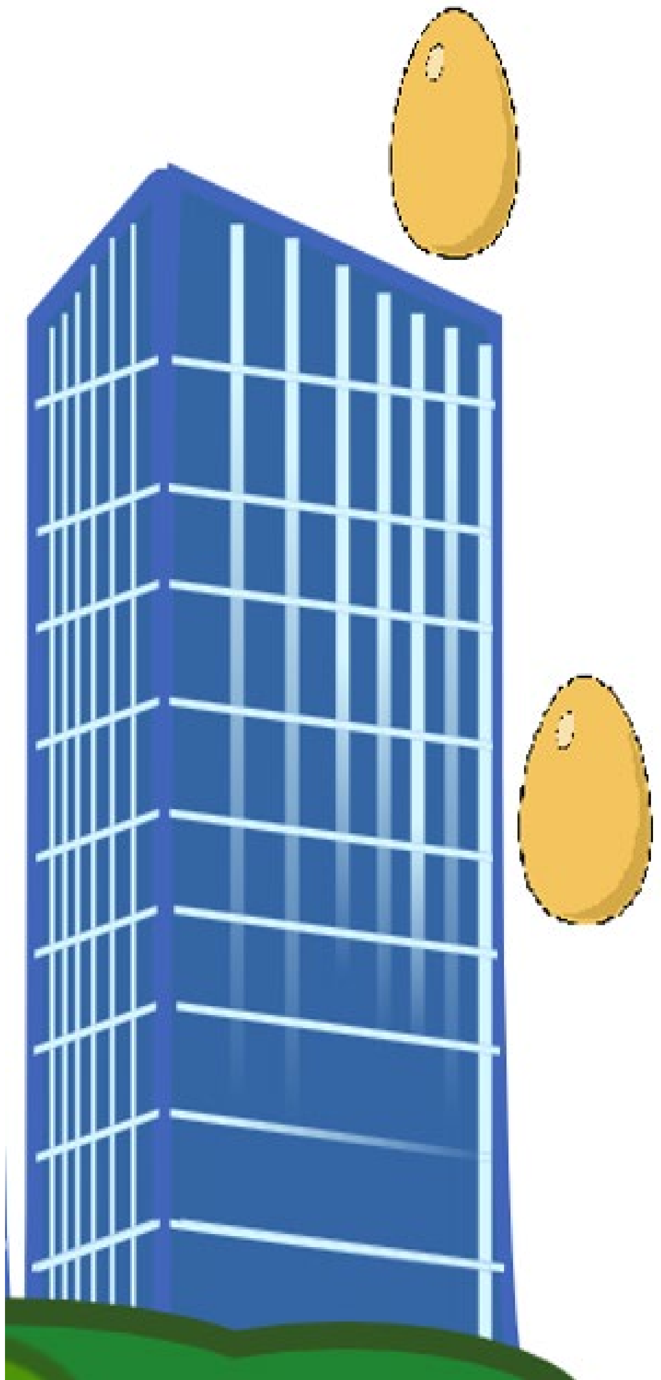


General Advice:

When given a hard problem:

- try to do an easier version first, and
- try to do specific values of parameters.

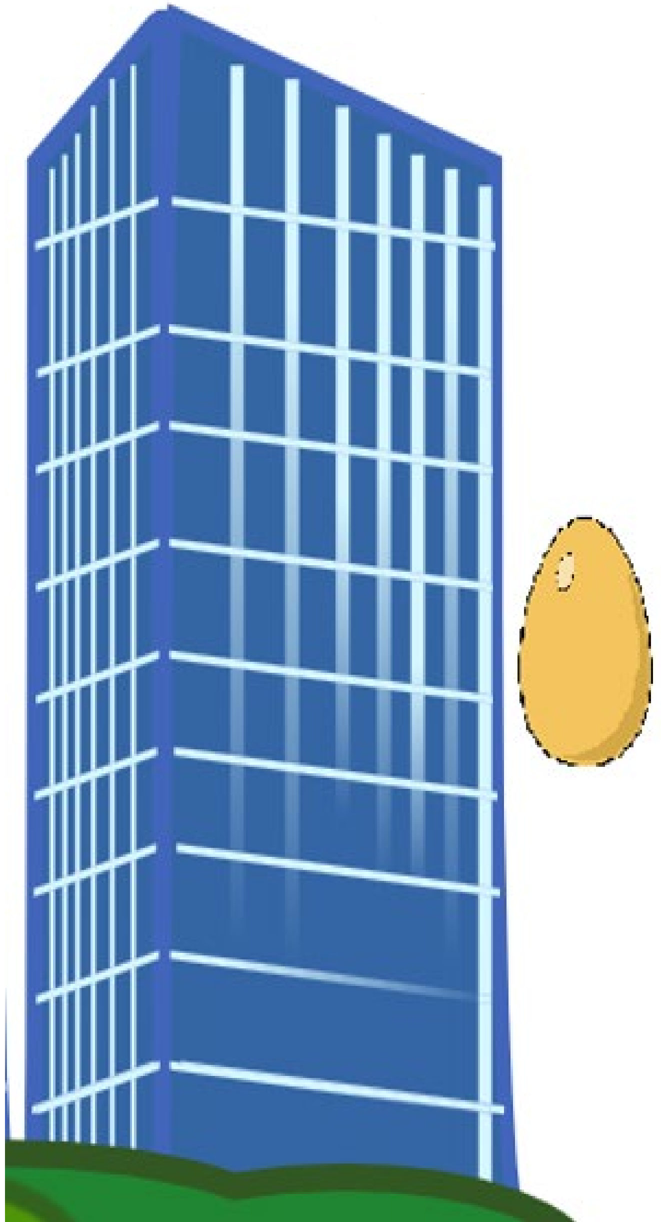
What is an easier problem?



Simple Case: 1 Egg

What is the solution?

Go 1, 2, 3, ... till crack

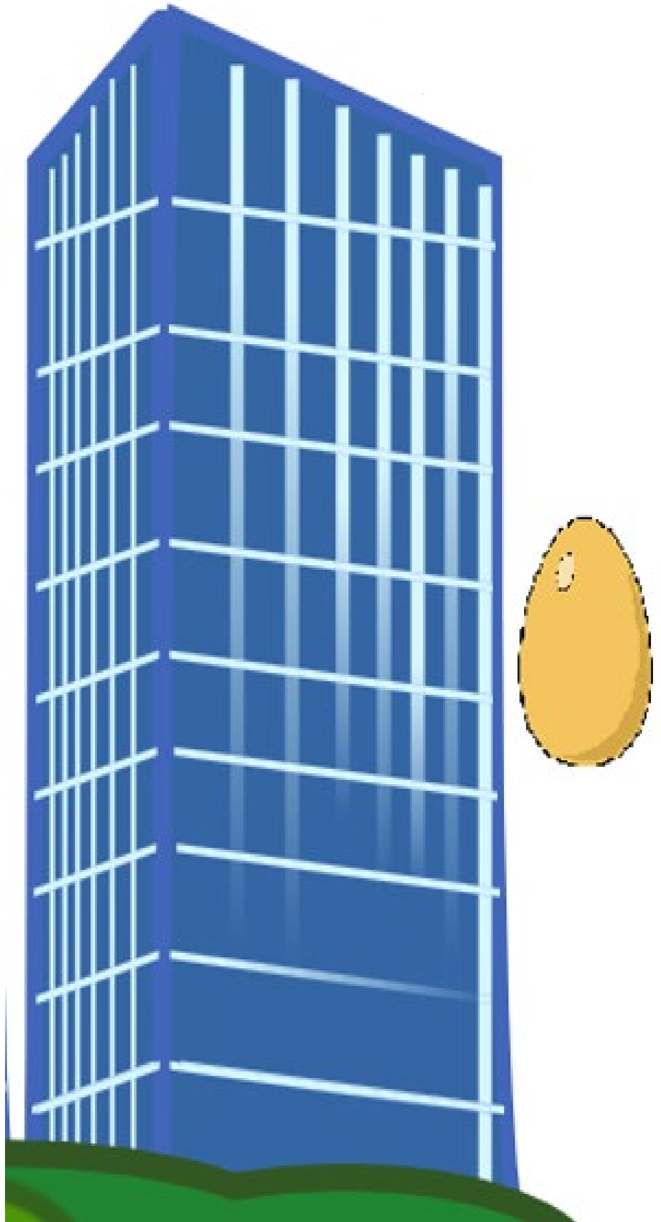


Simple Case: 1 Egg

What is the solution?

Only possibility is go 1, 2, 3, … till break.

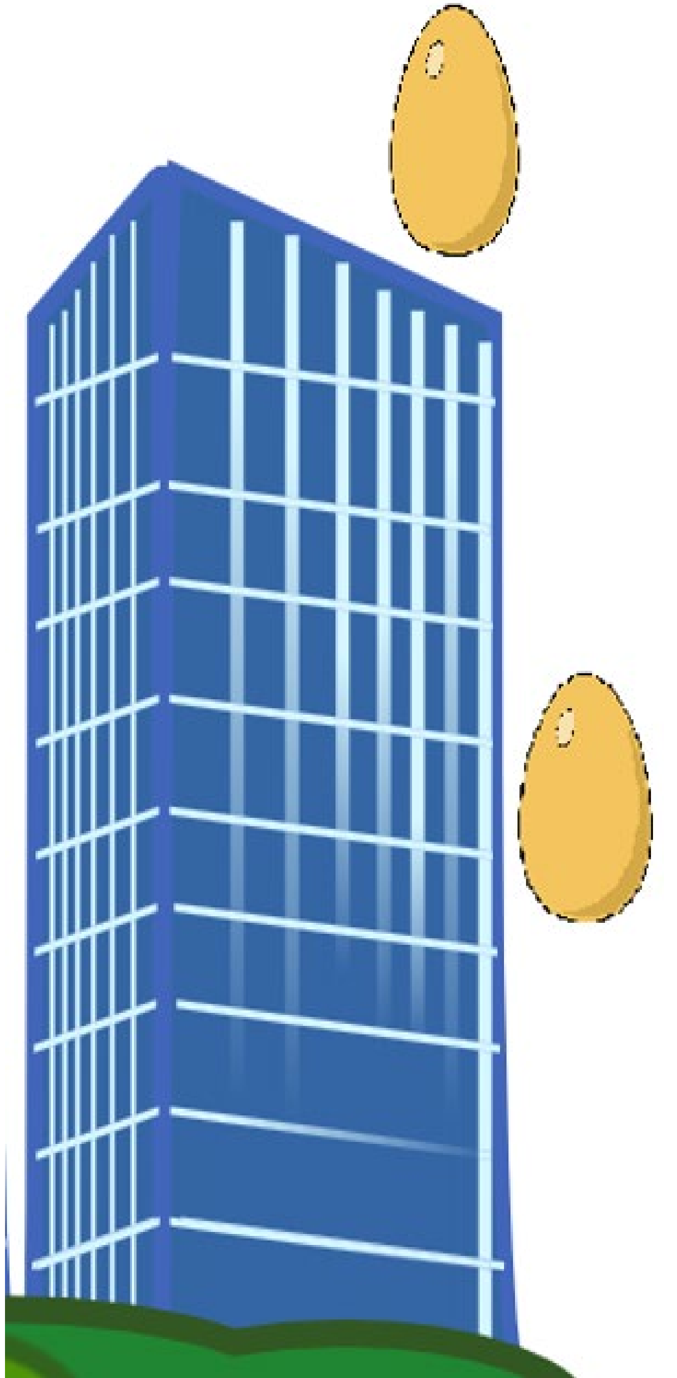
Worse case is order N drops.



Next Case: 2 Eggs

Once one cracks, reduced to 1 egg case.

What are possible strategies?



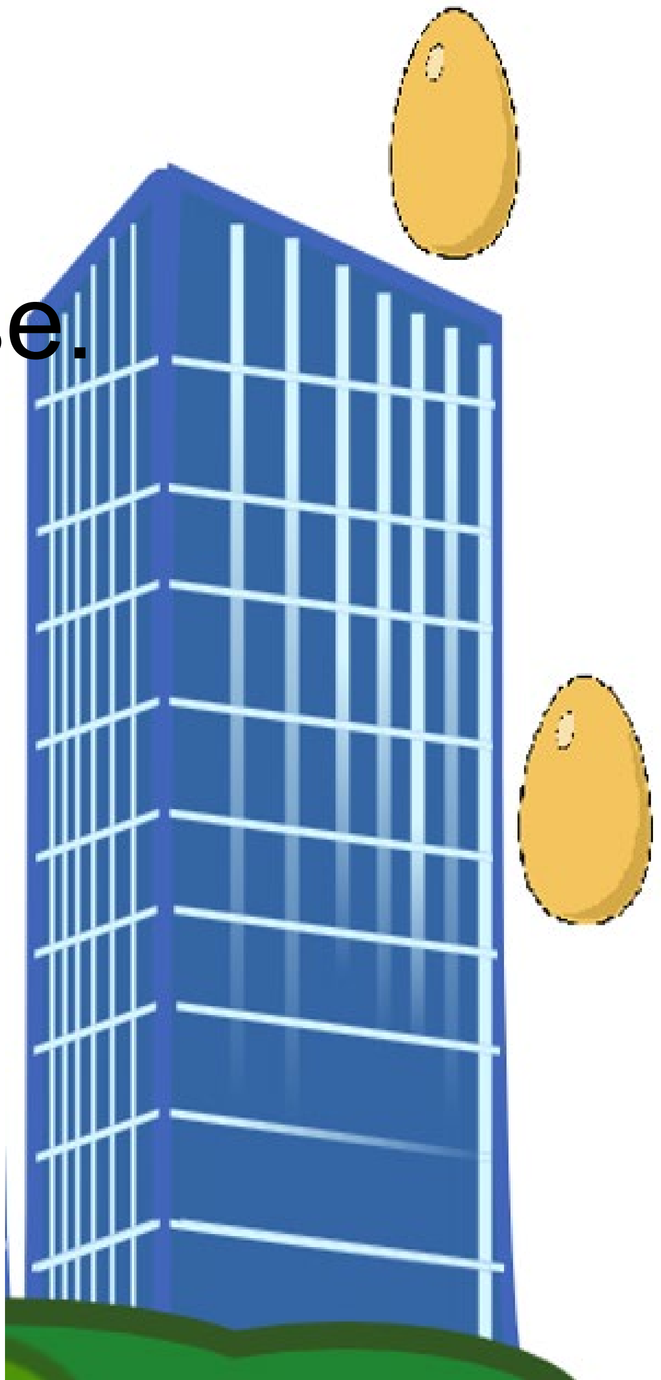
Next Case: 2 Eggs

Once one cracks, reduced to 1 egg case.

What are possible strategies?

Extreme cases:

- Drop every 2nd floor.
- Drop at N/2.
- (more generally drop every x)



Worse case: every 2: basically $\frac{N}{2}+1$

half-way: basically $1+\left(\frac{N}{2}-1\right)$

Do every x floors:

Worse Case: $\underbrace{\frac{N}{x}+x-1}$ (or $1+(x-1)$ if x is huge relative to N)

$$\left(\frac{N}{x}+x\right)-1$$

Competing Influences

Drop every 2nd floor.

- Once first breaks fast, but could take many drops.
- #Drops = N/2 + 1

Drop at N/2

- If doesn't crack eliminate a lot, when crack lot to check.
- #Drops = 1 + (N/2 – 1).

Both basically on the order of N/2 drops….

Competing Influences: Balance

Drop every x floors.

Competing Influences: Balance

Reduced to choosing x to minimize

$$\frac{N}{x} + x.$$

Competing Influences: Balance

Reduced to choosing x to minimize

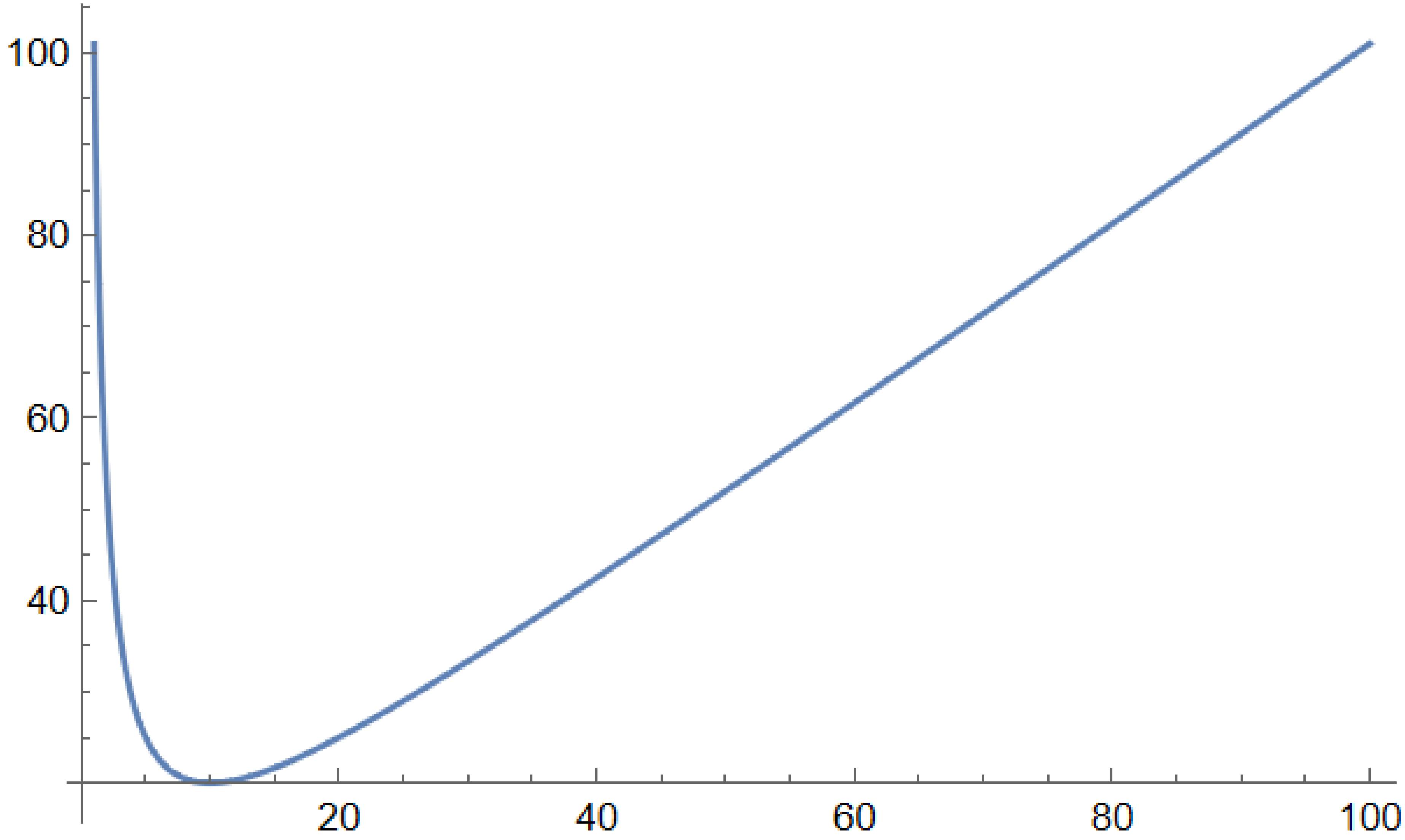
$$\frac{N}{x} + x.$$

Set two terms equal to each other to balance:

$$\frac{N}{x} = x \quad so \quad N = x2^{2} \quad or \quad x = N^{1/2}.$$

Gives #Drops = $\frac{N}{N^{1/2}} + N^{1/2}$ - 1 or about 2 $N^{1/2}$.

```
Plot[100 / x + x, {x, 1, 100}]
```



Write $x = t\,N^{1/2}$ in $\#\text{Drops} = \frac{N}{x} + x - 1$.

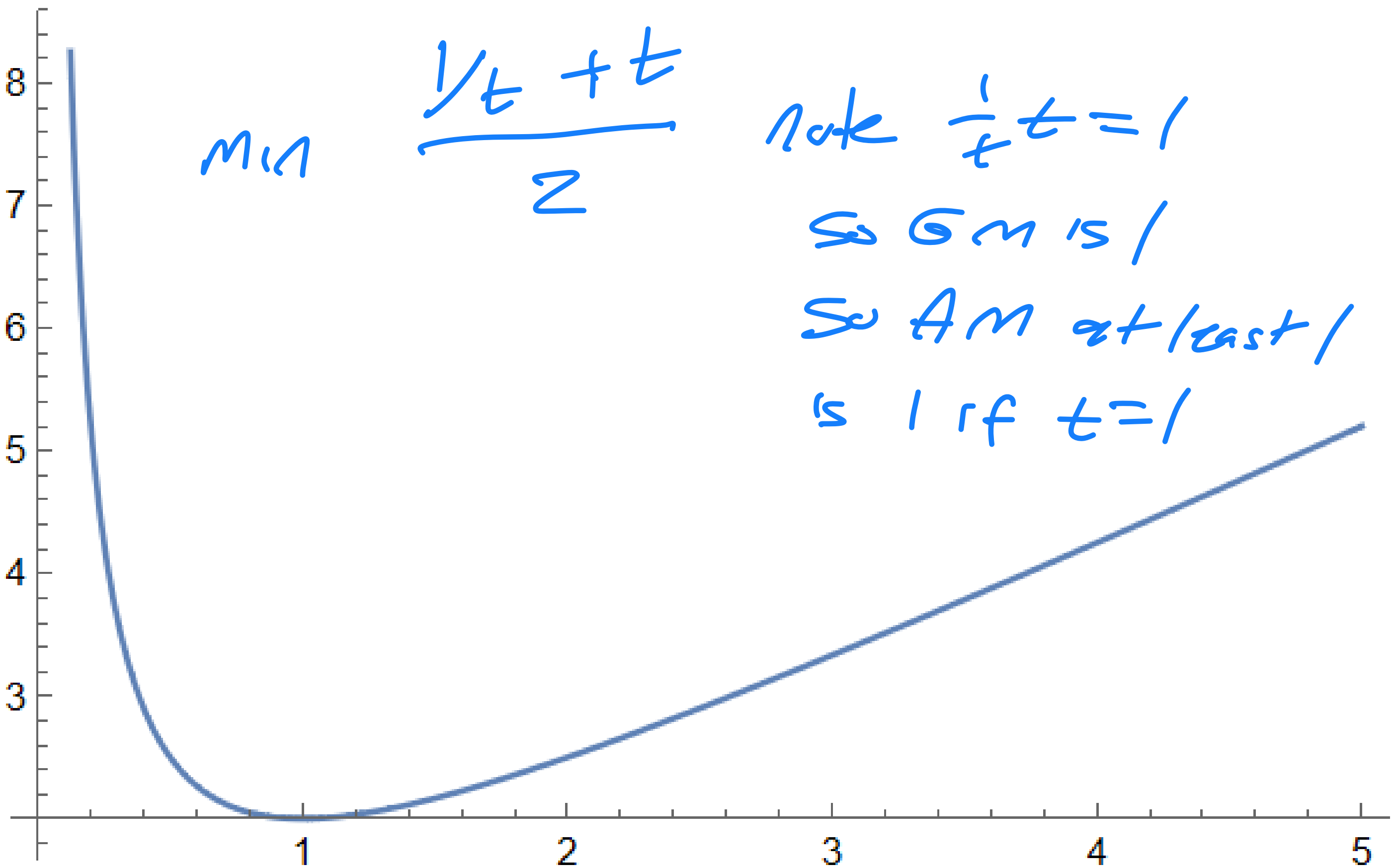
Cost is ≈ $2\sqrt{n}$ drops

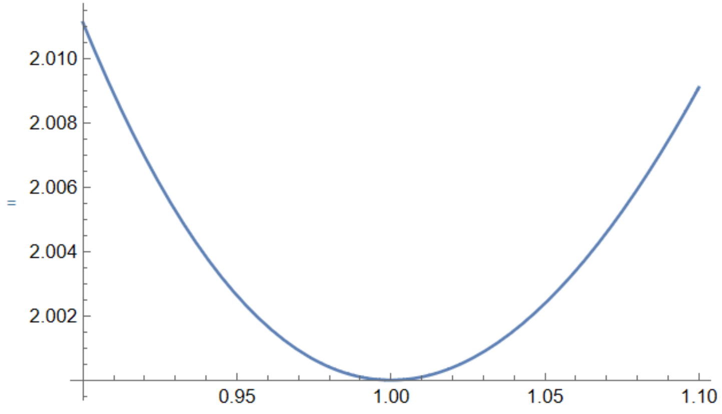
Gives $\#\text{Drops} = \frac{N}{t\,N^{1/2}} + t\,N^{1/2} - 1$.

This is just $N^{1/2}\,(\frac{1}{t} + t)$,

so on the order of $N^{1/2}$

```
Plot[1 / t + t, {t, 0, 5}]
```





If know calculus: want to minimize $f(x) = N/x + x$:

- Endpoints: $f(1)$ and $f(N)$ are of order N.
- $f'(x) = -N/x^2 + 1$, critical point $f'(x) = 0$ or $x = N^{1/2}$.
- Easily see minimum, or note $f''(x) = 2N/x^3 > 0$.

# eggs	"worse" case	(using easy x)
1	N	✓
2	$2N^{1/2}$	✓
3	$3N^{1/3}$	?

Balancing Application

Imagine have two algorithms:

- One always takes 1000 seconds.
- One takes 1 second except one in a million inputs take 1,000,000,000 seconds.

Both take on average approximately 1000 seconds….

Balancing Application

Imagine have two algorithms:

- One always takes 1000 seconds.
- One takes 1 second except one in a million inputs take 1,000,000,000 seconds.

Both take on average approximately 1000 seconds….

…but what if run algorithm 1 and if takes more than 2 seconds on an input switch to first?

Average of about 1 second!

Improving Strategy with 2 Eggs
Consider triangular numbers and dynamic rescaling.

- Do not move in constant steps of x floors.
- Do x, then x-1 if doesn't crack, then x-2….
  - Advantage is always same number of drops!
  - Basically if doesn't crack doing 2 egg problem but now with N-x floors (after first drop).

Improving Strategy with 2 Eggs
Consider triangular numbers and dynamic rescaling.

- Do not move in constant steps of x floors.
- Do x, then x-1 if doesn't crack, then x-2….
  - Advantage is always same number of drops!
  - Basically if doesn't crack doing 2 egg problem but now with N-x floors (after first drop).

Example: N = 105 = 14 + 13 + 12 + … + 1:

(1 + 13) or (2 + 12) or (3 + 11) ….

All are 14 drops, better than 2 * $105^{1/2}$ (about 20).

What if we have 3 Eggs? Or k eggs?

Drop every x floors till crack, then 2 egg problem

Worse Case: $\frac{N}{x} + WC(2 \text{ eggs and } x-1 \text{ floors})$

$\frac{N}{x} + 2x^{1/2}$ need to minimize

if $N/x = 2x^{1/2} \Rightarrow N = 2x^{3/2}$

so $x = (N/2)^{2/3}$

worse case is $2 \cdot 2\left(\left(\frac{N}{2}\right)^{2/3}\right)^{\frac{1}{2}} = \frac{2^2 \cdot N^{1/3}}{2^{1/3}} = 2^{\frac{5}{3}} N^{\frac{1}{3}}$

What if we have 3 Eggs? Or k eggs?

For 3 eggs: once one cracks, 2 egg problem.
If do every x it would be, worse case:

$$f(x) = \frac{N}{x} + 2x^{1/2}$$

$$f'(x) = -N/x^2 + x^{-1/2}$$ endpoints not minima

$$f'(x) = 0 \implies N/x^2 = x^{-1/2}$$

or $N = x^{3/2}$ so $x = N^{2/3}$

Yields worse case: $\frac{N}{N^{2/3}} + 2\left(N^{2/3}\right)^{\frac{1}{2}} = 3N^{1/3}$

Prove $\exists$ constants c_k such that with k eggs worse case $\leq c_k \cdot N^{1/k}$

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Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 10: September 27, 2024: https://youtu.be/eV1Kd0t0ewI
Project Euler Problems

Multiples of 3 or 5

Problem 1

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get $3, 5, 6$ and 9. The sum of these multiples is 23.

Find the sum of all the multiples of 3 or 5 below 1000.

Even Fibonacci Numbers

Problem 2

Each new term in the Fibonacci sequence is generated by adding the previous two terms. By starting with 1 and 2, the first 10 terms will be:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$$

By considering the terms in the Fibonacci sequence whose values do not exceed four million, find the sum of the even-valued terms.

Largest Prime Factor

Problem 3

The prime factors of 13195 are $5, 7, 13$ and 29.

What is the largest prime factor of the number 600851475143?

Largest Palindrome Product

Problem 4

A palindromic number reads the same both ways. The largest palindrome made from the product of two 2-digit numbers is $9009 = 91 \times 99$.

Find the largest palindrome made from the product of two 3-digit numbers.

Smallest Multiple

Problem 5

2520 is the smallest number that can be divided by each of the numbers from 1 to 10 without any remainder.

What is the smallest positive number that is **evenly divisible** by all of the numbers from 1 to 20?

Sum Square Difference

Problem 6

The sum of the squares of the first ten natural numbers is,

$$1^2 + 2^2 + \ldots + 10^2 = 385.$$

The square of the sum of the first ten natural numbers is,

$$(1 + 2 + \ldots + 10)^2 = 55^2 = 3025.$$

Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sum is $3025 - 385 = 2640$.

Find the difference between the sum of the squares of the first one hundred natural numbers and the square of the sum.

10001st Prime

Problem 7

By listing the first six prime numbers: $2, 3, 5, 7, 11$, and 13, we can see that the 6th prime is 13.

What is the $10\,001$st prime number?

Largest Product in a Series

Problem 8

The four adjacent digits in the 1000-digit number that have the greatest product are $9 \times 9 \times 8 \times 9 = 5832$.

```
73167176531330624919225119674426574742355349194934
96983520312774506326239578318016984801869478851843
85861560789112949495459501737958331952853208805511
12540698747158523863050715693290963295227443043557
66896648950445244523161731856403098711121722383113
62229893423380308135336276614282806444486645238749
30358907296290491560440772390713810515859307960866
70172427121883998797908792274921901699720888093776
65727333001053367881220235421809751254540594752243
52584907711670556013604839586446706324415722155397
53697817977846174064955149290862569321978468622482
83972241375657056057490261407972968652414535100474
82166370484403199890008895243450658541227588666881
16427171479924442928230863465674813919123162824586
17866458359124566529476545682848912883142607690042
24219022671055626321111109370544217506941658960408
07198403850962455444362981230987879927244284909188
84580156166097919133875499200524063689912560717606
05886116467109405077541002256983155200055935729725
71636269561882670428252483600823257530420752963450
```

Find the thirteen adjacent digits in the 1000-digit number that have the greatest product. What is the value of this product?

Special Pythagorean Triplet

Problem 9

A Pythagorean triplet is a set of three natural numbers, $a < b < c$, for which,

$$a^2 + b^2 = c^2.$$

For example, $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

There exists exactly one Pythagorean triplet for which $a + b + c = 1000$.
Find the product abc.

Summation of Primes

Problem 10

The sum of the primes below 10 is $2 + 3 + 5 + 7 = 17$.

Find the sum of all the primes below two million.

Find abc where $a^2+b^2=c^2$, $a+b+c=1000$

a, b, c pos integers $a<b<c$

$$c = 1000 - a - b$$

$$a^2+b^2 = (1000-(a+b))^2 = 10^6 - 2\cdot 10^3(a+b) + a^2+2ab+b^2$$

$$10^6 = 2\cdot 10^3(a+b) - 2ab$$

Find $(a,b) \Longrightarrow$ get $c \Longrightarrow abc$

Or: Know $2ab = 2\cdot 10^3(a+b) - 10^6$: can we get to abc?

$$\hookrightarrow \quad 2abc = \left(2\cdot 10^3(a+b) - 10^6\right)\left(10^3-(a+b)\right)$$

$x = a+b \Longrightarrow$ disc is a perfect square

$1 \leq a < b < c$ and a, b, c sum to 1000

$c = 1000 - (a+b)$

Range of b: $2 \to 498$

Range of a: $1 \to b-1$

Is $\sqrt{a^2+b^2}$ an integer?

Better: does $a^2+b^2 = (1000-(a+b))^2$

Is ω an integer, $\omega = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} = \cos(1)$

$\frac{1}{0!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\cos(1) = 1 - 1/2! + 1/4! - 1/6! + 1/8! - \cdots \in \mathbb{Z}$$

$$2\cos\left(\frac{\pi}{3}\right) = 1 = 2\left(1 - \frac{1}{2!}\left(\frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{3}\right)^4 - \frac{1}{6!}\left(\frac{\pi}{3}\right)^6 + \cdots\right)$$

$$\in \mathbb{Z}$$

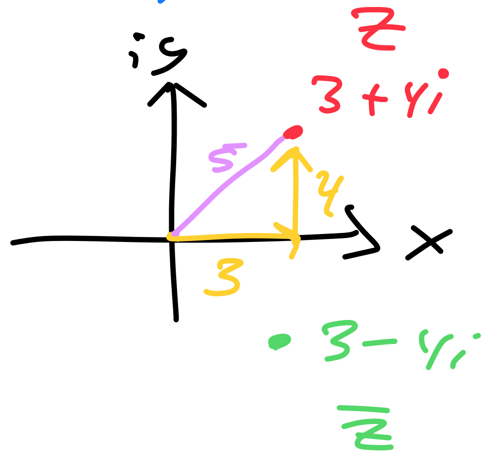
$c^2 = a^2+b^2$ and x^2+y^2 : smallest c?

$z = x+iy \qquad w = a+ib$

$$zw = (x+iy)(a+ib) = (xa-yb) + i(xb+ya)$$

$$|z|^2 = x^2+y^2 = z\cdot\bar{z} = (x+iy)(x-iy)$$

length$(z) = \sqrt{z\cdot\bar{z}}$



$$|w|^2 = a^2+b^2$$

$$|zw|^2 = (xa-yb)^2+(xb+ya)^2 = (x^2+y^2)(a^2+b^2)$$

$$(xa-yb)^2+(xb+ya)^2=(x^2+y^2)(a^2+b^2)$$

Fix $x^2+y^2 \to x+iy$, $\quad y+ix$

Fix $a^2+b^2 \to a+ib$, $\quad b+ia$

$5^2=3^2+4^2 \to 3+4i$, $\quad 4+3i$

$13^2=5^2+12^2 \to 5+12i$, $\quad 12+5i$

$x=3 \quad y=4 \quad a=5 \quad b=12$	$x=3 \quad y=4 \quad a=12 \quad b=5$
$(15-48)^2+(36+20)^2$	$(36-20)^2+(15+48)^2$
33^2+56^2	16^2+83^2
$=4225$	$=4225$

$15 \quad 5^2 * 13^2$

$$(xa-yb)^2+(xb+ya)^2=(x^2+y^2)(a^2+b^2)$$

$x=4 \quad y=3 \quad a=12 \quad b=5$

$$(48-15)^2+(20+36)^2$$

$$=33^2+56^2$$

Find smallest S' st $S'=A^2+B^2=X^2+Y^2$
and $A,B,X,Y \neq 0$ and S' is a square

11

Hint Euclid's parameterization of the Pythagorean triples (*Elements*, Book X, Proposition XXIX) is:

$$a = k(m^2 - n^2), \qquad b = 2kmn, \qquad c = k(m^2 + n^2),$$

where $m > n > 0$ and m, n coprime and not both odd.

Substituting in our condition gives

$$1000 = a + b + c = 2km(m + n),$$

and clearing the constant leaves

$$500 = km(m + n). \qquad (*)$$

Now, notice that (1) $500 = 2^2 5^3$ has only two distinct prime factors, and (2) since m and n are coprime, so are m and $m + n$.

Reveal spoiler

Share Cite Follow

edited Jun 4, 2020 at 11:38
mathlove
146k 9 120 297

answered Oct 2, 2019 at 19:45
Travis Willse
104k 13 131 264

```
>>> ck = [0]*5000;
>>> for i in range(1, 100):
...         for j in range (i + 1, 100):
...                 if i**2 + j**2 < 5000:
...                          ck[i**2+j**2] = ck[i**2 + j**2] + 1;
...
>>> for i in range(1, 5000):
...         if ck[i] > 1:
...                 print(i);
...
65
85
125
130
145
170
185
205
221
```

```
numbsumtwosquares[max_, power_, printfirst_] := Module[{},
 For[n = 0, n <= 2 max^2 + 1, n++,
  {
  listreps[n] = {};    numreps[n] = 0;
  }];
 For[y = 1, y <= max, y++,
  For[x = 1, x  <=   y, x++,
   {
   sum = x^2 + y^2;
   numreps[sum] = numreps[sum]+1;    listreps[sum] = AppendTo[listreps[sum], {x,y}];
   }];
  ];
 notfound = 0; k = 1;
 While[notfound == 0 && k < 2 max^2 + 1,
  {
  If[numreps[k^power] > 1,
   {
   Print["Number is ", k^power, " and have ",numreps[k^power], " reps."];
   Print["Reps are ", listreps[k^power]];
   If[printfirst == 1, notfound = 1];
   }];
  k = k + 1;
  }];
 ]
```

https://proofwiki.org/wiki/Sum_of_2_Squares_in_2_Distinct_Ways

Sequence

The sequence of positive integers which can be expressed as the sum of two square numbers in two or more different ways begins:

$$
\begin{aligned}
50 &= 7^2 + 1^2 &&= 5^2 + 5^2 \\
65 &= 8^2 + 1^2 &&= 7^2 + 4^2 \\
85 &= 9^2 + 2^2 &&= 7^2 + 6^2 \\
125 &= 11^2 + 2^2 &&= 10^2 + 5^2 \\
130 &= 11^2 + 3^2 &&= 9^2 + 7^2 \\
145 &= 12^2 + 1^2 &&= 9^2 + 8^2 \\
170 &= 13^2 + 1^2 &&= 11^2 + 7^2
\end{aligned}
$$

```
numbsumtwosquares[100, 2, 0]

Number is 625 and have 2 reps.

Reps are {{15, 20}, {7, 24}}

Number is 2500 and have 2 reps.

Reps are {{30, 40}, {14, 48}}

Number is 4225 and have 4 reps.

Reps are {{39, 52}, {33, 56}, {25, 60}, {16, 63}}

Number is 5625 and have 2 reps.

Reps are {{45, 60}, {21, 72}}

Number is 7225 and have 4 reps.

Reps are {{51, 68}, {40, 75}, {36, 77}, {13, 84}}

Number is 10 000 and have 2 reps.

Reps are {{60, 80}, {28, 96}}
```

Jacobi's two-square theorem [edit]

Jacobi's two-square theorem states

> The number of representations of n as a sum of two squares is four times the difference between the number of divisors of n congruent to 1 modulo 4 and the number of divisors of n congruent to 3 modulo 4.

proved by Gauss using quadratic forms and Jacobi using elliptic functions.[4] Hirschhorn gives a short proof derived from the Jacobi triple product.[5] https://en.wikipedia.org/wiki/Sum_of_two_squares_theorem

The number of ways to represent n as the sum of four squares is eight times the sum of the divisors of n if n is odd and 24 times the sum of the odd divisors of n if n is even (see divisor function), i.e.

$$r_4(n) = \begin{cases} 8\displaystyle\sum_{m|n} m & \text{if } n \text{ is odd,} \\[2ex] 24\displaystyle\sum_{\substack{m|n \\ m \text{ odd}}} m & \text{if } n \text{ is even.} \end{cases}$$

Equivalently, it is eight times the sum of all its divisors which are not divisible by 4, i.e.

$$r_4(n) = 8\sum_{\substack{m|n, \\ 4\nmid m}} m.$$

https://en.wikipedia.org/wiki/Jacobi%27s_four-square_theorem

Math 331: The little Questions (Fall 2024)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 11: September 30, 2024: https://youtu.be/vCjXRTuFbFQ
Sums of Squares

Find

$a_1^2 + b_1^2 = a_2^2 + b_2^2$ wlog $a_i \leq b_i$ and $a_1 \neq a_2$

each positive integer

Clock Arithm

$10 + 5 = 3$ on a clock with 12 hours

$a + b \equiv c \mod n$ means $\exists$ integer k

st $a + b = c + kn$

Sums of Squares

$m = 4$

n	0	1	2	3
n^2	0	1	0	1

Sums of two squares

$$\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}$$

not all numbers sums of two squares

↳ never get 3 mod 4

miss at least 25%

had at most 3/4

$m = 8$

n	0	1	2	3	4	5	6	7	8
n^2	0	1	4	1	0	1	4	1	0

Sums of two squares

$$\begin{Bmatrix} 0 \\ 1 \\ 4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 5 \end{Bmatrix}$$

not all numbers are sums of two squares.

↳ miss 3, 6, 7 mod 8

miss at least 37.5%

at most 5/8

$m = 16$

```
sumsquareoptions[modulus_, printme_] := Module[{},
  (* variables: modulus is the modulus studying; if printme = 1 we print results, else just compute *)
  list = {}; (* stores the squares of 0 to modulus modulo the modulus *)
   For[n = 0, n ≤ modulus / 2, n++,
    {
     temp = Mod[n^2, modulus];
     If[MemberQ[list, temp] == False, list = AppendTo[list, temp]]; (* only add to list if new *)
    }];
  list = Sort[list]; (* sort to make easier to read *)
  sumlist = {}; (* store the sum of pairs of squares here *)
  For[i = 1, i ≤ Length[list], i++,
   For[j = 1, j ≤ i, j++,
     {
      sum = Mod[list[[i]] + list[[j]], modulus];
      If[MemberQ[sumlist, sum] == False, sumlist = AppendTo[sumlist, sum]];
     }];];
  percent = 100.0 Length[sumlist] / modulus;
  (* calculates percent of numbers from 1 to  modulus that are a sum of two squares mod the modulus *)
  If[printme == 1 || percent < 50, (* prints if printme < 1 OR under 50% *)
   {
    Print["Modulus = ", modulus, "."];
    Print["Distinct squares are ", list];
    Print[sumlist]; Print[percent, "%."];
   }];
  Return[percent]; (* outputs/returns the percent *)
 ] (* end of function *)
```

```
modtester[min_, max_, printvalue_] := Modulus[{},
  percentlist = {}; (* stores modulus and percent here *)
  underfifty  = {}; (* stores both here as well if percent < 50% *)
  For[m = min, m ≤ max, m++,
   {
    temppercent = sumsquareoptions[m, printvalue];
    percentlist = AppendTo[percentlist, {m, temppercent}];
    (* if percent < 50 saves in underfifty and also factors the modulus *)
    If[temppercent < 50, underfifty = AppendTo[underfifty, {m, temppercent, FactorInteger[m]}]];
   }
  ];
  (* listplot and includes a solid line at 50% so can easily see when below *)
  Print[Show[ListPlot[percentlist], Plot[50.0, {x, min, max}, PlotStyle → {Red}]]];
  Print[underfifty];
  (*Print[percentlist];*)
 ]
```

Pigeonhole or Box Principle (Dirichlet)

n boxes, $n+1$ pigeons.

if each pigeon is assigned exactly one box, at least one box has at least 2 pigeons.

if not, then each box has at most 1 pigeon, only accounting for at most n pigeons.

Choose some modulus n

look at sums of squares mod n, assume sum $\leq X$

Saw some moduli miss a lot of residues

Find a "good" modulus where miss a lot of residues.

Hope is then find pairs go to same thing....

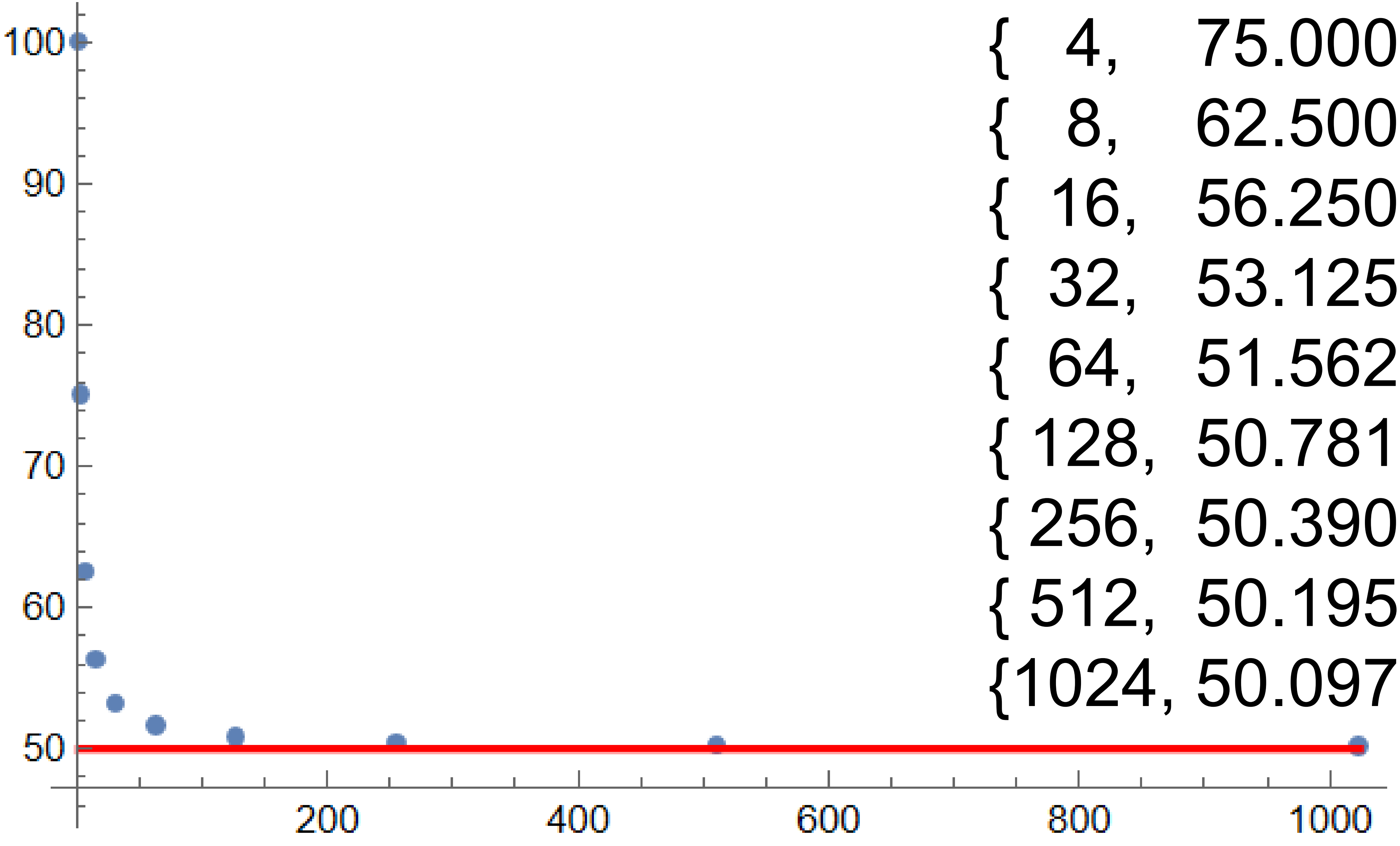
Question: (a,b) with $1 \leq a \leq b \leq \sqrt{X}$ and $a^2+b^2 \leq X$

How many pairs?

Upper bound: $a \leq \sqrt{X}$, $b \leq \sqrt{X}$ so at most X pairs.

lower bound...?

```
powertwomodtester[1, 10, 0]
```



{2, 100.0000}
{4, 75.0000}
{8, 62.5000}
{16, 56.2500}
{32, 53.1250}
{64, 51.5625}
{128, 50.7813}
{256, 50.3906}
{512, 50.1953}
{1024, 50.0977}

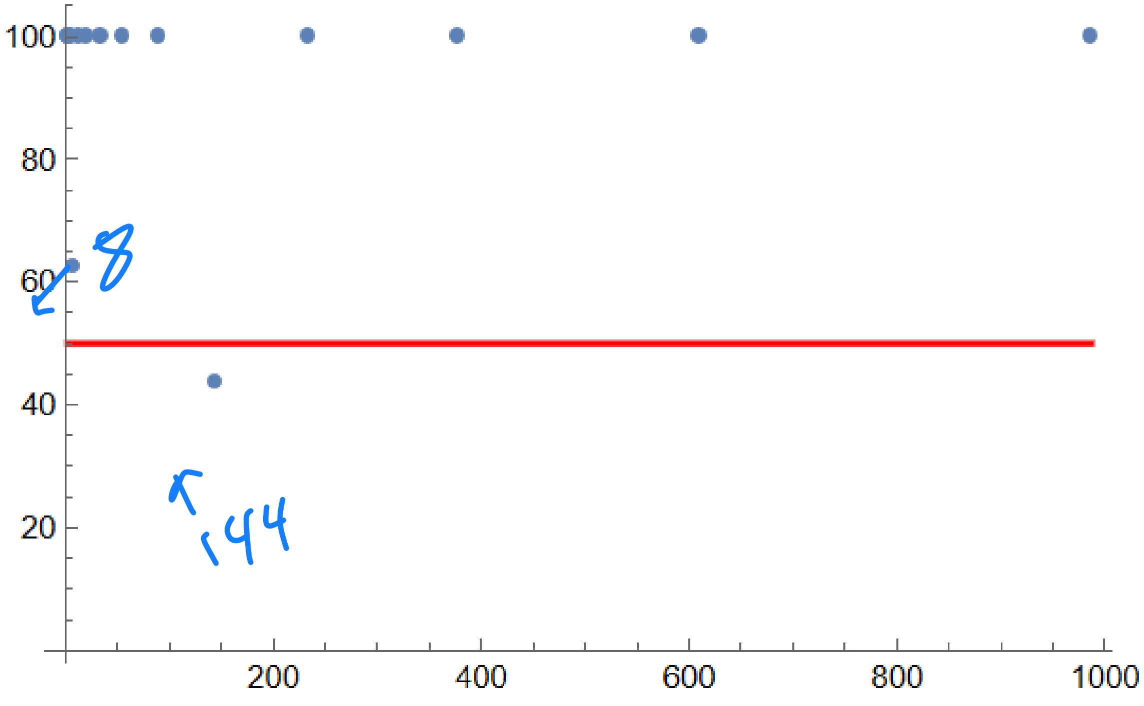
```
fibonaccimodtester[3, 16, 0]
```

Modulus = 144.

Distinct squares are {0, 1, 4, 9, 16, 25, 36, 49, 52, 64, 73, 81, 97, 100, 112, 121}

{0, 1, 2, 4, 5, 8, 9, 10, 13, 18, 16, 17, 20, 25, 32, 26, 29, 34, 41, 50, 36, 37, 40, 45, 52, 61, 72, 49, 53, 58, 65, 74, 85, 98,
56, 68, 77, 88, 101, 104, 64, 73, 80, 89, 100, 113, 116, 128, 82, 109, 122, 125, 137, 81, 90, 97, 106, 117, 130, 133, 136, 112, 12

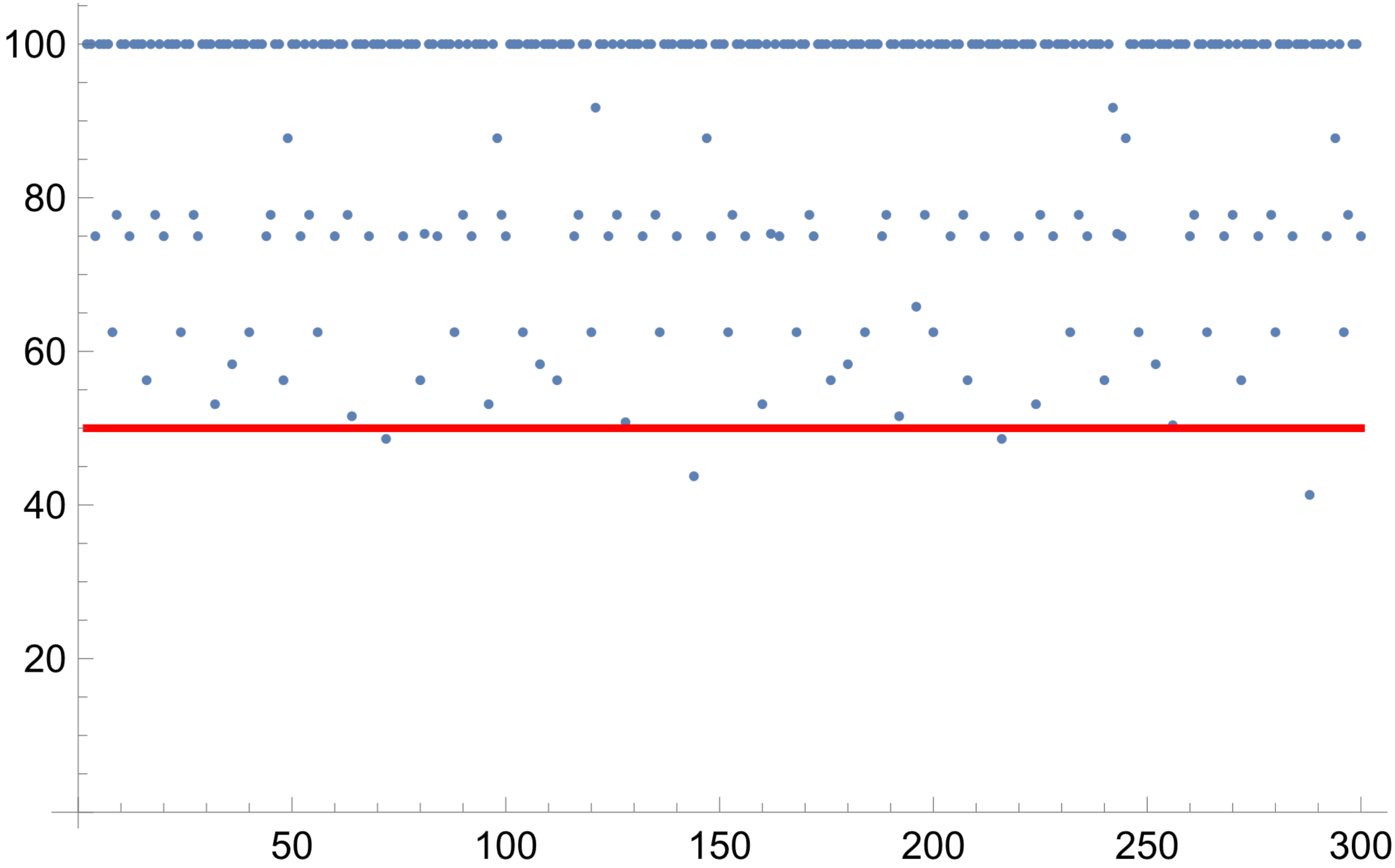
43.75%.



{{144, 43.75, {{2, 4}, {3, 2}}}}

Fibonaccis:

1, 2, 3, 5, 8, 13, 21,
34, 55, 89, 144, ...


100
80
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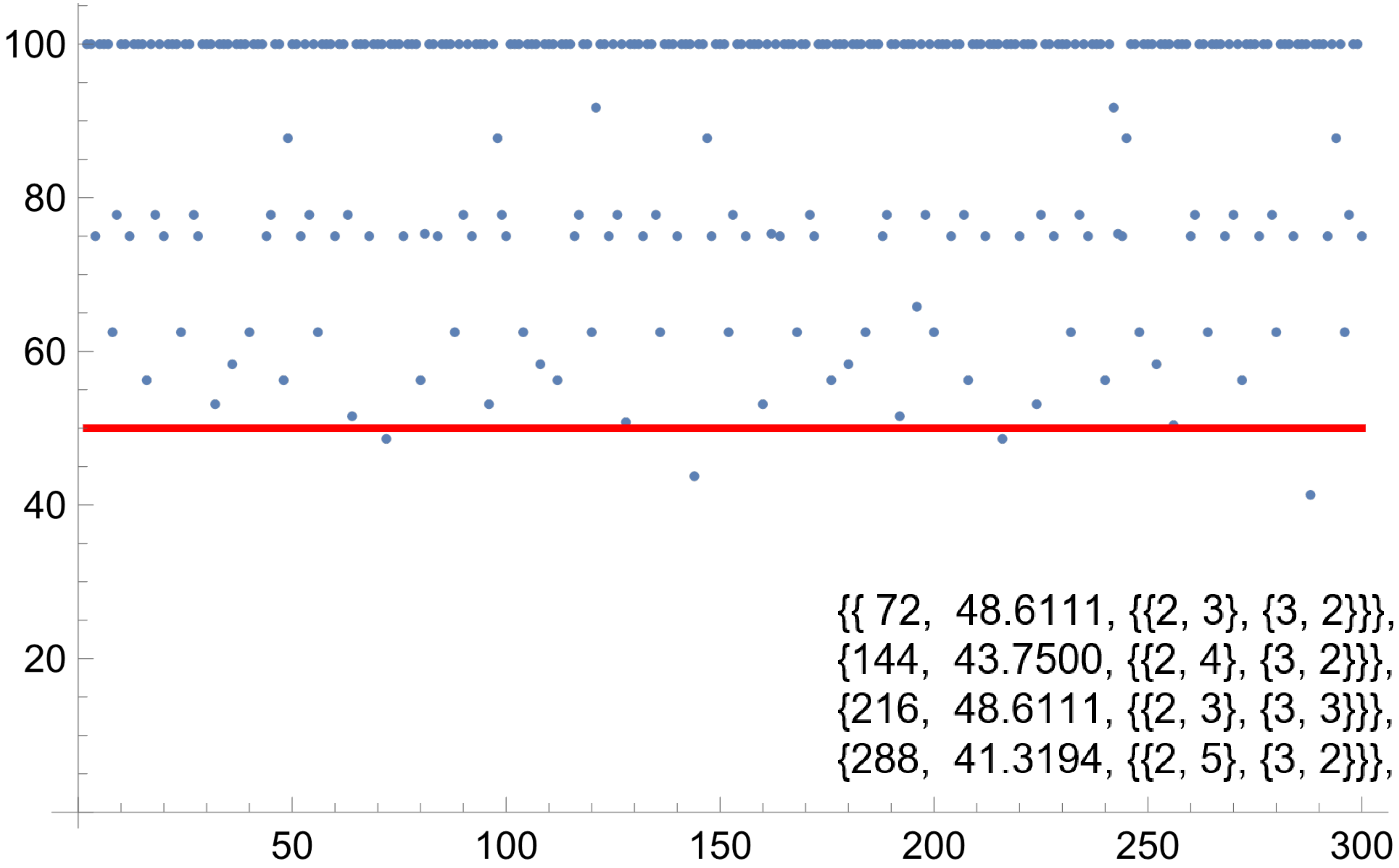
50
100
150
200
250
300



100
80
60
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50
100
150
200
250
300
{{ 72, 48.6111, {{2, 3}, {3, 2}}},
{144, 43.7500, {{2, 4}, {3, 2}}},
{216, 48.6111, {{2, 3}, {3, 3}}},
{288, 41.3194, {{2, 5}, {3, 2}}},

#pairs (a,b) with $1 \le a \le b \le \sqrt{x}$ and $a^2+b^2 \le x$

$$\frac{1}{2}\sum_{b=1}^{\sqrt{x}} \sum_{a=1}^{\sqrt{x-b^2}} 1 \approx \frac{1}{2}\sum_{b=1}^{\sqrt{x}} \sqrt{x-b^2} \approx \frac{1}{2}\int_{b=1}^{\sqrt{x}} (x-b^2)^{1/2}\, db$$

Slight issue if $a = b \le \sqrt{x}$ but only $\sqrt{x}$ such items, lower order

Let $b = t\sqrt{x}$ so $db = \sqrt{x}\, dt$ $\quad b: 1 \to \sqrt{x}$ $\quad t: 0 \to 1$

$$\approx \frac{1}{2}\int_{t=0}^{1} \sqrt{x - xt^2}\; x^{\frac{1}{2}} dt = \frac{1}{2} x \int_0^1 \sqrt{1-t^2}\, dt$$

$t = \sin\theta$ $\quad t: 0 \to 1$ becomes $\theta: 0 \to \pi/2$ $\quad dt = \cos\theta\, d\theta$

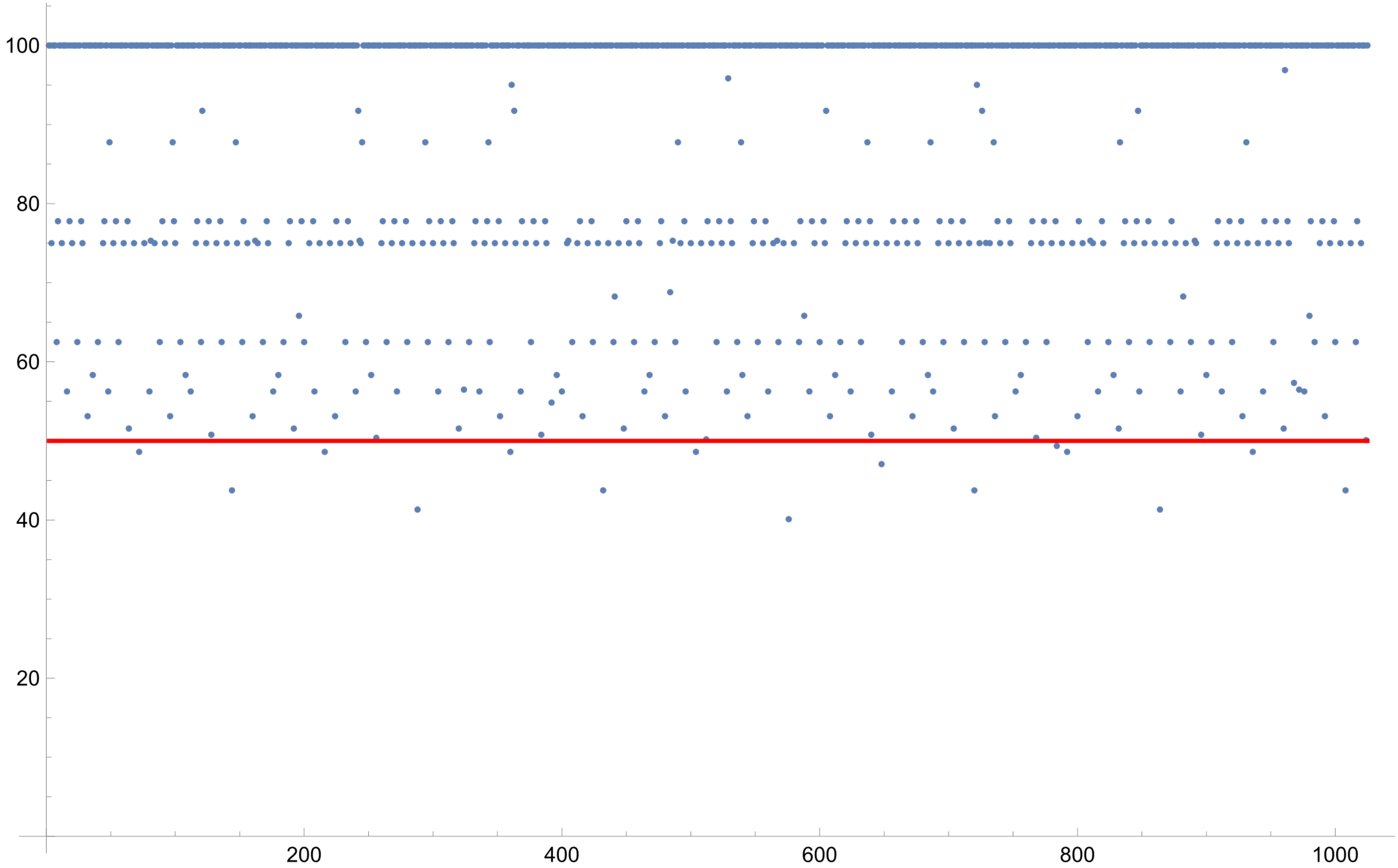
$$\approx \frac{1}{2} x \int_{\theta=0}^{\pi/2} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta = \frac{1}{2} x \int_0^{\pi/2} \cos^2\theta\, d\theta$$

use $\cos 2\theta = 2\cos^2\theta - 1$ or $\cos^2\theta + \sin^2\theta = 1$

$$\approx \frac{1}{2} x \cdot \frac{1}{2} \int_0^{\pi/2} [\cos^2\theta + \sin^2\theta]\, d\theta = \frac{1}{2} x \cdot \frac{\pi}{4}$$

Found the

Missing a factor of 2 $\pi/8 \approx .392699$



100
80
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200
400
600
800
1000

```
{{ 72,  48.6111, {{2, 3}, {3, 2}}},
{144,  43.7500, {{2, 4}, {3, 2}}},
{216,  48.6111, {{2, 3}, {3, 3}}},
{288,  41.3194, {{2, 5}, {3, 2}}},
{360,  48.6111, {{2, 3}, {3, 2}, {5, 1}}},
{432,  43.7500, {{2, 4}, {3, 3}}},
{504,  48.6111, {{2, 3}, {3, 2}, {7, 1}}},
{576,  40.1042, {{2, 6}, {3, 2}}},
{648,  47.0679, {{2, 3}, {3, 4}}},
{720,  43.7500, {{2, 4}, {3, 2}, {5, 1}}},
{784,  49.3622, {{2, 4}, {7, 2}}},
{792,  48.6111, {{2, 3}, {3, 2}, {11, 1}}},
{864,  41.3194, {{2, 5}, {3, 3}}},
{936,  48.6111, {{2, 3}, {3, 2}, {13, 1}}},
{1008,43.7500, {{2, 4}, {3, 2}, {7, 1}}}}
```

Math 331: The little Questions (Fall 2024)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 12: October 2, 2024: https://youtu.be/cgpxfVRE-vQ
Sums of Squares II

```
{{ 72,  48.6111, {{2, 3}, {3, 2}}},
{144,  43.7500, {{2, 4}, {3, 2}}},
{216,  48.6111, {{2, 3}, {3, 3}}},
{288,  41.3194, {{2, 5}, {3, 2}}},
{360,  48.6111, {{2, 3}, {3, 2}, {5, 1}}},
{432,  43.7500, {{2, 4}, {3, 3}}},
{504,  48.6111, {{2, 3}, {3, 2}, {7, 1}}},
{576,  40.1042, {{2, 6}, {3, 2}}},
{648,  47.0679, {{2, 3}, {3, 4}}},
{720,  43.7500, {{2, 4}, {3, 2}, {5, 1}}},
{784,  49.3622, {{2, 4}, {7, 2}}},
{792,  48.6111, {{2, 3}, {3, 2}, {11, 1}}},
{864,  41.3194, {{2, 5}, {3, 3}}},
{936,  48.6111, {{2, 3}, {3, 2}, {13, 1}}},
{1008,43.7500, {{2, 4}, {3, 2}, {7, 1}}}}
```

```
100 Pi / 8.

39.2699
```

```
lessthanfiftyunderonethousand = {72, 144, 216, 288, 360, 432, 504, 576, 648, 720, 784, 792, 864, 936, 1008};
```


THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

72, 144, 216, 288, 360, 432, 504, 576, 648, 720 | Search | Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:72,144,216,288,360,432,504,576,648,720**

Sorry, but the terms do not match anything in the table.

The following advanced matches exist for the numeric terms in your query.

Beatty sequence

a(n) = the Beatty sequence floor(n*z) with 72/1 <= z < 721/10.

The next few terms would be .

Polynomial

a(n) = 72 x

The next few terms would be 792, 864, 936.
Sequence is a trivial polynomial. Did not search for other matches.

```
{{ 72,  48.6111, {{2, 3}, {3, 2}}},
{144,  43.7500, {{2, 4}, {3, 2}}},
{216,  48.6111, {{2, 3}, {3, 3}}},
{288,  41.3194, {{2, 5}, {3, 2}}},
{360,  48.6111, {{2, 3}, {3, 2}, {5, 1}}},
{432,  43.7500, {{2, 4}, {3, 3}}},
{504,  48.6111, {{2, 3}, {3, 2}, {7, 1}}},
{576,  40.1042, {{2, 6}, {3, 2}}},
{648,  47.0679, {{2, 3}, {3, 4}}},
{720,  43.7500, {{2, 4}, {3, 2}, {5, 1}}},
{784,  49.3622, {{2, 4}, {7, 2}}},
{792,  48.6111, {{2, 3}, {3, 2}, {11, 1}}},
{864,  41.3194, {{2, 5}, {3, 3}}},
{936,  48.6111, {{2, 3}, {3, 2}, {13, 1}}},
{1008,43.7500, {{2, 4}, {3, 2}, {7, 1}}}}
```

```
100 Pi / 8.

39.2699
```

Modulus 2304
39.1927%

Is every n a sum of two squares?

Mod 4 squares are 0, 1, so sum of two squares is 0, 1 or 2 mod 4, so never get 3 mod 4.

Say $a, b \le x$ and $a^2+b^2 \le x$ and $a \le b$

$\hookrightarrow \underbrace{\frac{1}{2}\sum_{b=1}^{\sqrt{x}} \sum_{a=1}^{\sqrt{x-b^2}} 1}_{\text{Const}\cdot x} - \underbrace{\sum_{b=1}^{\sqrt{x}} 1}_{\text{const}\cdot\sqrt{x}}$

Choose $a, b \le \sqrt{x}$: distinct, $a < b$: $\binom{\sqrt{x}}{2} \approx \frac{\sqrt{x}(\sqrt{x}-1)}{2}$

Choose $a = b \le \sqrt{x}$: $\binom{\sqrt{x}}{1} = \sqrt{x}$

sum $\approx \frac{\sqrt{x}(\sqrt{x}+1)}{2}$

$\approx x/2$

Is every n a sum of three squares?

$a^2+b^2+c^2 \le x$

$$\frac{1}{6}\sum_{c=1}^{\sqrt{x}} \sum_{b=1}^{\sqrt{x-c^2}} \sum_{a=1}^{\sqrt{x-(b^2+c^2)}} 1$$

Counting "distinct" triples, deal with lower order of 2 the same, or 3 the same

Can permute (a,b,c) how many ways? $3!$ or 6 ways

all same: $a^2+a^2+a^2 \le x \Rightarrow$ # choices is $\sqrt{x/3}$

$a \le b \le c \le \sqrt{x/3}$ Then $a^2+b^2+c^2 \le x$

Lower bound on triples!

Choose 3 distinct integers from $\{1, 2, \ldots, \sqrt{x/3}\}$

↳ ans: $\binom{\sqrt{x/3}}{3} \approx \frac{(\sqrt{x/3})^3}{3!} \approx \frac{x^{3/2}}{\sqrt{27}\cdot 6} = \frac{x^{1/2}}{\sqrt{27}\cdot 6}\, x$

large

Look at sums of squares mod 8

n :	0	1	2	3	4	5	6	7	8
n^2 :	0	1	4	1	0	1	4	1	0

$$\begin{Bmatrix} 0 \\ 1 \\ 4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \\ 4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{Bmatrix}$$

missing 7

Example A.4.2. *If we choose a subset S from the set $\{1, 2, \ldots, 2n\}$ with $|S| = n+1$, then S contains at least two elements a, b with $a|b$.*

distinct

Exercise A.4.3. *If we choose 55 numbers from $\{1, 2, 3, \ldots, 100\}$ then among the chosen numbers there are two whose difference is ten (from [Ma]).*

Exercise A.4.4. *Let $a_1, \ldots, a_{n+1}$ be distinct integers in $\{1, \ldots, 2n\}$. Prove two of them add to a number divisible by $2n$.* Note: these two numbers need not be distinct. If instead we required the two numbers to be distinct, then $a_1, \ldots, a_{n+1}$ would have to be $n+1$ distinct numbers from $\{1, ..., 2n-1\}$.

Exercise A.4.5. *Let $a_1, \ldots, a_n$ be integers. Prove that there is a subset whose sum is divisible by n.*

What are the boxes?

What are the pigeons?

Choose 55 distinct #s from $\{1, 2, \dots, 100\}$, Claim two differences 10

How many differences (positive) do we have?

↳ $\binom{55}{2}$ chose a pair of numbers (x, y) with $x < y$

↳ This is $\frac{55 \cdot 54}{2} = 55 \cdot 27 = 1485$

How many distinct differences? Can get 99, 98, 97, ..., 2, 1 or 99 possible

$a_7 - a_6 = a_{11} - a_8$ so ......

Break into congruence classes mod 10: 10 classes

remainder 0 : 10 - 30 - 50 - 70 - ~~90~~ 100

remainder 1 : 1 - 21 - 41 - 61 - ~~81~~ 91

⋮

remainder 9 : 9 - 29 - 49 - 69 - 89

accounts for 50

Math 331: The little Questions (Fall 2024)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 13: Oct 4, 2024: M&M Game: https://youtu.be/-plcuC64a6E (slides pdf)
Lecture 14: October 7, 2024: https://youtu.be/vWYXyXN6YSE
Introduction to Games: Triangle Game, Pirates

Rules for Triangle Game

Take an equilateral triangle, label corners 0, 1 and 2.

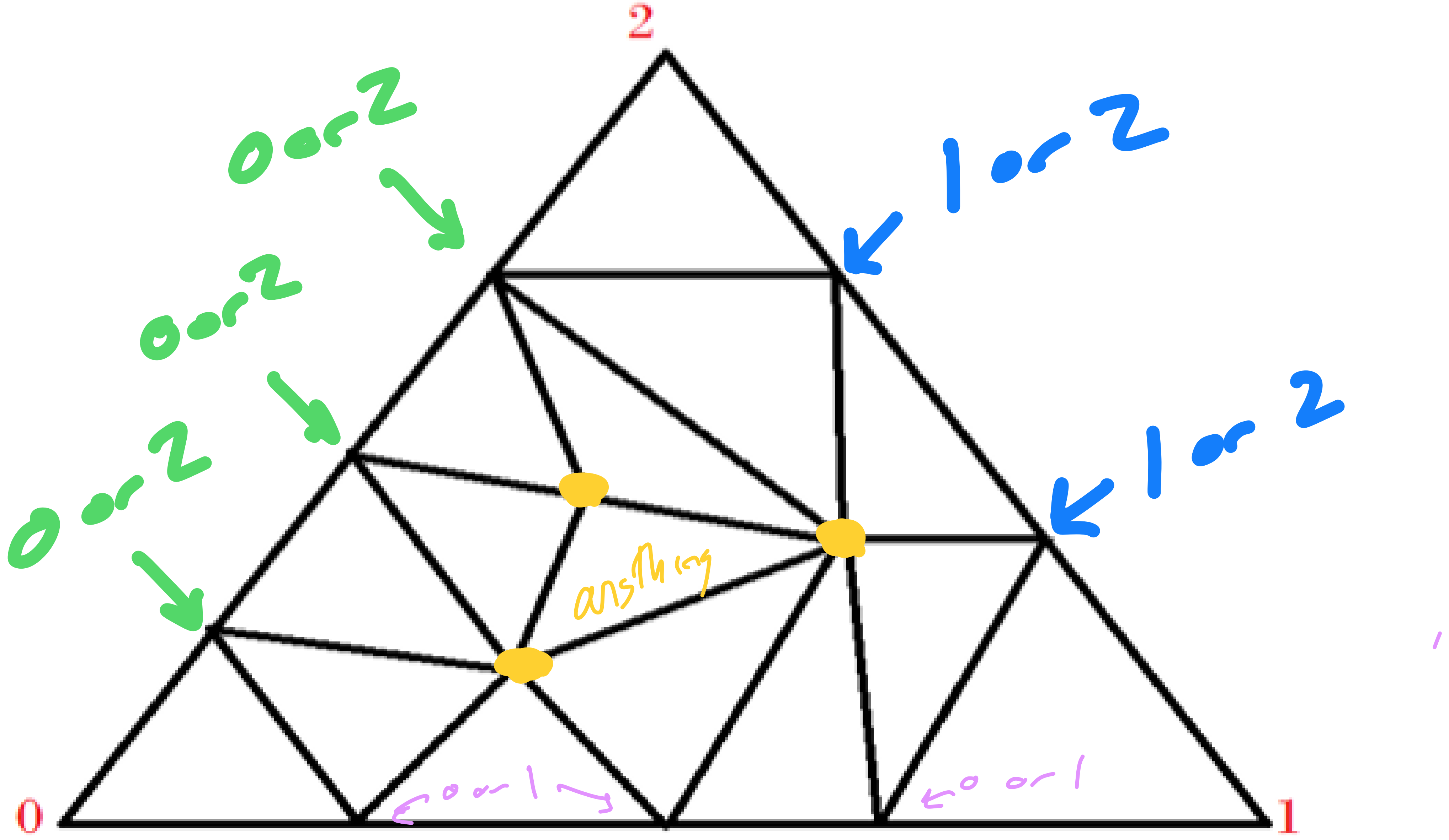
Subdivide however you wish into triangles.

Add labels, if a sub-triangle labeled 0–1–2 then Player 1 wins, else Player 2.

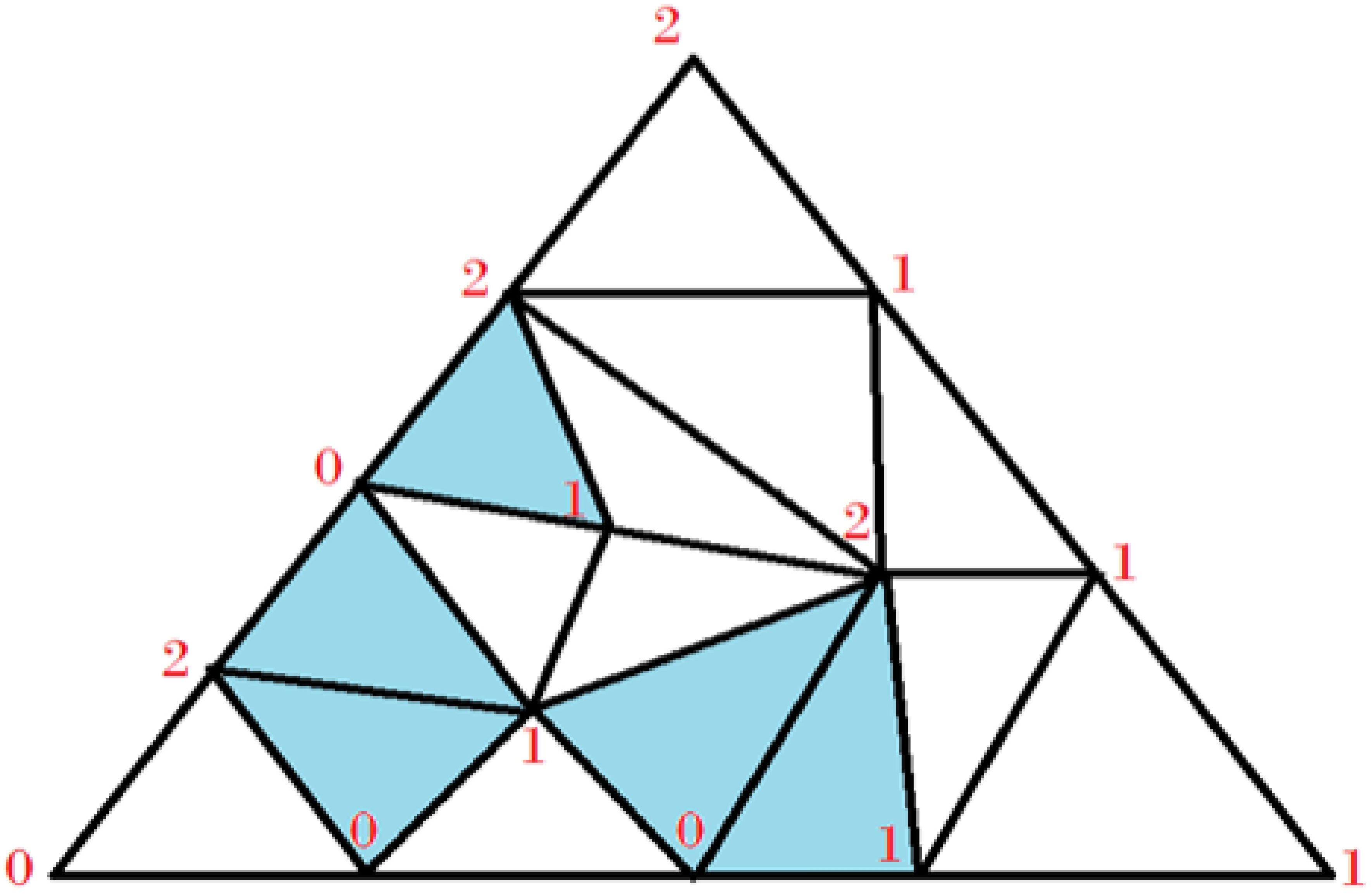
Take turns adding labels, subject to:
On 0–1 boundary must use 0 or 1
On 1–2 boundary must use 1 or 2
On 0–2 boundary must use 0 or 2

Who has the winning strategy? What is it?

Rules for Triangle Game



Rules for Triangle Game

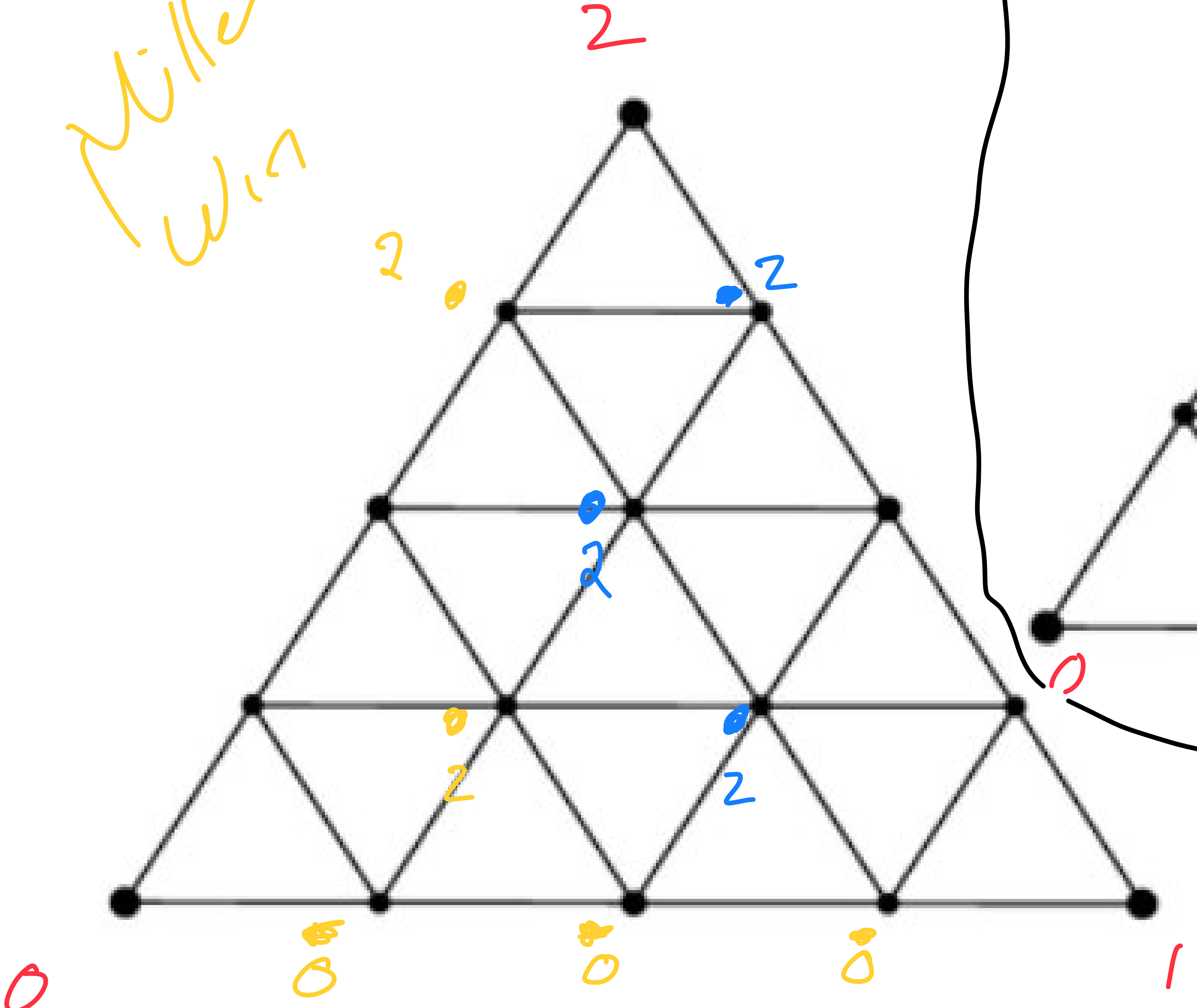


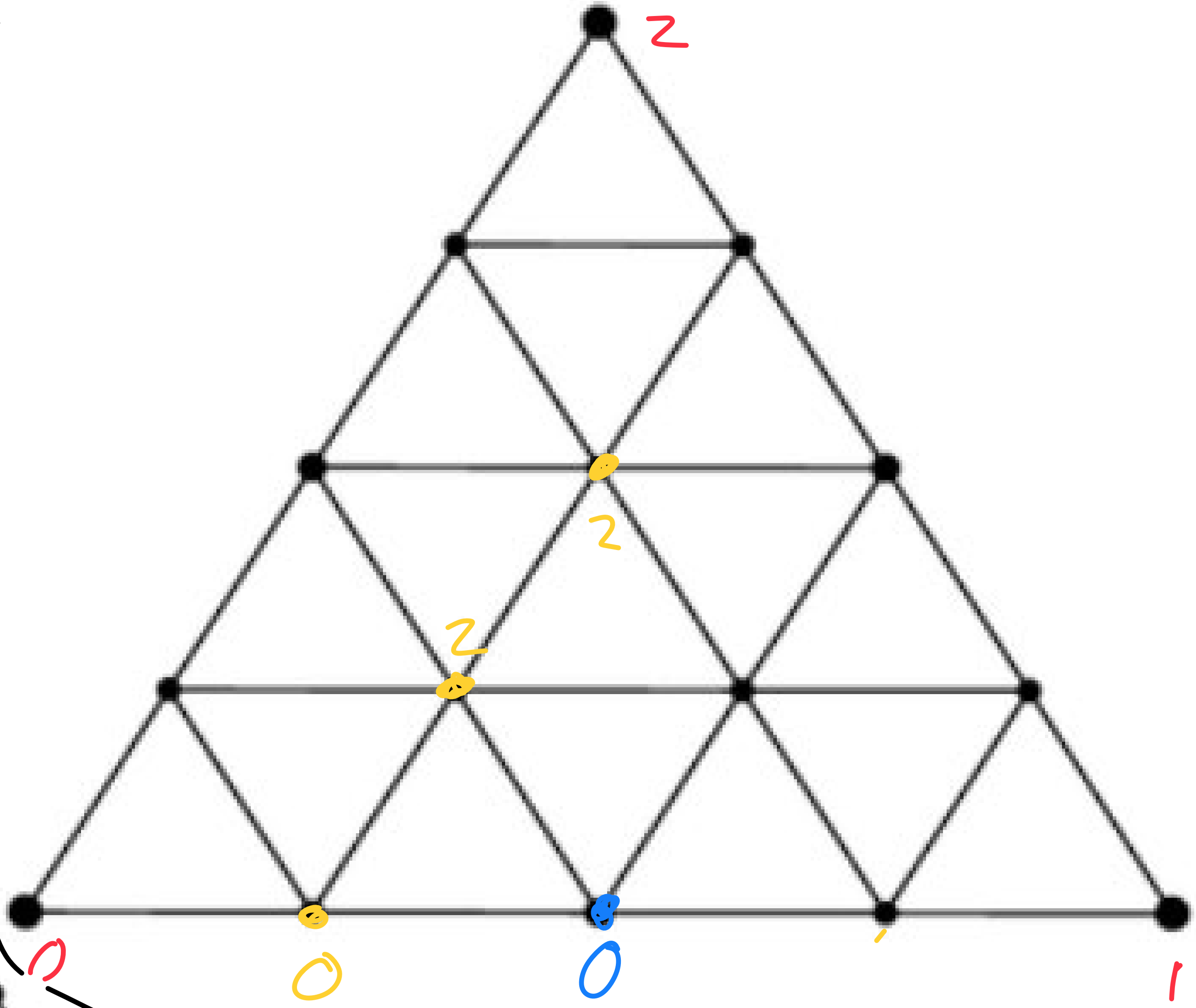
Payout Options:

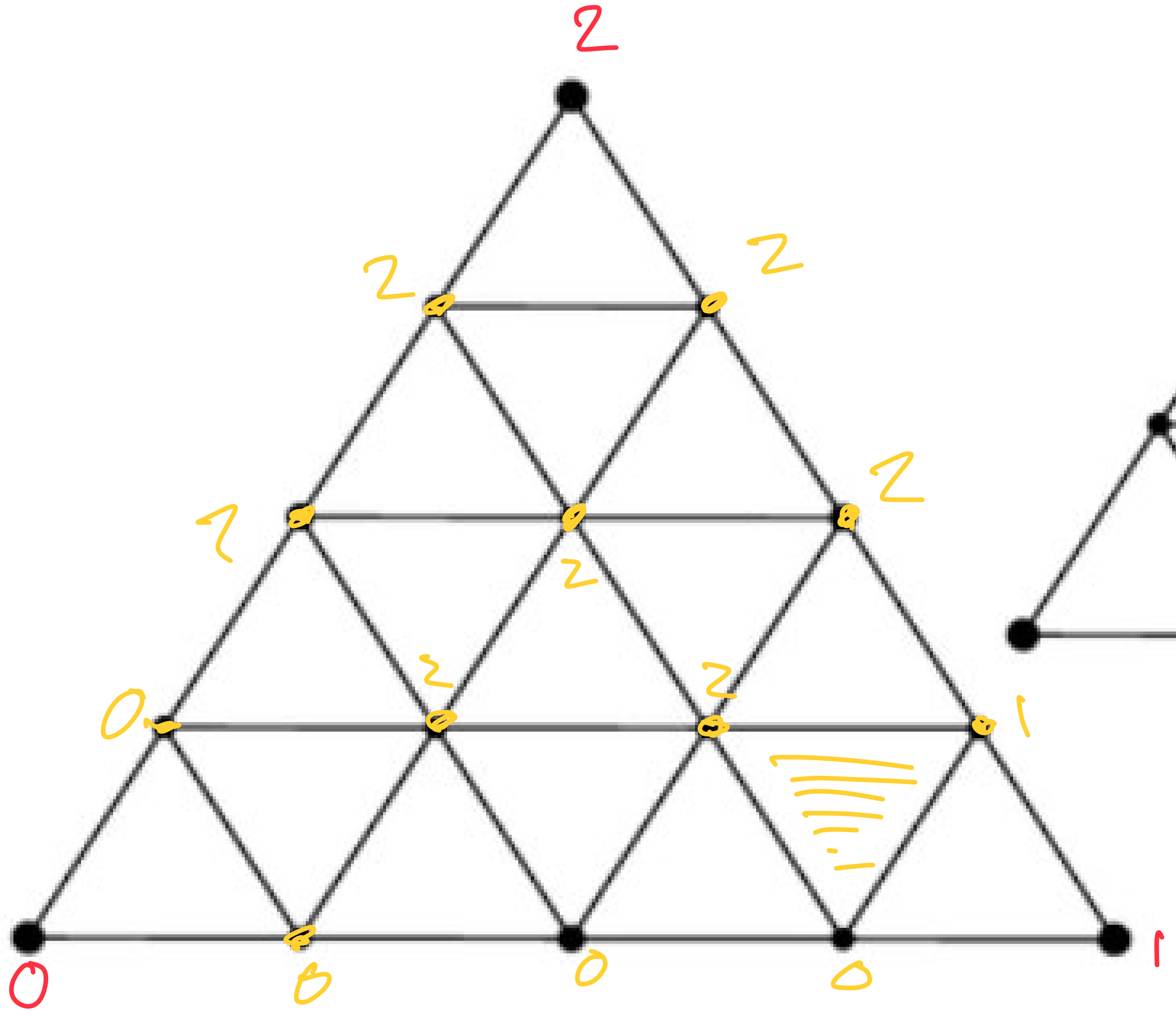
- I get 1 move for every 1 of yours, if you win you get $64.
- I get 1 move for every 2 of yours, if you win you get $32.
- I get 1 move for every 3 of yours, if you win you get $16.
- I get 1 move for every 4 of yours, if you win you get $8.
- I get 1 move for every 5 of yours, if you win you get $4.
- I get 1 move for every 6 of yours, if you win you get $2.

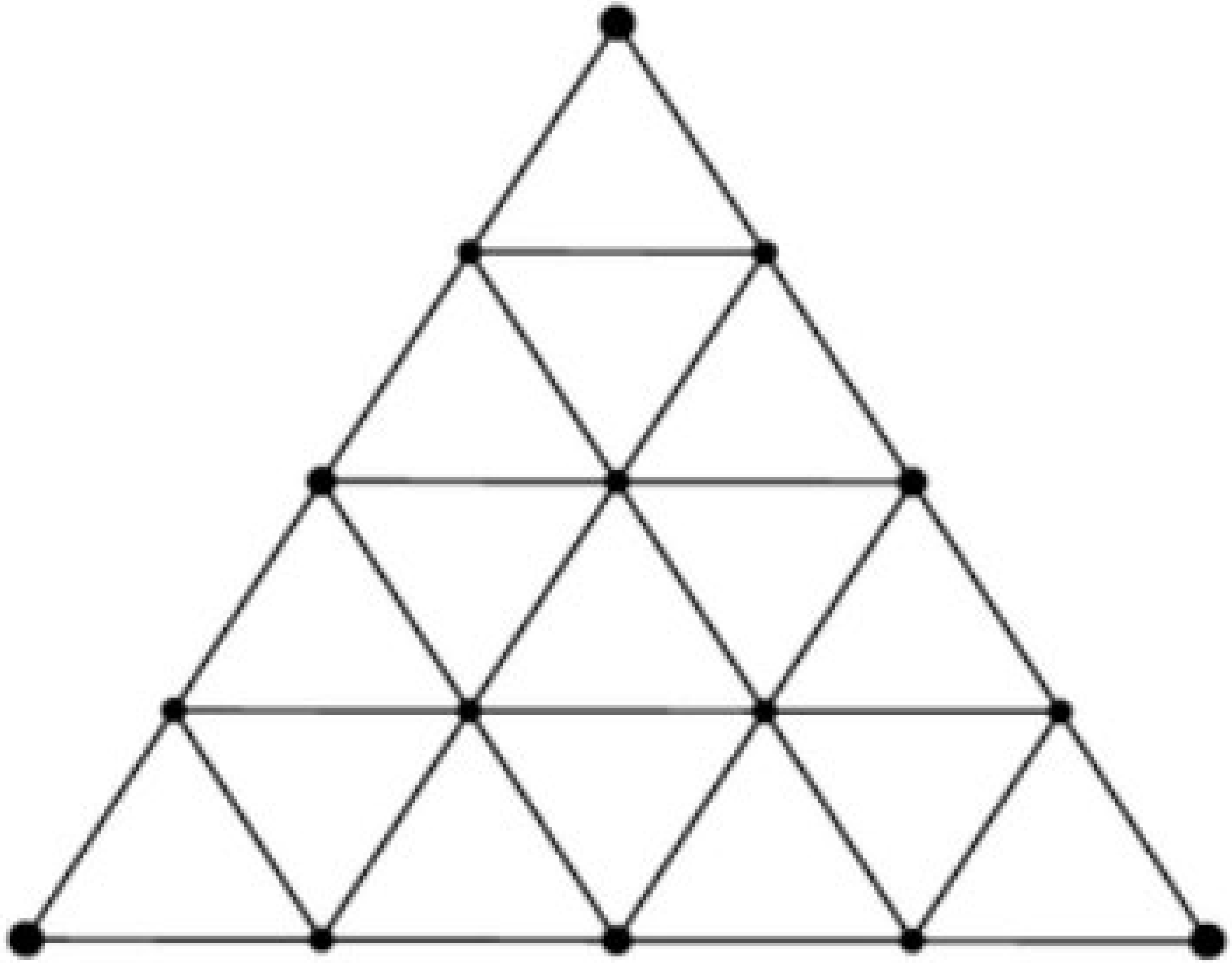
- (you can bank moves if you want…)

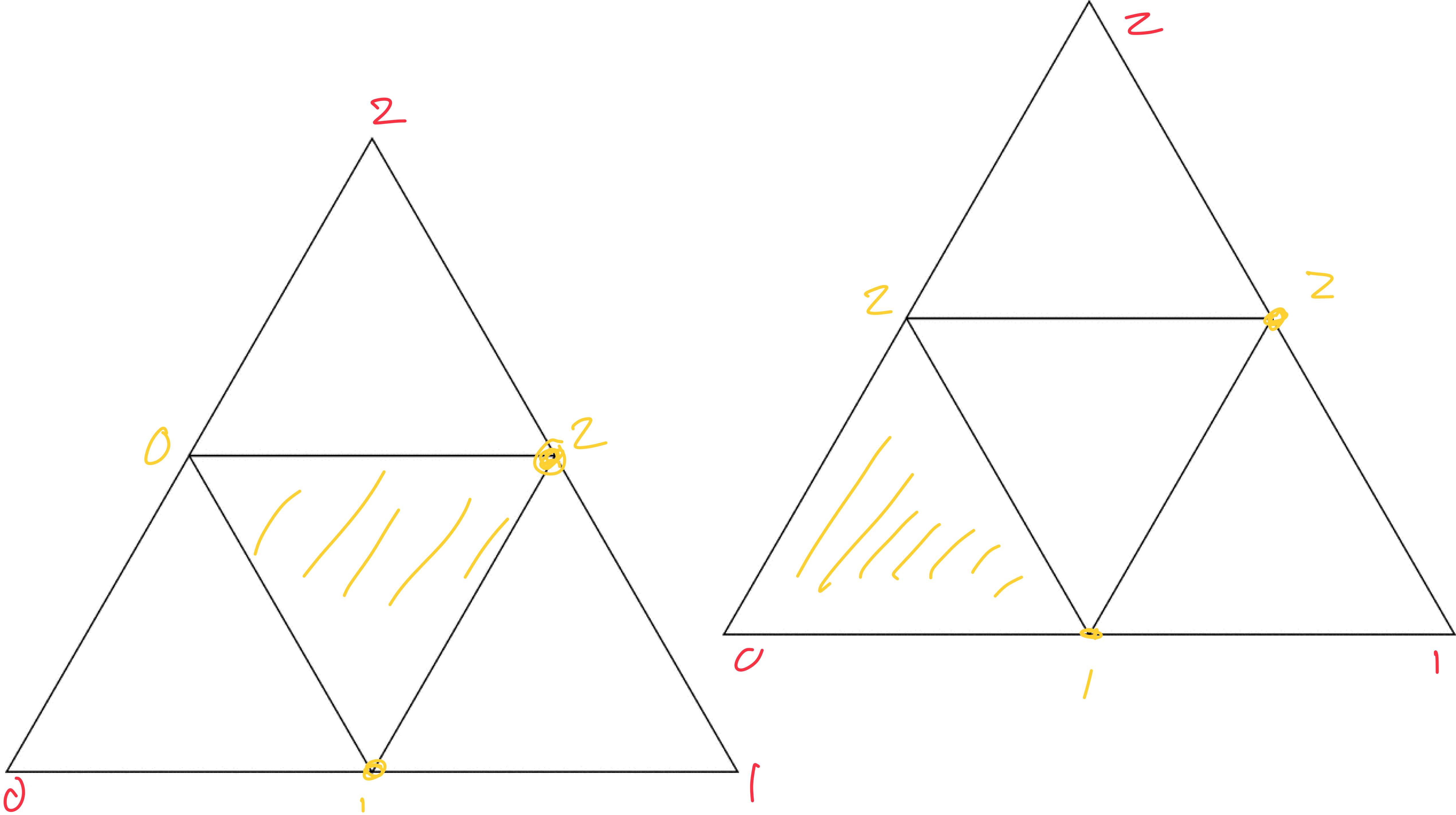
Miller
Win





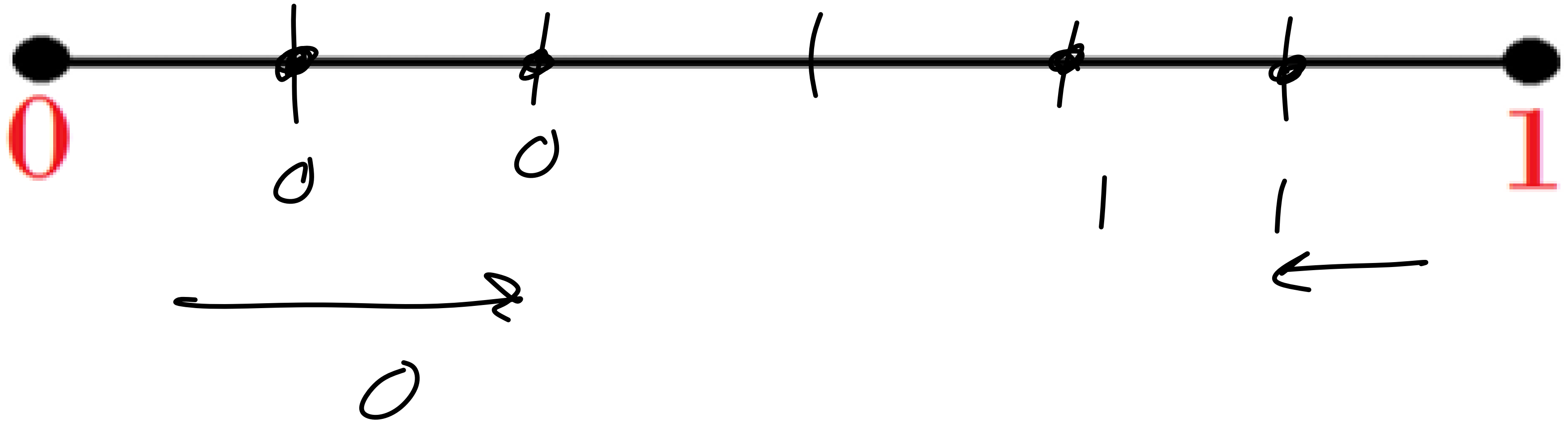

2
2 2
2 2 2
2 2
0 0 1
0 0 0 0 1



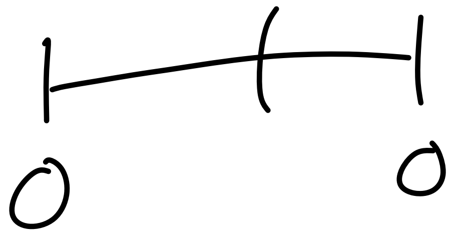


2
0
2
0
1
1
2
2
2
2
0
1
1

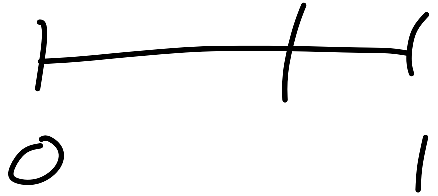
Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.



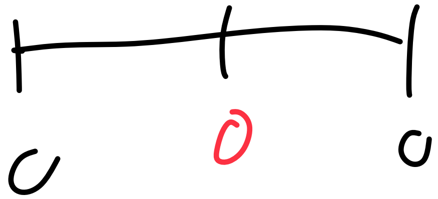
Always a 0-1 segment



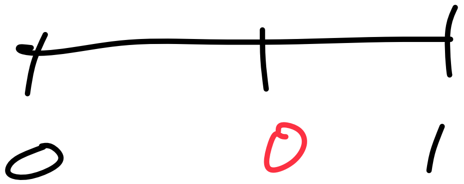
0 segments



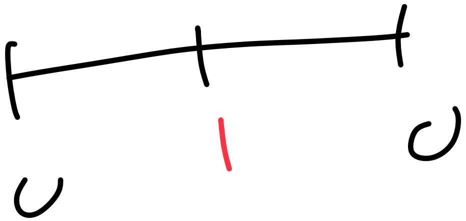
1 segment



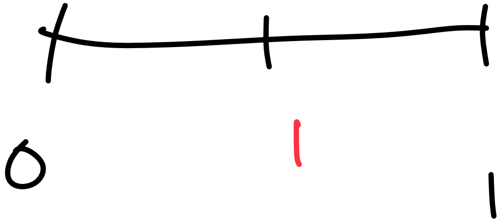
0 segments



1 segment



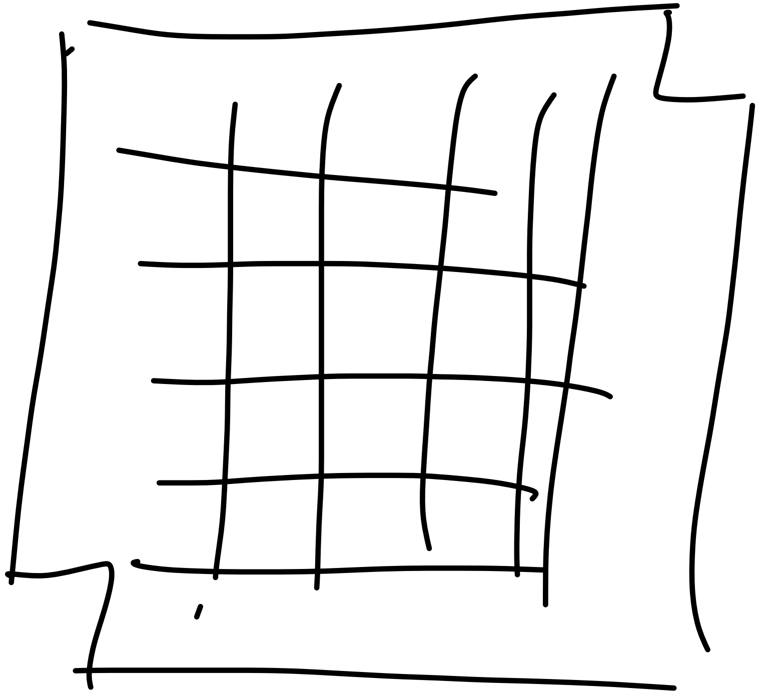
2 segments



1 segment

PARITY UNCHANGED!

Chess board



Remove two opposite corners: Can you cover the 8x8 board minus those with 1x2 tiles?

No: Start with 30 and 32 for the two colors, each tile removes one of each color

Pirate Riddle

There are five pirates, named 1, 2, 3, 4 and 5.
They must split 100 gold coins.
The lowest named pirate alive proposes a division.

- If 50% or more vote for the plan it passes and the coins are so divided.
- If 50% or more vote against the plan it fails, the proposer walks the plank and dies, and the lowest named surviving pirate proposes a plan.

We repeat till a plan is accepted.
Assume the pirates are intelligent – what is the final division?

Start with 1 pirate : Gets all 100

Have pirates 1 and 2 : Pirate 1 gets 100

Have pirates 1, 2, 3 : if Pirate 1's proposal fails then dies and reduces to two pirates.

Pirate 1 proposes (99, 0, 1)

(0, 0, 100)

(50, 0, 50)

(n, 100 - n, 0)

Credible Threat

next time?

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Steven J Miller
Williams College
sjm1@williams.edu

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https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 15: October 9, 2024: https://youtu.be/F2_h4cwwuKg
Pigeonhole Principle II

Example A.4.2. *If we choose a subset S from the set $\{1, 2, \ldots, 2n\}$ with $|S| = n + 1$, then S contains at least two elements a, b with $a|b$.*

Exercise A.4.3. *If we choose 55* distinct *numbers from $\{1, 2, 3, \ldots, 100\}$ then among the chosen numbers there are two whose difference is ten (from [Ma]).*

Exercise A.4.4. *Let $a_1, \ldots, a_{n+1}$ be distinct integers in $\{1, \ldots, 2n\}$. Prove two of them add to a number divisible by $2n$.* Note: these two numbers need not be distinct. If instead we required the two numbers to be distinct, then $a_1, \ldots, a_{n+1}$ would have to be $n + 1$ distinct numbers from $\{1, ..., 2n - 1\}$.

Exercise A.4.5. *Let $a_1, \ldots, a_n$ be integers. Prove that there is a subset whose sum is divisible by n.*

What are the boxes?

What are the pigeons?

Choose $n+1$ elements from $1, 2, \ldots, 2n$, have at least one $a | b$.

Relation b/w $n+1$ and $2n$

If choose n elements, maybe claim fails for special choice

↳ maybe key is having more than half the elements

if have $2n$ objects paired, have n pairs

if choose $n+1$ objects, at least one pair has two elements chosen.

Boxes: special set of pairs of #s from 1 to $2n$

Pigeons: $n+1$ numbers chosen

Gather Data

$n=1$ $\{1, 2\}$ either $\{1\}$ or $\{2\}$ works for no $a|b$

$n=2$ $\{1,2,3,4\}$ $\cancel{\{1,2\}}$ $\cancel{\{1,3\}}$ $\cancel{\{1,4\}}$

$\{2,3\}$ $\cancel{\{2,4\}}$

$\{3,4\}$

$n=3$ $\{1,2,3,4,5,6\}$ eliminate 1's

$\binom{6}{3} = \frac{6!}{3!\,3!} = 20$

$\cancel{\{2,3,4\}}$ $\{2,3,5\}$ $\{2\cancel{3}6\}$, $\cancel{\{2\,4\,5\}}$ $\{2\cancel{4}6\}$ $\{2\cancel{5}6\}$

middle $\{345\}$ $\{3\cancel{4}6\}$ $\{3\cancel{5}6\}$

$\{456\}$ last 1

Conj 1: "Middle" works

$\{1, 2, \ldots, 2m, 2m+1, \ldots, 2n\}$ $n = 2m$

$2n = 4m$

$m+1, \ldots, 2m, 2m+1, \ldots 3m$

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$

$n = 8$ $2n = 16$

$n = 2m$ so $m = 4$

Note $6 \mid 12$

consider $m+2$ and $2m+4$

or $m+3$ and $2m+6$

less than $3m$ if m large

FALSE!

Conj 2: Last n work

$\{n+1, n+2, \ldots, 2n\}$

If $a \mid b$ then $\exists m$ st $am = b$

Smallest m is 2

so $2a \leq b$ as $am = b$ and $m \geq 2$

any a in list has $2a$ at least $2n+2$, so no $a \mid b$. $\blacksquare$

TRUE

Boxes try $(a, 2a)$

$(1, 2)$, $(2, 4)$ using 2 twice

FAIL

$(1, n+1)$ $(2, n+2)$ $(3, n+3)$, ... $(n, 2n)$

Ex: $n = 8$ $(1, 9)$, $(2, 10)$, $(3, 11)$, . . .

FAIL

Smaller not always dividing larger.

Stupid

Impossible:

If $p > n$ prime then no pair (p, b) or (a, p).

we have a problem

(can do once with $a = 1$ then trouble)

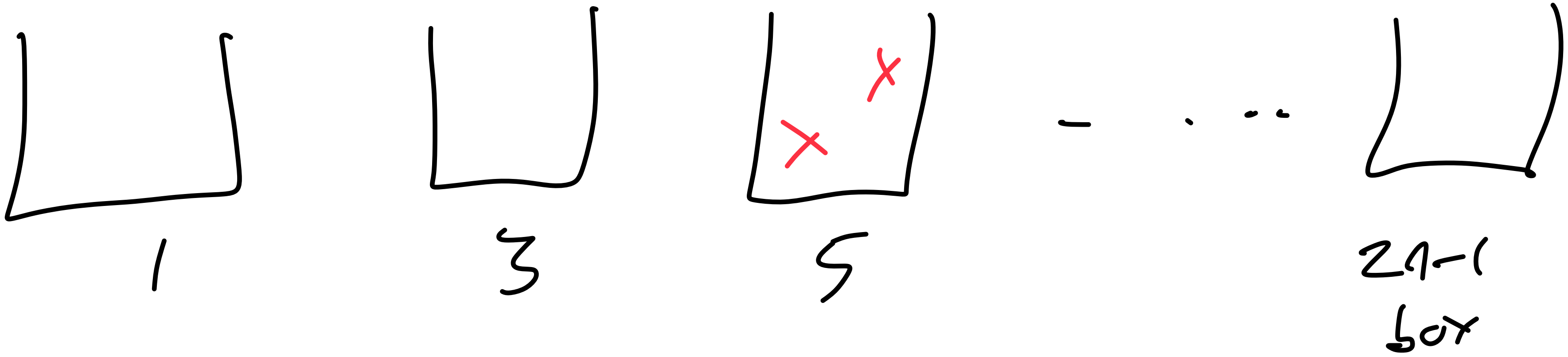
Observation: $x = 2^k(2m+1)$

We have n odd numbers

If $x_1 = 2^{k_1}(2m+1)$

$x_2 = 2^{k_2}(2m+2)$ wlog $k_1 < k_2$

$\Longrightarrow x_1$ divides x_2



n boxes
choose n+1
numbers:
one box
chosen twice

Research Qs:

- If take n items from $2n$, what is the percent of times have $a \mid b$ for some a, b in set?

↳ does this go to 100% as $n \to \infty$?

There are $\binom{2n}{n} = \frac{(2n)!}{n!^2}$

Stirling's Formula: $m! \approx m^m e^{-m} \sqrt{2\pi m}$

$$\frac{1}{2n+1} 2^{2n} \le \binom{2n}{n} \le (1+1)^{2n} = 2^{2n}$$

↳ if yes, how quickly?

Structure of "BAD" sets

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Steven J Miller
Williams College
sjm1@williams.edu

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Lecture 16: October 11, 2024: https://youtu.be/xx8f8z8zzfQ

No class, watch: 3/1/2017: Irrationality Proofs, Morley's Theorem, Pigeon Hole Problems

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Steven J Miller
Williams College
sjm1@williams.edu

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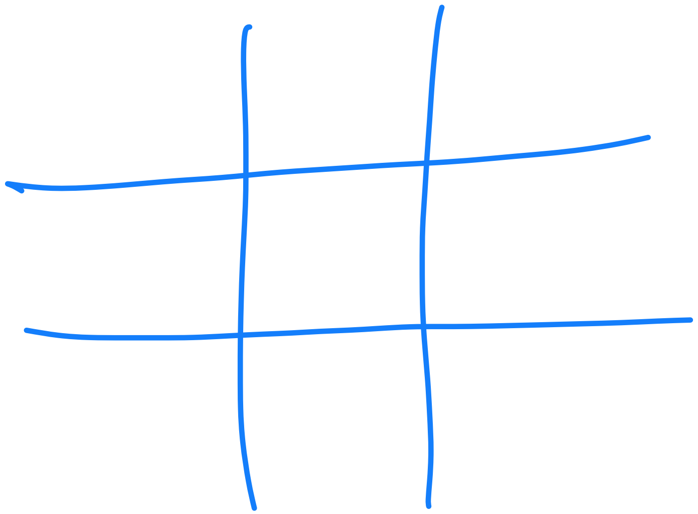
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 17: October 16, 2024: https://youtu.be/weGmZ3YmRqE
Game Theory II: Tic-Tac-Toe, Candy Bar Games
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/CoronaGamesTicZombieFibDots_ME2019Lec3.pdf

Tic-Tac-Toe



$$\left.\begin{array}{l}9 \text{ openings}\\ 8 \text{ responses}\end{array}\right\} 72$$

Symmetry: 3 openings

1	2	1
2	X	2
1	2	1

2 +

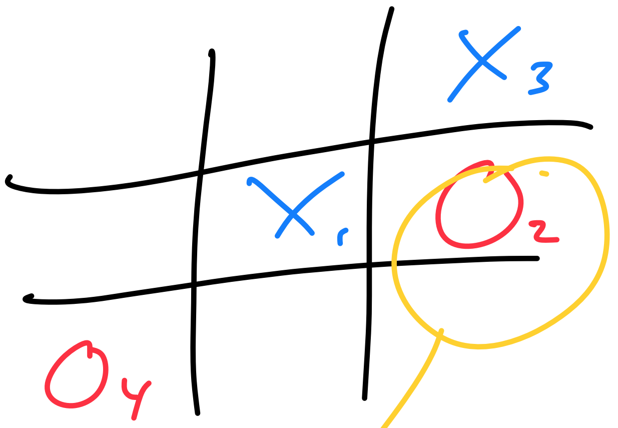
X	1	3
1	2	4
3	4	5

5 +

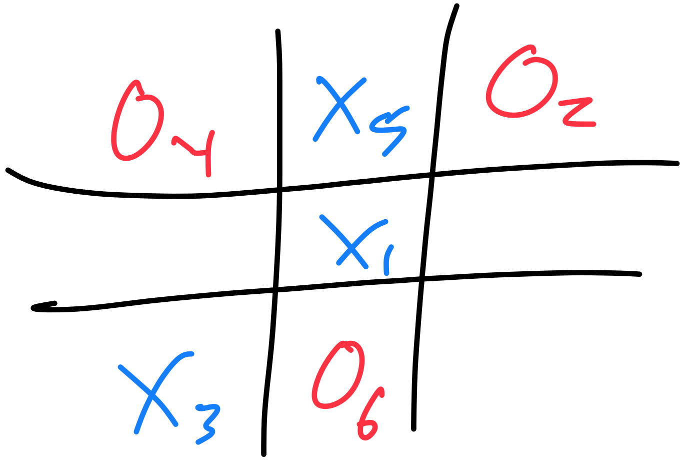
1	X	1
3	2	3
4	5	4

5 = 12

Analyze T-T-T



mistake



Random

Tic

Tac

Toe

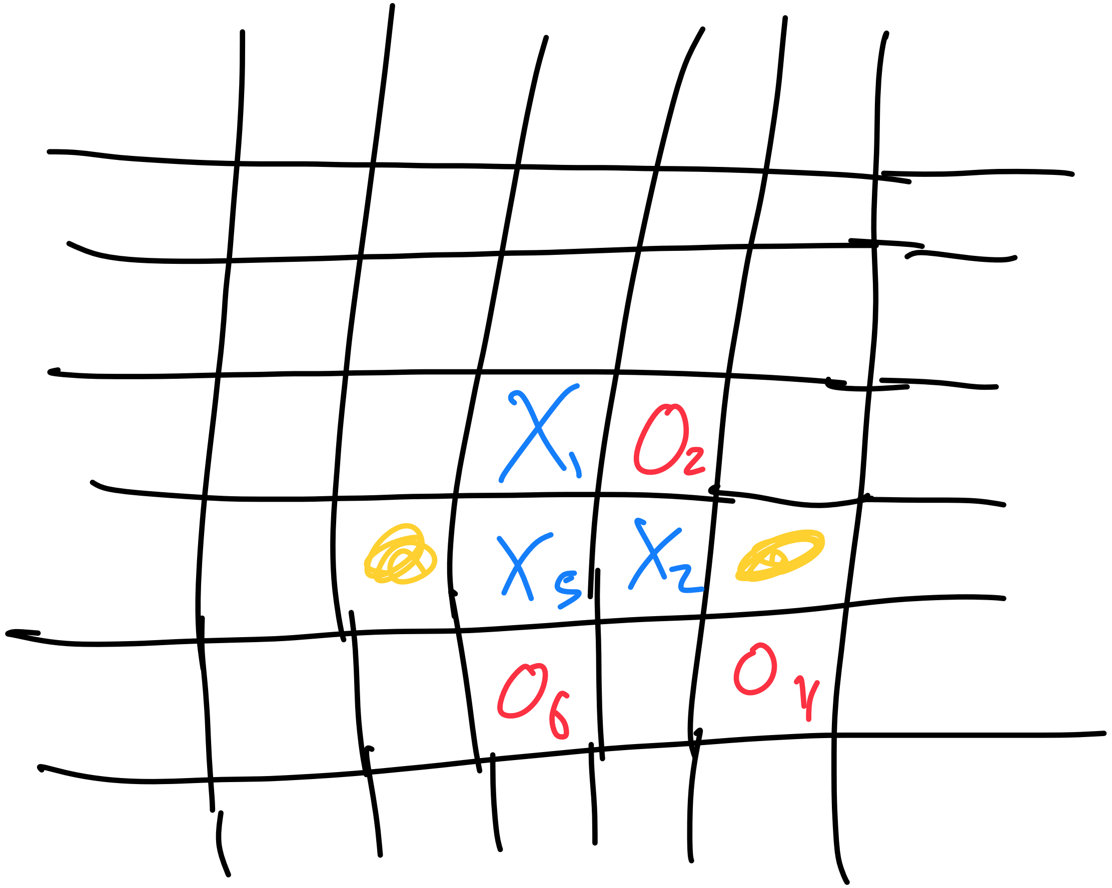
0) Choose randomly among legal moves

1) Secure a win, else 0)

2) secure win, else block, else 0)

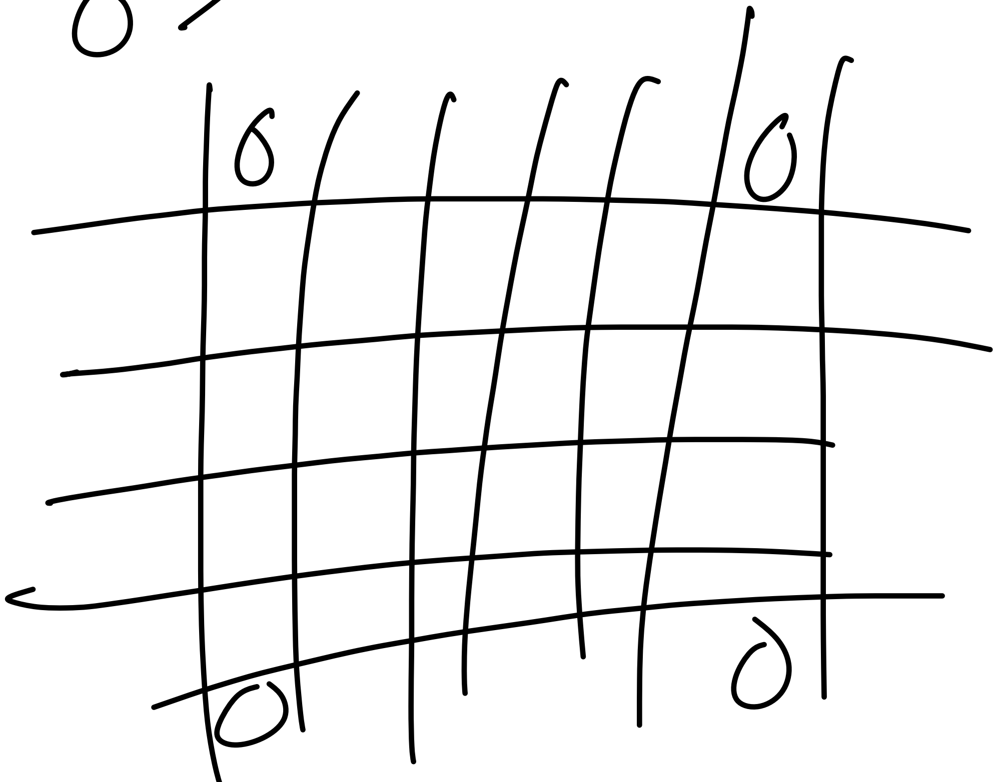
3) secure win, block, threaten 3 in a row 2 ways, 0)

Change Size

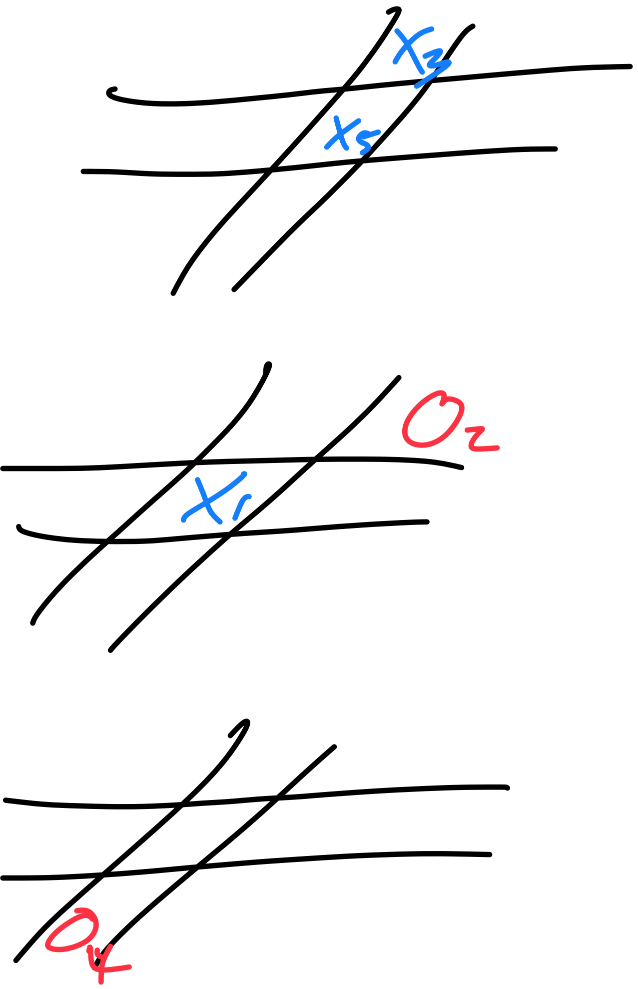


3 in a row: 0 dead
4 in a row

Pre-place O's on the board



3-D T-T-T



Russian Doll T-T-T

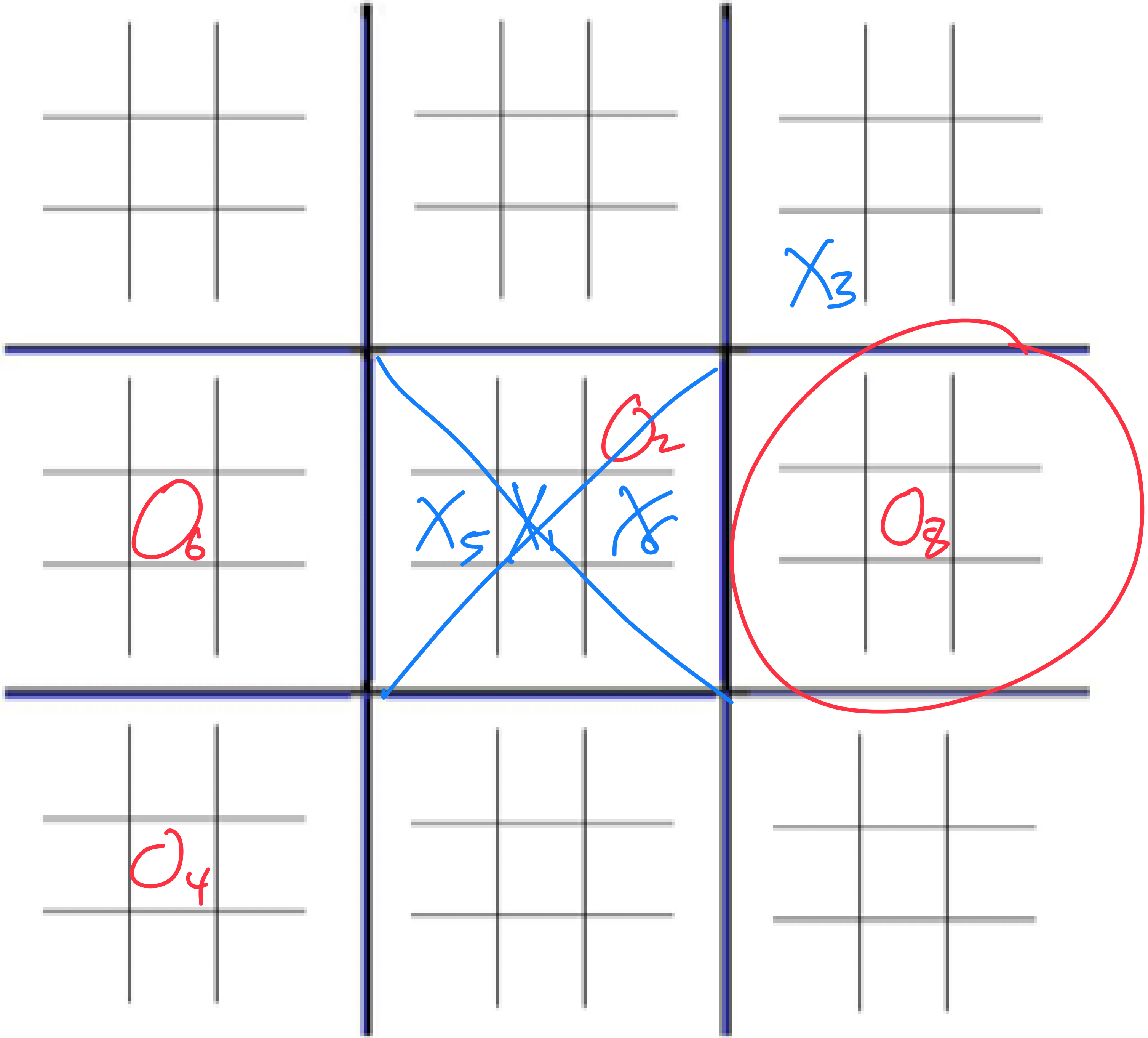
(Gobble)

small — 2x

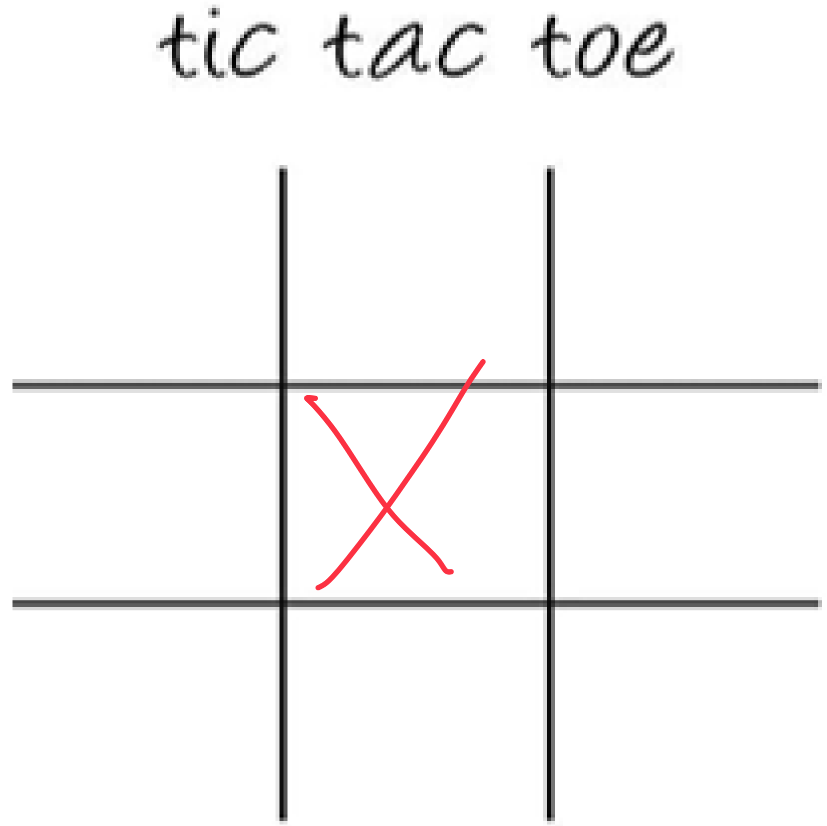
medium — 2x

large — 2x

Ultimate Tic-Tac-Toe



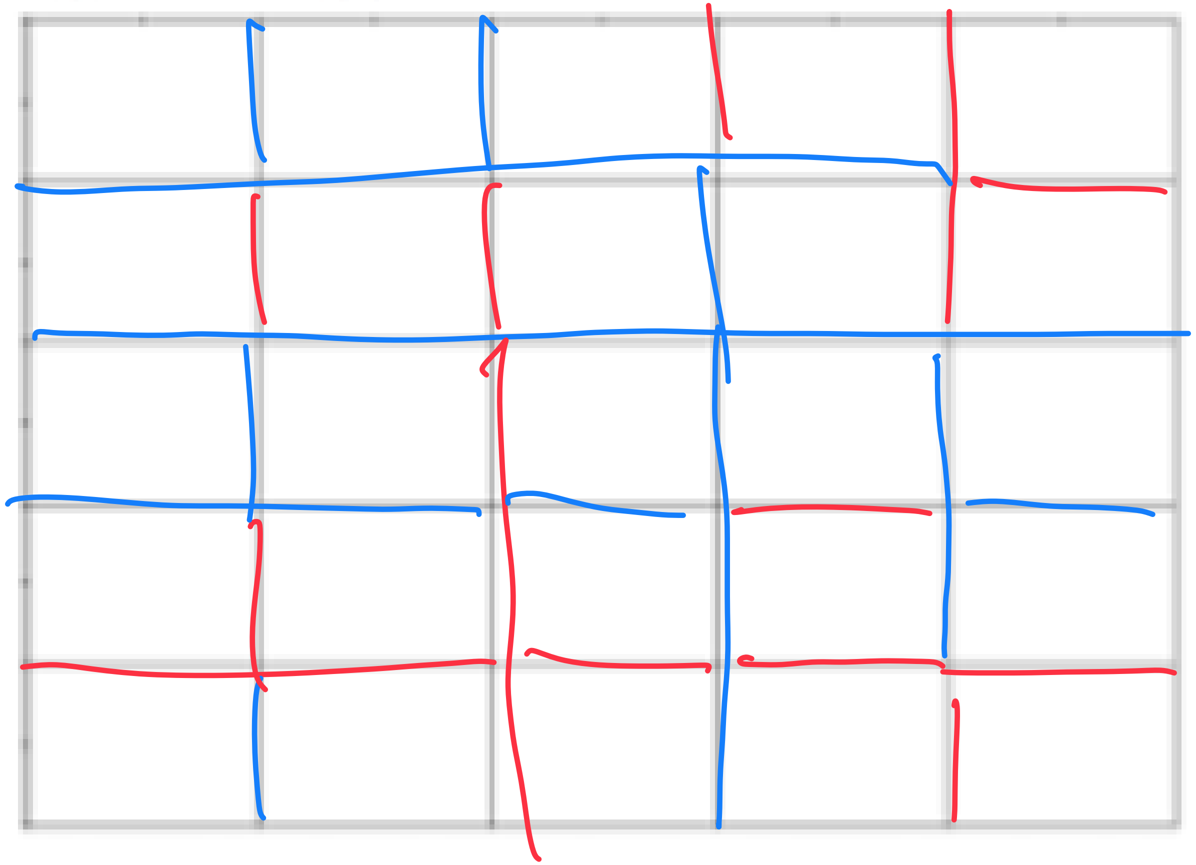
Bidding Tic-Tac-Toe



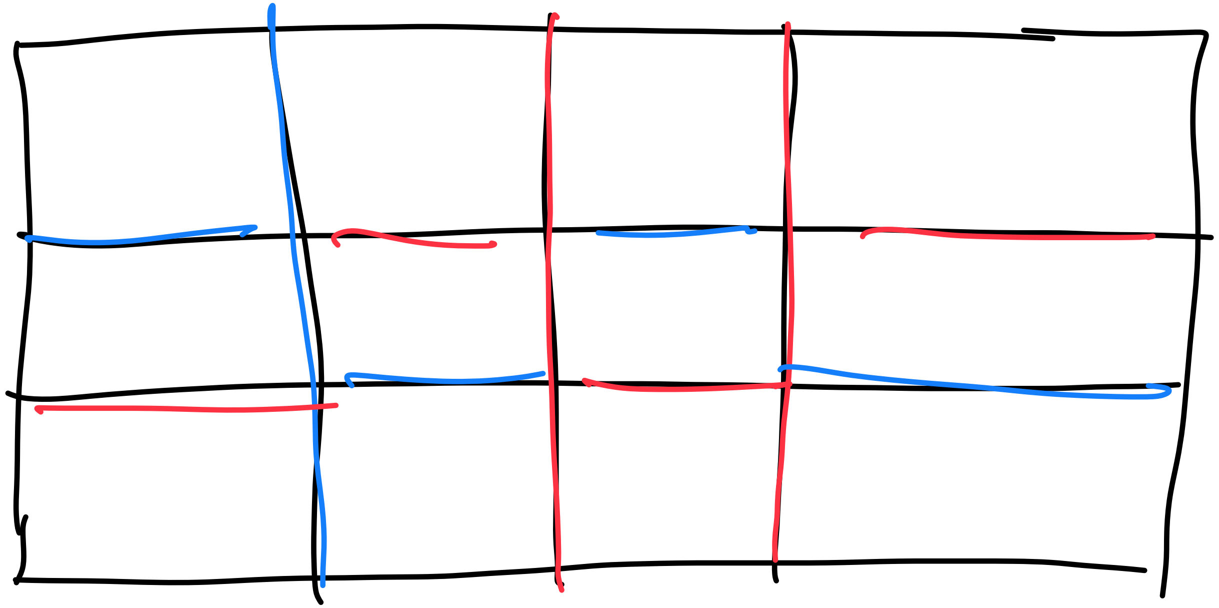
Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	800	400
2	200	1800		
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Player 2 wins



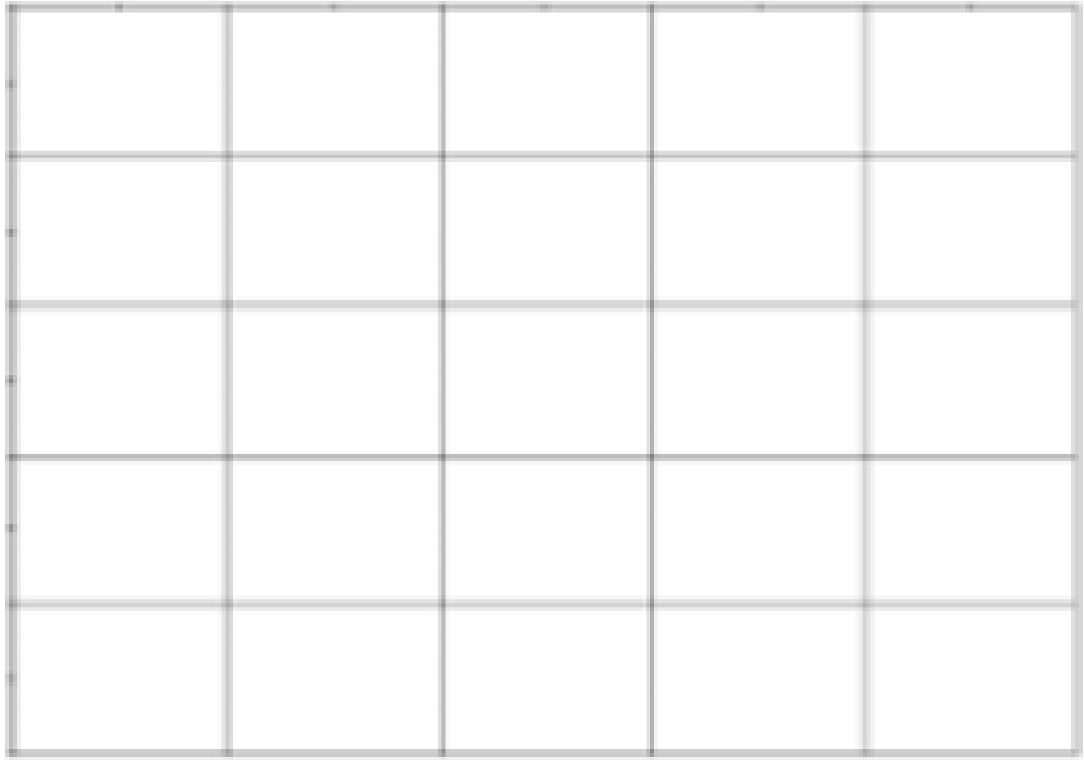
Player One

$0

Player Two

$0

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Length	Width	Winner
2	2	--1--
2	3	--1--
3	3	--2--
2	4	--1--
3	4	--1--
4	4	--1--
3	5	--2--

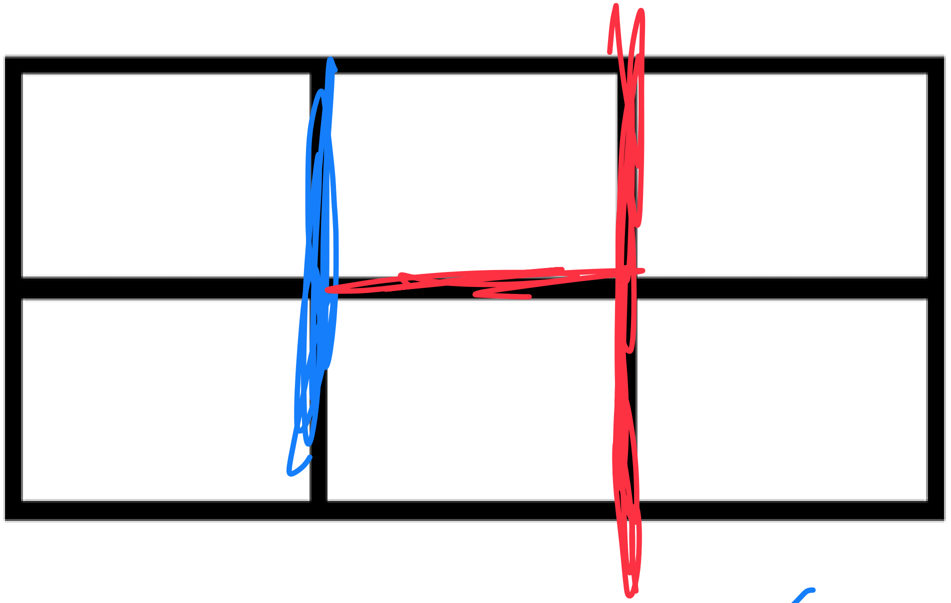
Figure: Do you see a pattern?

PARITY

A mono-variant is a quantity that moves on only one direction (either non-decreasing or non-increasing).

Idea: Associate a mono-variant to the rectangle game....

Every time move, increase number of pieces by 1!

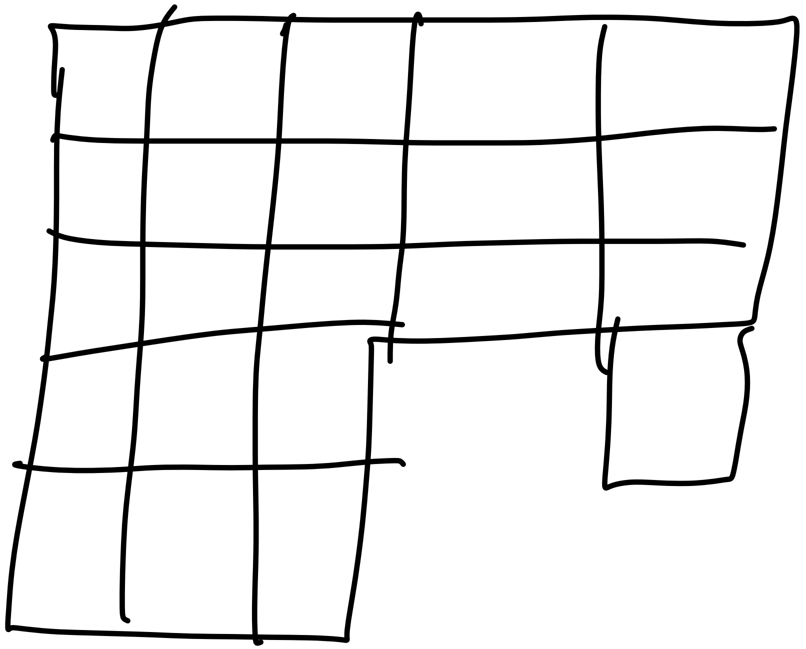


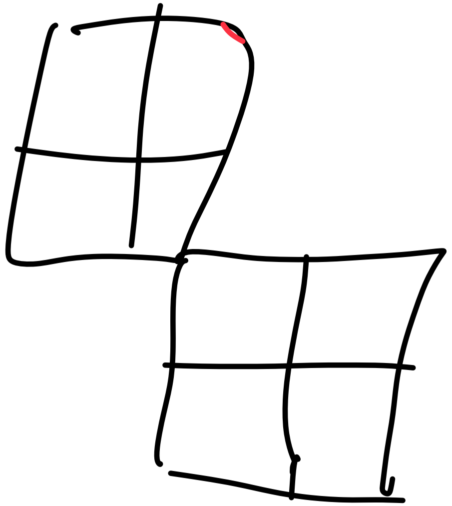
Move #	Number of pieces
0	1
1	2
2	3
3	4

4 → 5

5 → 6

2×W have 2*W−1 moves till done





even # of squares

odd # of moves

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Williams College
sjm1@williams.edu

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Lecture 18: October 18, 2024: No class, work on Projects
Lecture 19: October 21, 2024: https://youtu.be/-ZujtK5k-fM
How Long / How Many?

	Retention	Percent of Incoming Graduating		
Institution	1st year	4 years	6 years	eventually
Faber College	96	91	93	94
Grand Lakes University	82	75	78	78
Monsters University	75	60	69	? 76
Springfield University	60	50	55	58
Starfleet Academy	100	100	100	100
Wossamotta University	65	53	56	57

Cultural extra credit (can ask friends but cannot use the internet): What movies / tv shows are these from? Must get at least four to qualify for any bragging rights….

Assume p percent of students graduate after 4 years, and then every subsequent year p percent of the remaining graduate. Does everyone eventually graduate?

(no one leaves before four years)

If $p=0$: NO!

If $p>0$: % students graduate: $p + (1-p)\,p + (\#\text{ left}) * p$

4yrs   5 years $1-p-(1-p)\,p$

$= (1-p)^2$

$$\sum_{k=0}^{\infty} (1-p)^n p = p \frac{1}{1-(1-p)} = \frac{p}{p} = 1$$

$r = 1-p$

after n years, % left is $(1-p)^n \to 0$

Assume p percent of students graduate and q percent leave after 4 years, and then every subsequent year p percent of the remaining graduate and q percent leave. What percent eventually graduates?

$d_{p,q}$ = % eventually get a diploma

$w_{p,q}$ = % eventually withdraw and never return.

$d_{p,q}$: as $p \uparrow$ should go up or stay same

as $q \uparrow$ should go down or stay same

and $p \leq d_{p,q} \leq 1-q$

$$d_{p,q} = \underbrace{p}_{4\text{yrs}} + \underbrace{(1-p-q)p}_{5\text{yrs}} + \underbrace{(1-p-q)^2 p}_{6\text{yrs}} + \underbrace{(1-p-q)^3 p}_{7\text{yrs}} + \cdots$$

$$= p\,\frac{1}{1-(1-p-q)} = \frac{p}{p+q}$$

$w_{p,q} = 1 - d_{p,q}$ or $= \dfrac{q}{q+p}$ by symmetry

Memoryless Process

$$d_{p,q} = p + (1-p-q)\,[\,\%\text{ graduate eventually}\,]$$

$$d_{p,q} = p + (1-p-q)\, d_{p,q}$$

$$\big(1-(1-p-q)\big)\, d_{p,q} = p \implies d_{p,q} = \frac{p}{p+q}$$

Bypasses infinite sum!

Assume p percent of students graduate and q percent leave after 4 years, and then every subsequent year $p^{f(n)}$ percent of the remaining graduate and $q^{g(n)}$ percent leave. What percent eventually graduates?

We did $f(n) = g(n) = 1$

Try $f(n) = g(n) = 2$ or any constant

Try $f(n) = g(n) = n$ or maybe $n+1$

Find something fun

On each turn, you have a probability of p of receiving a prize. If all turns are independent, on average how many turns till you get it?

% 1 turn   % 2 turns ... % n turns

p $(1-p)$ $(1-p)^{n-1}p$

$$\text{Expected Weight} = 1 \cdot p + 2 \cdot (1-p)p + \cdots + n(1-p)^{n-1}p + \cdots$$

$$= \frac{p}{1-p}\left[0 \cdot (1-p)^0 + 1 \cdot (1-p) + 2(1-p)^2 + \cdots + n(1-p)^n + \cdots\right]$$

using next page with $x = 1-p$ the sum is $\frac{x}{(1-x)^2}$

$$= \frac{p}{1-p} \cdot \frac{1-p}{p^2} = \frac{1}{p}$$

Differentiating Identities

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if } |x|<1$$

$$xf'(x) = x\frac{d}{dx}\sum_{n=0}^{\infty} x^n = x\sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} nx^n = x\frac{d}{dx}\frac{1}{1-x}$$

$$\begin{aligned}((f+g)+h)' &= (f+g)' + h' \\ &= (f'+g') + h' \\ &= f'+g'+h'\end{aligned}$$

Thus $\sum_{n=0}^{\infty} nx^n = \dfrac{x}{(1-x)^2}$

Let w_p = wait be how long wait on average

$$w_p = 1 \cdot p + (1-p)[\text{how long wait on average} + 1]$$

$$w_p = p + (1-p)(w_p + 1)$$

$$w_p = p + (1-p)w_p + 1 - p$$

$$w_p = 1 + (1-p)w_p$$

$$(1 - (1-p))w_p = 1$$

$$p\, w_p = 1 \implies w_p = 1/p$$

There are c different prizes, and each box is equally likely to have one and only one of them. How many boxes do you expect to need before you have at least one of each prize (boxes are independent)?

$$W_c = \frac{c}{c} + \frac{c}{c-1} + \frac{c}{c-2}$$

$$P_1 = \frac{c}{c} \qquad P_2 = \frac{c-1}{c} \qquad P_3 = \frac{c-2}{c}$$

$$W_c = c\left(\frac{1}{c} + \frac{1}{c-1} + \dots + \frac{1}{1}\right)$$

or $c\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{c}\right)$

$= c H_c$ the c^{th} harmonic number

and $H_n \simeq n \log(n)$

https://www.alamy.com/aggregator-api/download?url=https://c8.alamy.com/comp/2D53X7C/the-original-cracker-jack-box-photographed-on-a-white-background-2D53X7C.jpg

$c = 3$ wait should on average be

$$3\left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

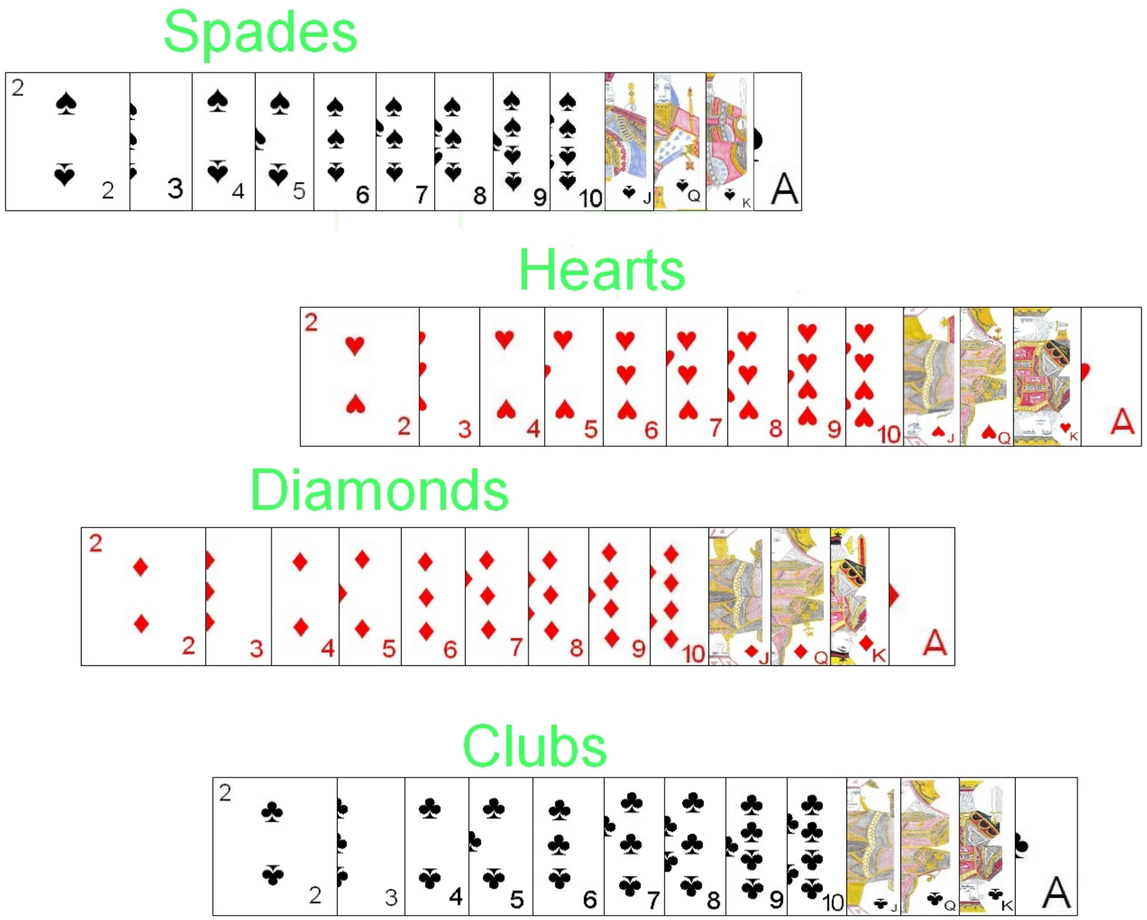
$$= 3 \cdot \frac{11}{6}$$

$$= \frac{11}{2}$$

$$= 5.5$$ (we had 8)

I deal you one card from a deck, you look at it and return, I shuffle and then deal you another. I continue till you've seen all the cards. How long do you expect to wait?

$S2 \; H_{52}$



https://www.catsatcards.com/CImages/Final/CompleteDeck.jpg

In bridge the 52 cards are dealt 13 to a player. How many deals are needed before a player expects to see each card at least once?

(Question from Kayla Miller, a few hours after a hand where she had 6 trump in opposition to Cameron Miller's 6!)

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Steven J Miller
Williams College
sjm1@williams.edu

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Lecture 20: October 23, 2024:
How Long / How Many: II?

In bridge the 52 cards are dealt 13 to a player. How many deals are needed before a player expects to see each card at least once?

(Question from Kayla Miller, a few hours after a hand where she had 6 trump in opposition to Cameron Miller's 6!)

4 Suits, c cards per suit

As, Ah, Ad, Ac

Classic prize problem if $c = 1$

wait $= 4\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = 4 H_4$

$C=2$: 8 cards: A, 2 s, A, 2 h A, 2 d A, 2 c

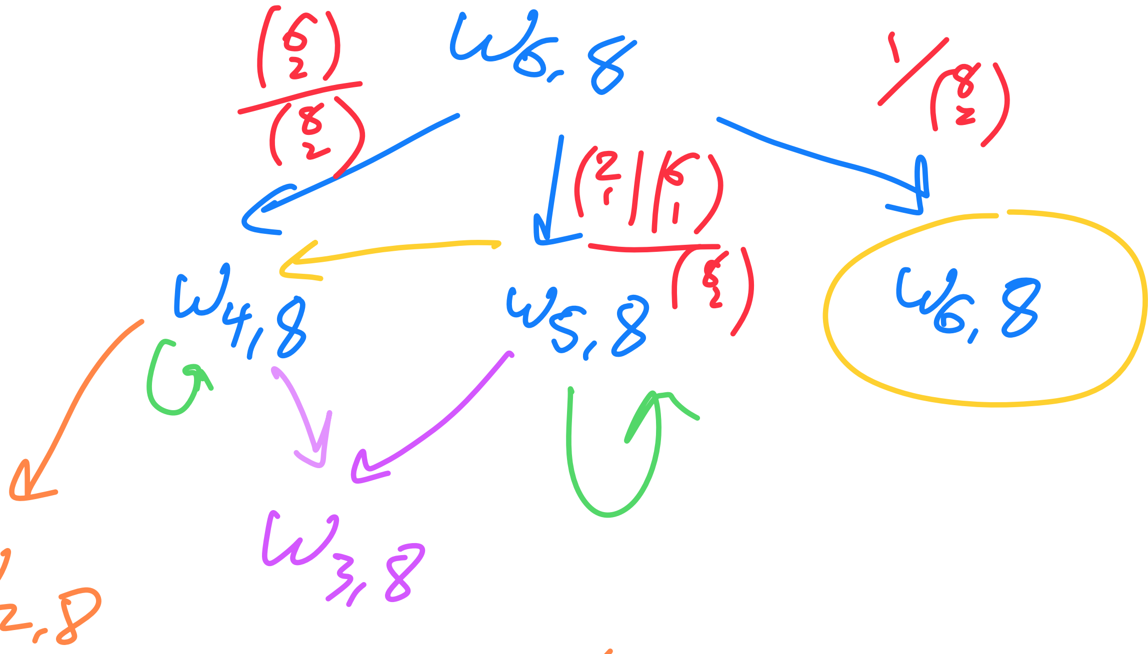
wlog get A, 2 spades on first deal

6 cards left, get two at a time

$W_{n;\#\text{cards}}$ = wait time if need n more cards and there are #cards total

#cards = 8

$$W_{8,8} = 1 + W_{6,8}$$



Start with $W_{1,8}$ and work up!

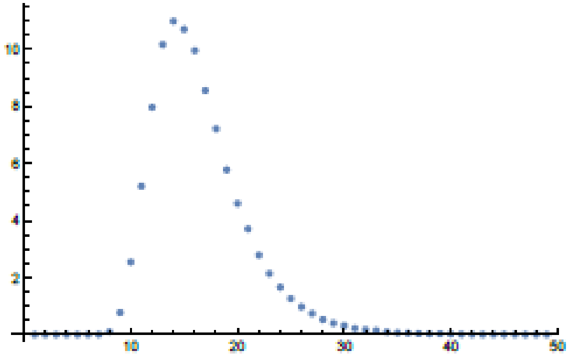
Bridge Hands : Getting all cards

Math 331 : Fall 2024 : sjm1@williams.edu

In bridge the 52 cards are dealt 13 to a player. How many deals are needed before a player expects to see each card at least once?
(Question from Kayla Miller, a few hours after a hand where she had 6 trump in opposition to Cameron Miller's 6!)

```
dealstillseeall[numcards_, numdo_] := Module[{},
  For[d = 1,
   d ≤ 80 * numcards * Log[8 * numcards] , d++, numdeals[d] = 0];
  (* numdeals records how many deals of length
   d were there before a given person saw everything.
    estimating that do not need to go
   further than numcards * Log[4*numcards] * 4 *)
  deck = {};
  For[d = 1, d ≤ 4 * numcards, d++, deck = AppendTo[deck, d]];
  (* creates deck *)
  For[n = 1, n ≤ numdo, n++,
   {
    For[d = 1, d ≤ 4 * numcards, d++, seen[d] = 0];
    (* initialize to have nothing *)
    currnumdeals = 0; (* start with no deals *)
    haveall = 0; (* check - do not have all *)
    While[haveall == 0,
     {
      hand = RandomSample[deck, numcards]; (* creates random hand *)
      currnumdeals = currnumdeals + 1; (* dealt another hand *)
      For[c = 1, c ≤ numcards, c++, seen[hand[[c]]] = 1];
      If[
       Sum[seen[i], {i, 1, 4 * numcards}] == 4 * numcards, haveall = 1];
      If[currnumdeals == 1000 * 4 * numcards,
       {
        haveall = 1; Print["Exiting - waiting too long."]
       }]; (* end of exit if statement *)
     }]; (* end of while loop on haveall = 0 *)
    numdeals[currnumdeals] = numdeals[currnumdeals] + 1;
   }]; (* end of n loop *)
  max = 1;
  For[d = 1, d ≤ 8 * numcards * Log[8 * numcards] , d++,
   If[numdeals[d] > 0, max = d]]; (* finds max observed *)
  For[d = 1, d ≤ max, d++, numdeals[d] = numdeals[d] * 100.0 / numdo];
  (* finds percentage *)
  numdealslist = {}; (* list to store values *)
  For[d = 1, d ≤ max, d++, numdealslist =
    AppendTo[numdealslist, {d, numdeals[d]}]]; (* creates list *)
  Print[ListPlot[numdealslist]];
  Print[numdealslist];
  mean = Sum[ numdeals[d] * d / 100.0, {d, 1, max}];
  stdev = Sqrt[Sum[ numdeals[d] * (d - mean) ^2 / 100.0, {d, 1, max}]];
  Print["Mean = ", mean, " and StDev = ", stdev, "."] ;
 ]
```

In[ ]:= `Timing[dealstillseeall[13, 100 000]]`

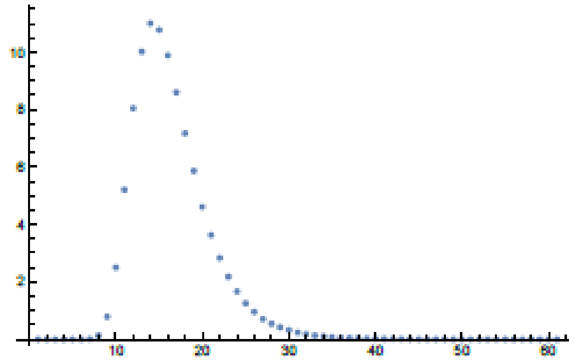



```
{{1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.}, {7, 0.006}, {8, 0.103}, {9, 0.773},
 {10, 2.546}, {11, 5.2}, {12, 7.971}, {13, 10.165}, {14, 10.985}, {15, 10.702},
 {16, 9.959}, {17, 8.554}, {18, 7.214}, {19, 5.781}, {20, 4.599}, {21, 3.701}, {22, 2.787},
 {23, 2.142}, {24, 1.663}, {25, 1.261}, {26, 0.97}, {27, 0.737}, {28, 0.526}, {29, 0.404},
 {30, 0.314}, {31, 0.221}, {32, 0.176}, {33, 0.15}, {34, 0.094}, {35, 0.068}, {36, 0.057},
 {37, 0.039}, {38, 0.03}, {39, 0.035}, {40, 0.02}, {41, 0.013}, {42, 0.006}, {43, 0.005},
 {44, 0.006}, {45, 0.007}, {46, 0.002}, {47, 0.003}, {48, 0.004}, {49, 0.001}}

Mean = 16.4162 and StDev = 4.3567.
```

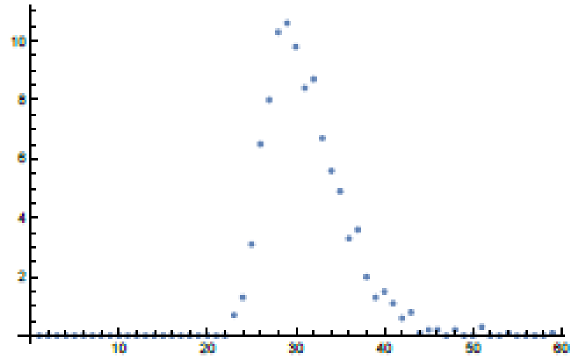
Out[ ]=

`{77.7969, Null}`

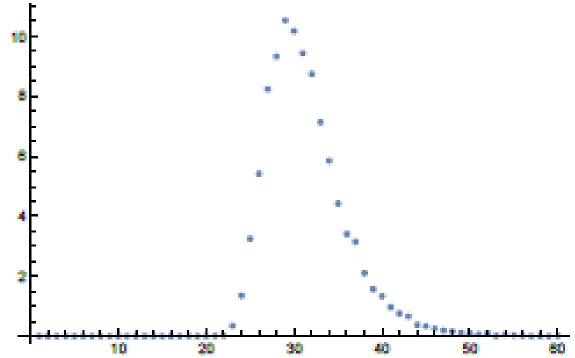
In[ ]:= `Timing[dealstillseeall[13, 1000 000]]`




```
{{1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.0003}, {7, 0.0076}, {8, 0.1243}, {9, 0.7812},
 {10, 2.5056}, {11, 5.2055}, {12, 8.0426}, {13, 10.0207}, {14, 11.0024}, {15, 10.7706},
 {16, 9.8814}, {17, 8.5987}, {18, 7.1714}, {19, 5.8688}, {20, 4.61}, {21, 3.6326}, {22, 2.8283},
 {23, 2.1793}, {24, 1.6625}, {25, 1.2464}, {26, 0.954}, {27, 0.7016}, {28, 0.5362}, {29, 0.4136},
 {30, 0.3196}, {31, 0.238}, {32, 0.1743}, {33, 0.1269}, {34, 0.0997}, {35, 0.0748}, {36, 0.0543},
 {37, 0.0414}, {38, 0.0314}, {39, 0.0235}, {40, 0.0185}, {41, 0.013}, {42, 0.0094},
 {43, 0.0074}, {44, 0.0054}, {45, 0.0051}, {46, 0.0031}, {47, 0.002}, {48, 0.0013},
 {49, 0.0009}, {50, 0.0009}, {51, 0.0011}, {52, 0.0003}, {53, 0.0009}, {54, 0.0003},
 {55, 0.0002}, {56, 0.0002}, {57, 0.}, {58, 0.0004}, {59, 0.}, {60, 0.}, {61, 0.0001}}
Mean = 16.4165 and StDev = 4.35662.

{646.172, Null}

Timing[dealstillseeall[1000, 1000]]
```


```
{{1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.}, {7, 0.}, {8, 0.}, {9, 0.}, {10, 0.},
 {11, 0.}, {12, 0.}, {13, 0.}, {14, 0.}, {15, 0.}, {16, 0.}, {17, 0.}, {18, 0.}, {19, 0.},
 {20, 0.}, {21, 0.}, {22, 0.}, {23, 0.7}, {24, 1.3}, {25, 3.1}, {26, 6.5}, {27, 8.},
 {28, 10.3}, {29, 10.6}, {30, 9.8}, {31, 8.4}, {32, 8.7}, {33, 6.7}, {34, 5.6}, {35, 4.9},
 {36, 3.3}, {37, 3.6}, {38, 2.}, {39, 1.3}, {40, 1.5}, {41, 1.1}, {42, 0.6}, {43, 0.8},
 {44, 0.1}, {45, 0.2}, {46, 0.2}, {47, 0.}, {48, 0.2}, {49, 0.}, {50, 0.}, {51, 0.3},
 {52, 0.}, {53, 0.}, {54, 0.1}, {55, 0.}, {56, 0.}, {57, 0.}, {58, 0.}, {59, 0.1}}
Mean = 31.236 and StDev = 4.51335.

Out[ ]=
{126.234, Null}

In[ ]:= Timing[dealstillseeall[1000, 10000]]
```




```
{{1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.}, {7, 0.}, {8, 0.}, {9, 0.}, {10, 0.}, {11, 0.},
 {12, 0.}, {13, 0.}, {14, 0.}, {15, 0.}, {16, 0.}, {17, 0.}, {18, 0.}, {19, 0.}, {20, 0.},
 {21, 0.01}, {22, 0.02}, {23, 0.35}, {24, 1.35}, {25, 3.25}, {26, 5.42}, {27, 8.24}, {28, 9.33},
 {29, 10.53}, {30, 10.18}, {31, 9.43}, {32, 8.74}, {33, 7.14}, {34, 5.85}, {35, 4.42}, {36, 3.41},
 {37, 3.15}, {38, 2.11}, {39, 1.57}, {40, 1.33}, {41, 0.97}, {42, 0.76}, {43, 0.66}, {44, 0.38},
 {45, 0.34}, {46, 0.25}, {47, 0.19}, {48, 0.16}, {49, 0.11}, {50, 0.09}, {51, 0.06}, {52, 0.09},
 {53, 0.01}, {54, 0.05}, {55, 0.03}, {56, 0.01}, {57, 0.}, {58, 0.}, {59, 0.}, {60, 0.01}}
Mean = 31.3678 and StDev = 4.50579.
```

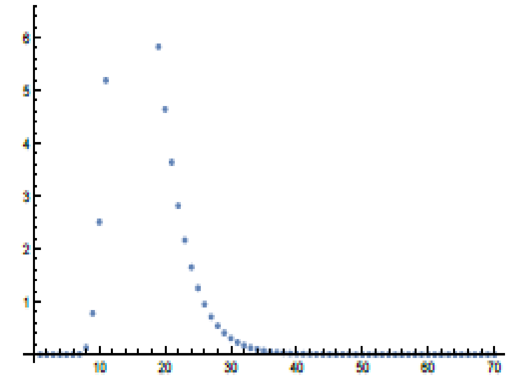
```
recurrencetofindwaittime[suits_, cards_, players_, hands_] := Module[{},
  s = suits; c = cards; p = players; h = hands; d = s * c;
  (* x[m] is average weight time when missing m;
  we find recursively, start knowing x[0] = 0 and x[1] = 1/p,
  both from the formula and there are p hands! *)
  (* it is convenient to set y[i] =
   x[i] for i < m if trying to find x[m] and y[m] = 0 *)
  (* want y[m] = 0 from bringing things over to solve for x[m] *)
  For[i = 0, i ≤ d + 1, i++,
   {
    x[i] = 0; y[i] = 0; (* initialize quantities to zero *)
   }];
  x[1] = p; y[1] = x[1];
  numhands = Binomial[d, h]; (* number of ways to choose h cards from deck of d *)
  For[m = 2, m ≤ d, m++,
   {
    x[m] = Sum[ Binomial[m, k] Binomial[d - m, h - k] (y[m - k] + 1) / numhands,
       {k, 0, Min[h, m]}] / (1 - (Binomial[d - m, h] / numhands));
    (* this is the recursive formula, do only for k at most min(h,m) *)
    y[m] = x[m]; (* update y[m] from 0 to x[m] *)
    Print["Wait time x[", m, "] = ", 1.0 x[m], "."];
   }]; (* end of n loop *)
  Print["x[", m - 1, "] = ", x[m - 1]];
 ]
```

```
recurrencetofindwaittime[4, 13, 4, 13]
```

```
Wait time x[2] = 5.73333.
Wait time x[3] = 6.90512.
Wait time x[4] = 7.78406.
Wait time x[5] = 8.48704.
Wait time x[6] = 9.07285.
Wait time x[7] = 9.57498.
Wait time x[8] = 10.0144.
Wait time x[9] = 10.4049.
Wait time x[10] = 10.7564.
Wait time x[11] = 11.0759.
Wait time x[12] = 11.3688.
Wait time x[13] = 11.6392.
Wait time x[14] = 11.8903.
Wait time x[15] = 12.1246.
Wait time x[16] = 12.3443.
Wait time x[17] = 12.5511.
Wait time x[18] = 12.7463.
Wait time x[19] = 12.9313.
Wait time x[20] = 13.1071.
Wait time x[21] = 13.2745.
Wait time x[22] = 13.4342.
Wait time x[23] = 13.587.
Wait time x[24] = 13.7335.
Wait time x[25] = 13.8741.
Wait time x[26] = 14.0093.
Wait time x[27] = 14.1395.
Wait time x[28] = 14.265.
Wait time x[29] = 14.3862.
Wait time x[30] = 14.5034.
Wait time x[31] = 14.6168.
Wait time x[32] = 14.7266.
Wait time x[33] = 14.8331.
Wait time x[34] = 14.9365.
Wait time x[35] = 15.0369.
Wait time x[36] = 15.1346.
Wait time x[37] = 15.2295.
Wait time x[38] = 15.322.
Wait time x[39] = 15.4122.
Wait time x[40] = 15.5.
Wait time x[41] = 15.5858.
Wait time x[42] = 15.6695.
Wait time x[43] = 15.7512.
Wait time x[44] = 15.8311.
Wait time x[45] = 15.9092.
Wait time x[46] = 15.9856.
Wait time x[47] = 16.0604.
Wait time x[48] = 16.1336.
Wait time x[49] = 16.2054.
Wait time x[50] = 16.2757.
Wait time x[51] = 16.3446.
Wait time x[52] = 16.4122.
```

```
x[52] =
 (545 928 847 452 258 491 492 389 829 320 450 626 160 841 197 170 151 499 543 576 764 239 417 023 255 451 861 085 -
   626 905 883 156 684 405 195 107 023 780 479 317 372 271 274 509 508 567 967 387 626 740 665 018 015 009 371 549 -
   436 149 407 597 300 658 313 676 235 368 117 130 572 713 596 038 893 838 341 598 601 195 428 485 082 034 721 825 -
   135 468 987 /
  33 263 651 815 411 301 455 093 132 853 409 898 491 173 712 201 961 171 863 009 501 862 172 850 150 290 107 365 -
   581 703 955 244 599 875 291 300 187 598 652 081 762 922 197 224 670 489 260 819 393 711 347 009 189 515 918 -
   586 716 637 050 967 309 053 192 156 474 313 569 986 447 499 658 535 026 680 206 545 307 025 594 664 981 127 -
   461 443 909 937 520)
```

```
Timing[dealstillseeall[13, 10 000 000]]
```




```
{{1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.00005}, {7, 0.00745}, {8, 0.12907}, {9, 0.78296},
 {10, 2.50582}, {11, 5.18848}, {12, 8.01843}, {13, 10.0628}, {14, 10.9782}, {15, 10.8006},
 {16, 9.90676}, {17, 8.59254}, {18, 7.19095}, {19, 5.82429}, {20, 4.63875}, {21, 3.63591},
 {22, 2.81515}, {23, 2.16187}, {24, 1.65213}, {25, 1.25848}, {26, 0.95041}, {27, 0.71555},
 {28, 0.54286}, {29, 0.40812}, {30, 0.307}, {31, 0.23202}, {32, 0.17344}, {33, 0.12906},
 {34, 0.09771}, {35, 0.07305}, {36, 0.05439}, {37, 0.042}, {38, 0.0314}, {39, 0.02348},
 {40, 0.01734}, {41, 0.01351}, {42, 0.00964}, {43, 0.00705}, {44, 0.00524}, {45, 0.00406},
 {46, 0.00319}, {47, 0.00228}, {48, 0.00205}, {49, 0.00124}, {50, 0.00076}, {51, 0.00071},
 {52, 0.00045}, {53, 0.00031}, {54, 0.00024}, {55, 0.00023}, {56, 0.00011}, {57, 0.00012},
 {58, 0.00004}, {59, 0.00005}, {60, 0.00004}, {61, 0.00007}, {62, 0.00003}, {63, 0.00002},
 {64, 0.00001}, {65, 0.00001}, {66, 0.}, {67, 0.}, {68, 0.}, {69, 0.00001}, {70, 0.00001}}

Mean = 16.4131 and StDev = 4.35065.

{7874.61, Null}
```

Math 331: The little Questions (Fall 2024)

Steven J Miller Williams College sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 21: October 25, 2024: German Tank Problem:
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/GermanTankProblem_Talk_PennState2020.pdf

Expanded Lecture from Math/Stat 341: Probability (Fall 2021):
- Lecture 06: 9/22/21: German Tank Problem I: Theory: https://youtu.be/APsubcDVl1s (slides)
- Lecture 08: 9/27/21: German Tank Problem II: Statistical Inference: https://youtu.be/JnaVkeO9qtc (slides)

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https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 22: October 28, 2024:

Why I love Monovariants: From Zombies to Conway's Soldiers to Fibonacci Games: pdf

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https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 23: October 30, 2024: https://youtu.be/w_SBDcZY1Ac
Project Euler Problems: https://projecteuler.net/archives

#16

$2^{15} = 32768$ digits sum to 26

What is the sum of digits of 2^{1000}?

Soln: Brute Force: $2^{10} \approx 10^3$

How bad? $2^{1000} = (2^{10})^{100} = (10^3)^{100} = 10^{300}$

Universe has less than 10^{100} sub-atomic stuff

a universe of universes is $(10^{100})^2 = 10^{200} << \cdots << 10^{300}$

Gather Data!

Sorry! Can brute force

only 300 digits

n	2^n	sum of digits
0	1	1
1	2	2
2	4	4
3	8	8
4	16	7
5	32	5
6	64	10
7	128	11
8	256	13

n	2^n	sum
9	512	8
10	1024	7
11	2048	14
12	4096	19
13	8192	20

$$2^{1000} = (2^{100})^{10}$$

$$(314)^2 = (3\cdot 100 + 2\cdot 10 + 4)^2 = 3^2\cdot 100^2 + \cdots$$

$$2^{10} = 1024 \quad \text{so} \quad 2^{1000} = (1024)^{100}$$

$$((1024)^{10})^{10}$$

if Σ of digits is $\left\{\begin{matrix}3\\9\end{matrix}\right.$ is divisible by $\left\{\begin{matrix}3\\9\end{matrix}\right.$

$429 \to 15$

use arrays

a[1]

a[0] = 1's digit

a[1] = 10's digit

a[2] = 100 digit

3x+1 Problem

$$a_{n+1} = \begin{cases} 3a_n + 1 & \text{if } a_n \text{ odd} \\ a_n/2 & \text{if } a_n \text{ even} \end{cases}$$

$2 \to 1 \to 4$

$12 \to 6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

27

#10: Sum of all primes up to 2,000,000

$$\sum_{n=0}^{N} n = \frac{N(N+1)}{2}$$

$S_N = 0 + 1 + \cdots + N$

$S_N = N + (N-1) + \cdots + 0$

$2S_N = N + \cdots + N$ $(N+1)$ times

Find all primes up to 2,000,000 and add

inclusion-exclusion but too much work

$\pi(x) = \#\{\text{primes} \le x\} \sim \frac{x}{\log x}$

If m is composite $m = a \cdot b$ with $a, b > 1$ and
if $a \le b$ then $a \le \sqrt{m}$

Primes: 2, 3, 5, ..., P_{max} $\quad$ P_{max} largest prime < 2,000,000

$a[n] = 1$ for $2 \le n \le 2{,}000{,}000$

For each prime, set $a[kp] = 0$ $\quad k: 1$ to $\frac{2{,}000{,}000}{p}$

Compute $\sum n\, a[n]$

$N\#$ (N primorial) is the product of all primes at most N

$5\# = 5 \cdot 3 \cdot 2 = 30$

prime $\underbrace{N\# + 2}_{\text{Composite}}$ $\quad \underbrace{N\# + 3}_{\text{Composite}}$ $\quad \underbrace{N\# + 4}_{\text{Composite}}$ $\quad \cdots \quad \underbrace{N\# + N}_{\text{Composite}}$ prime

desert of length at least $N-2$

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Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

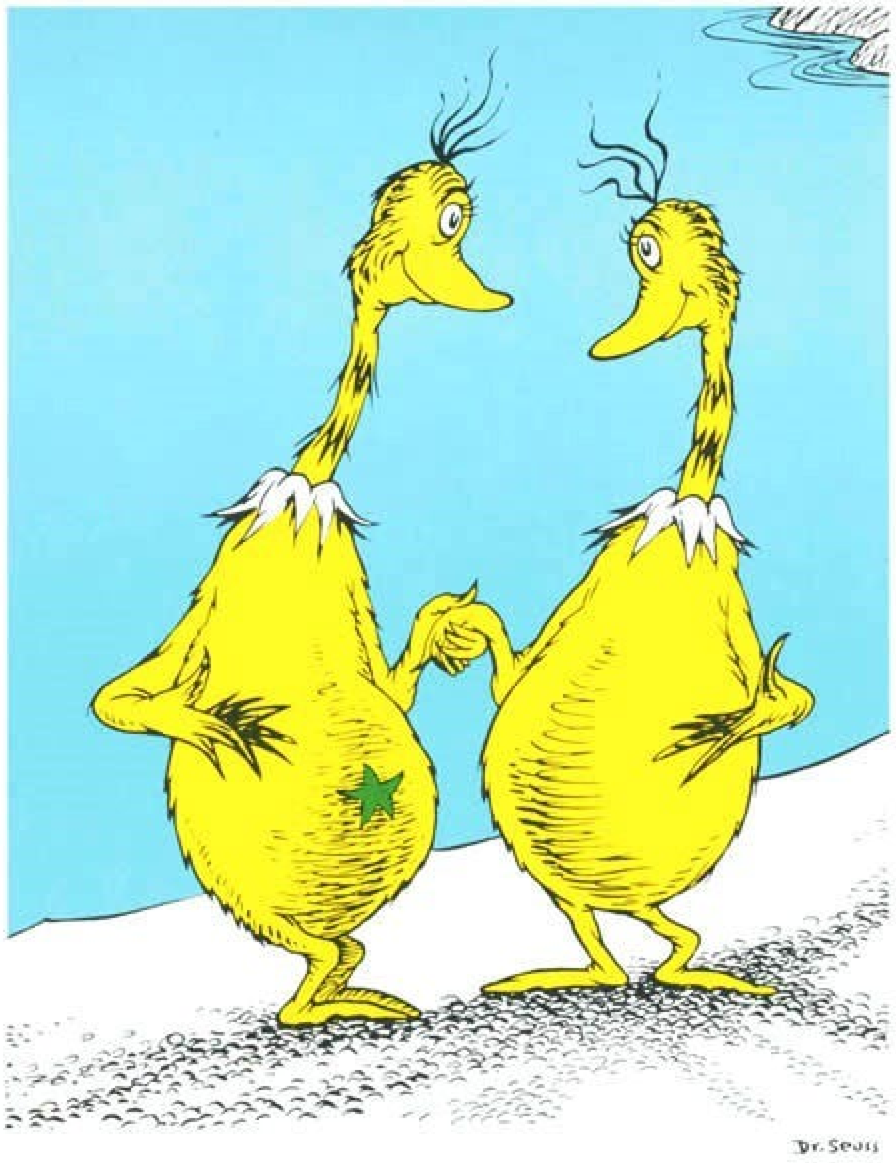
Lecture 24: November 1: Work on Project
Lecture 25: November 4, 2024: https://youtu.be/uhP6HusxCdc
Proof by Story, Putnam A1 2017 Problem:
https://www.math.uh.edu/~torok/Putnam/problems_A1.pdf

Proof By Story:

- Sneetches $\binom{n}{k} = \binom{n}{n-k}$
- Pascal's Identity $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$
- Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- Sum of Squares of Binomial Coefficients
- Cookie Problem
- Generalized Cookie Problem

Application: Zeckendorf's Theorem

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \text{nice}, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$



https://images-na.ssl-images-amazon.com/images/S/compressed.photo.goodreads.com/books/1496420330i/35287690.jpg

Sneetukes Theorem: $\binom{n}{k} = \binom{n}{n-k}$ with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$n! = n(n-1)\cdots 3\cdot 2\cdot 1$ and $0! = 1$ "$\left(-\frac{1}{2}\right)! = \sqrt{\pi}$"

Prove: $\binom{n}{k} = \binom{n}{n-k}$

Compare $\frac{n!}{k!\,(n-k)!}$ vs $\frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!}$ $\blacksquare$

Same

Story proof: choosing exactly k from n to take is the same as choosing exactly $n-k$ from n to exclude.

Pascal: $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

Proof: Compare $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-(k+1))!}$ vs $\frac{(n+1)!}{(k+1)!(n+1-(k+1))!}$

multiply thru by $\frac{k!(n-(k+1))!}{n!}$

yields $\frac{1}{n-k} + \frac{1}{k+1}$ vs $\frac{n+1}{(k+1)(n-k)}$

$$\frac{k+1+n-k}{(n-k)(k+1)} = \frac{n+1}{(k+1)(n-k)} \quad \blacksquare$$

Story: n sox fans, 1 Yankee fan, need to choose k+1 people

$$\binom{n+1}{k+1} = \binom{1}{0}\binom{n}{k+1} + \binom{1}{1}\binom{n}{k} \quad \blacksquare$$

Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \underbrace{\binom{n}{k} x^k y^{n-k}}$ or $\underbrace{\binom{n}{k} x^{n-k} y^k}$

Proof by Story

$$(x+y)^n = (x+y)(x+y) \cdots (x+y)(x+y)$$

Expand have only terms of form $x^k y^m$

with $k+m=n$ and $0 \le k, m \le n$

$$= \sum_{k=0}^{n} a_{k,n}\, x^k y^{n-k}$$

choose exactly k of the n $x+y$ terms for the x, remaining give y: #ways $= \binom{n}{k}\binom{n-k}{n-k} = \binom{n}{k}$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{n-k} x^k y^{n-k}$$

Note: $x=y=1$ then $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} = 2^n$

Socializing Theorem

n Williams

n Amherst-Harvard-Michigan Consortium

What is $\sum_{k=0}^{n} \binom{n}{k}^2$? GATHER DATA

$n=0$ ANS: 1

$n=1$ ANS: 2

$n=2$ ANS: 6 $\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 = 1^2 + 2^2 + 1^2$

$n=3$ ANS: 20 $1^2 + 3^2 + 3^2 + 1^2$

$n=4$ ANS: 70 $1^2 + 4^2 + 6^2 + 4^2 + 1^2$

Pascal's Triangle

row 0 → 1 ← n=0

1 1

row 2 → 1 2 1 ← n=1

1 3 3 1

row 4 → 1 4 6 4 1 ← n=2

1 5 10 10 5 1

row 6 → 1 6 15 20 15 6 1 ← n=3

1 7 21 35 35 21 7 1

row 8 → 1 8 28 56 70 56 28 8 1 ← n=4

CONJECTURE: $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Prove that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

n Williams

n AHMC

Proof: $\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \underbrace{\binom{n}{k}\binom{n}{n-k}}$ by Sneetches

Choose exactly k from Williams

Choose exactly $n-k$ from AHMC

Choosing exactly n people from $2n$ people, # ways is $\binom{2n}{n}$

What's next? Is it $\sum_{k=0}^{n} \binom{n}{k}^3$ or something else?

A1 ('17) Let S be the smallest set of positive integers such that

(a) 2 is in S,

(b) n is in S whenever n^2 is in S, and

(c) $(n+5)^2$ is in S whenever n is in S.

Which positive integers are not in S?

(The set S is "smallest" in the sense that S is contained in any other such set.)

Which numbers matter?

2

2

5

what happens if change?

Start by Gathering Data

$$S = \{2, 49, 7, 54^2, 54,$$

$$2 \to 49 \to 7 \to 144 \to 12 \to 17^2 \to 17$$

$$144 \to 144^2$$

$$17^2 \to (17^2+5)^2$$

$$49 \to 54^2 \to 54$$

$$54^2 \to (54^2+5)^2$$

Arithmetic Progression: 2, 7, 12, 17, 22, 27, 32, ...

↳ are any of these a 4th power?

say $2 + 5n = a^4 = (a^2)^2$ so $a^2 \in S$, $a \in S$

Congruence arguments

Note the LHS is always 2 mod 5

a	0	1	2	3	4
a^4 mod 5	0	1	1	1	1

No Solution

Math 331: The little Questions (Fall 2024)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 26: November 6, 2024: https://youtu.be/bLTh48U6VTg
Putnam A1 2017 Problems: 2017, 1989, 1994:
https://www.math.uh.edu/~torok/Putnam/problems_A1.pdf

A1 ('94) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \le a_{2n} + a_{2n+1}$ for all $n \ge 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

A1 ('89) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

A1 ('23) For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.

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A1 ('89) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

Gather data: 1, 101, 10101, 1010101, 101010101, ...

not prime Composite: digit sum is a multiple of 3

If composite: $n = xy$ with $x \le y$ and $x \le \sqrt{n}$

Let d = digit sum, assume d is composite

Ex. $d = 10$: $\underbrace{1010101010}\underbrace{1010101}= \underbrace{1010101} * 10\cdots001$

reduces to digit sum is prime

A1 ('89) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

$x =$

$$10101\cdots01 = \sum_{k=0}^{n} (10^2)^k = \frac{u^{n+1}-1}{u-1} = \frac{u^{n+1}-1}{99}$$

$u = 10^2$

~~where $u = 10$~~

$$100x = 1010101\cdots0100$$
$$x = 10101\cdots01$$
$$99x = 1000\cdots0-1$$

Recall: $a^2-b^2=(a-b)(a+b)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$$x = \frac{(u-1)(u^{2n}+u^{2n-1}+\cdots+u+1)}{99}$$

$$x = \frac{\cancel{9}\cdot(10^{2n}+10^{2n-1}+\cdots+10+1)}{\cancel{99}\ 11}$$

Should be $u=100$

A1 ('89) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

$$X = 101010\cdots101 = 1+10^2+10^4+\cdots+10^{2n}$$

let $u = 10^2$ so $X = \sum_{k=0}^{n} u = \frac{1-u^{n+1}}{1-u} = \frac{u^{n+1}-1}{u-1}$

$$X = \frac{(\cancel{u-1})(1+u^2+\cdots+u^{2n})}{\cancel{u-1}}$$ Back where we started

$$100X - X = 10^{2n+2} - 1 = (10^{n+1})^2 - 1$$

$$99X = (10^{n+1})^2 - 1 = (10^{n+1}-1)(10^{n+1}+1)$$

if each at least 2 then X is composite

$$X = \frac{10^{n+1}-1}{a} \cdot \frac{10^{n+1}+1}{b}$$ when $ab = 99$

Q: what makes $\frac{10^{n+1}-1}{a}$ small? $\frac{10^{n+1}+1}{b}$ small?

↳ if n is small or a is large

worse case is $a=99$

as long as $n \geq 2$, $\frac{10^{n+1}-1}{a} > 1$ thus at least 2

↳ second factor is at least 2 if $n \geq 1$

only chance for x to be prime is $n \in \{0, 1\}$

$n=0 \Rightarrow 1$ unit

$n=1 \Rightarrow 101$ prime

A1 ('94) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

$$a_1 \qquad \underbrace{a_2 \qquad a_3}_{\geq\, a_1} \qquad \underbrace{a_4 \qquad a_5}_{\geq\, a_2} \qquad \underbrace{a_6 \qquad a_7}_{\geq\, a_3} \qquad a_8$$

$$a_1 \leq a_2 + a_3$$

$$\underbrace{a_2 + a_3}_{\geq\, a_1}$$

Dyadic Decomposition: March in powers of 2

Harmonic Series : $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$

Calculus: Integral test: $1 + 1/2 + \cdots + 1/n \simeq \int_{x=1}^{n} \frac{1}{x}\,dx = \ln(n)$

$= \log(n)$

$$1 \quad \frac{1}{2} \quad \underbrace{\frac{1}{3} \quad \frac{1}{4}}_{\geq 2 \cdot \frac{1}{4}} \quad \underbrace{\frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8}}_{\geq 4 \cdot \frac{1}{8}} \quad \underbrace{\frac{1}{9} \cdots \frac{1}{16}}_{\geq 8 \cdot \frac{1}{16}}$$

Choose k so that $2^k \leq n < 2^{k+1}$ so $k \approx \log_2 n = \frac{\log(n)}{\log 2}$

So $H_n \geq \frac{1}{2} \cdot \frac{1}{\log 2} \cdot \log(n)$

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Lecture 27: November 8, 2024: No class, work on project.
Lecture 28: November 11, 2024: No class, work on project.
Lecture 29: November 13: Putnam A1 Problems: https://youtu.be/uwvFu57JJBc

https://www.math.uh.edu/~torok/Putnam/problems_A1.pdf

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PRIMEary Colors

9 comments

We define the coloring number of a graph of V vertices and E edges to be the smallest number of colors such that no two vertices that share an edge are colored the same. Consider the following graph:

1. The vertices are the integers 2, 3, 4, 5, 6, ..., 9,998, 9,999, and 10,000
2. Two vertices are connected by an edge if they share a common factor. So there isn't an edge between 2 and 3, but there is between 2 and 6, another between 3 and 6, but none between 2 and 5, between 3 and 5, ...

Prove that the coloring number of this graph is at least 13.

Problem created by Steven Miller as a practice test question for Princeton's Discrete Math 341, Fall '98.

Reverse Tic-Tac-Toe (I prefer Toe-Tac-Tic): Cameron, Elizabeth and Steven Miller:

In standard Tic-Tac-Toe, whomever gets three in a row (vertically, horizontally or diagonally) wins. A simple calculation (though tedious if you don't use symmetry to combine cases) shows that if both play optimally, the game will always end in a tie. Consider now *Reverse Tic-Tac-Toe*, where the object is to force the other person to get three in a row. Does one of the players have a winning strategy (and if so who and what is it), or will the game always end in a tie if each play optimally?

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$80\% = \frac{4}{5} = \frac{a}{b}$

for any p?

Consid $60\% = 3/5$

60% M M M H H H H H H : Not true $\forall p$

80% = 4/5

M M M H H H H H H H H H H H H H

$\frac{10}{13} < \frac{4}{5}$

$\frac{11}{14} < \frac{4}{5}$

$\frac{12}{15} : \frac{4}{5}$

A1 ('04) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

Assume on shot n go from below 80% to above 80%. Must show no such n exists.

$$\frac{h}{n-1} \longrightarrow \frac{h+1}{n-1+1}$$

Shot n — % is < 80%

Shot $n+1$ — % is > 80%

$$\frac{h}{n-1} < \frac{4}{5} \left(= \frac{a}{b}\right) < \frac{h+1}{n}$$

$5h < 4n-4$ and $4n < 5h+5$

$$5h+4 < 4n < 5h+5$$

adjacent: impossible

$$\frac{h}{n-1} < \frac{a}{b} < \frac{h+1}{n}$$

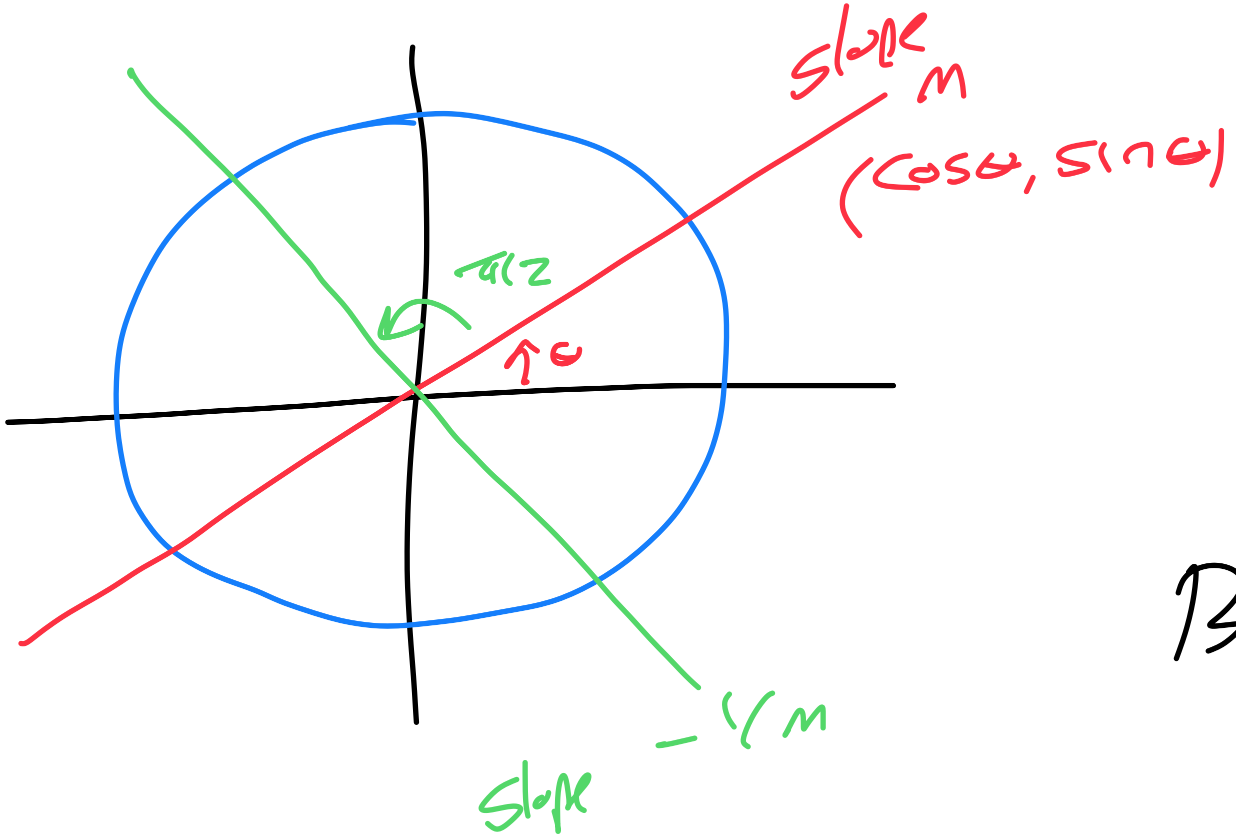
$bh < an-a$ and $an < bh+b$

$$\underbrace{bh+a < an < bh+b}_{\text{adjacent}}$$

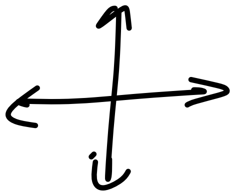
so $a = b-1$

% is $\frac{b-1}{b}$

Aside: $0 \cdot \infty = -1$



Thm: product of slopes equals -1.

But  $0 \cdot \infty = -1$

A1 ('23) For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.

year: probably not too important

Which numbers matter?

important

Second derivative

Note: $$f_n''(0) = \sum_{m=1}^{n}\left[-m^2\cos(m\,0)\right]\prod_{\substack{l=1\\ l\neq m}}^{n}\cos(l\,0)$$

$$= -\sum_{m=1}^{n} m^2 = \frac{n(n+1)(2n+1)}{6}$$

$n=1$ $\quad 1=1$

$n=2$ $\quad 5=5$

Find n so that $\frac{n(n+1)(2n+1)}{6} > 2023$

Solve n st $n(n+1)(2n+1) > 6 \cdot 2023 = 12138$

But LHS $> 2n^3$ so if $2n^3 > 12138$ clearly ok

so $n^3 > 6100$ ok 6100 replace with $(2 \cdot 10)^3$

so if $n > 20$ ok

All trig comes from the Exponential Function

$$e^x = 1 + x + x^2/2! + \cdots = \sum_{n=0}^{\infty} x^n/n!$$

$$\cos x = 1 - x^2/2! + \cdots = \sum_{n=0}^{\infty} (-x^2)^n/(2n)!$$

$$\sin x = x - \frac{x^3}{3!} + \cdots = \sum \_\_\_\_\_\_$$

Note: $e^{ix} = \cos x + i \sin x$

Also: $z = a + ib$ then $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$

with $\bar{z} = a - ib$

$$e^{ix} e^{-ix} = 1 = (\cos x + i\sin x)(\cos x - i\sin x)$$

using $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

$$\Rightarrow \quad 1 = \cos^2 x + \sin^2 x$$

Pythagoras!

$$e^{ix} e^{iy} = e^{i(x+y)}$$

so $(\cos x + i\sin x)(\cos y + i\sin y) = \cos(x+y) + i\sin(x+y)$

$$(\cos x \cos y - \sin x \sin y)$$

$$+ \; i(\sin x \cos y + \cos x \sin y)$$

$$\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x \lim_{h\to 0} \underbrace{\frac{\cos(0+h) - \cos(0)}{h}}_{\cos'(0)} + \cos x \lim_{h\to 0} \underbrace{\frac{\sin(0+h) - \sin 0}{h}}_{\sin'(0)}$$

$$\sin^2 x + \cos^2 x = 1$$

$$2 \sin x \sin' x + 2 \cos x \cos' x = 0 \qquad \text{take } x = 0$$

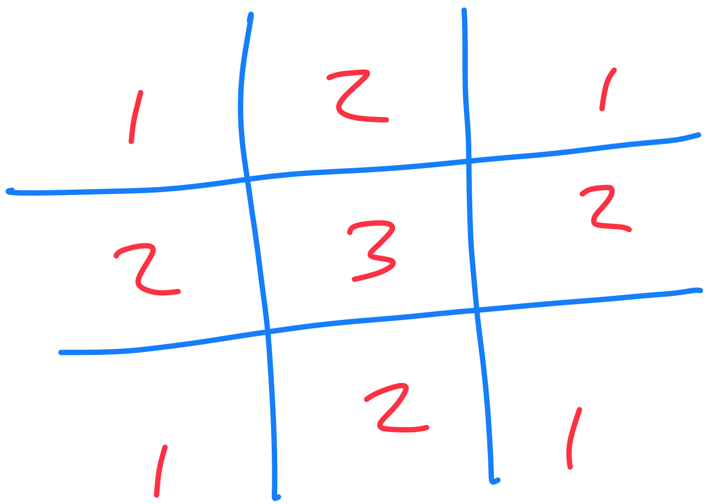
$$2 \underbrace{\sin 0}_{=\,0} \sin'(0) + 2 \underbrace{\cos 0}_{=\,1} \cos' 0 = 0$$

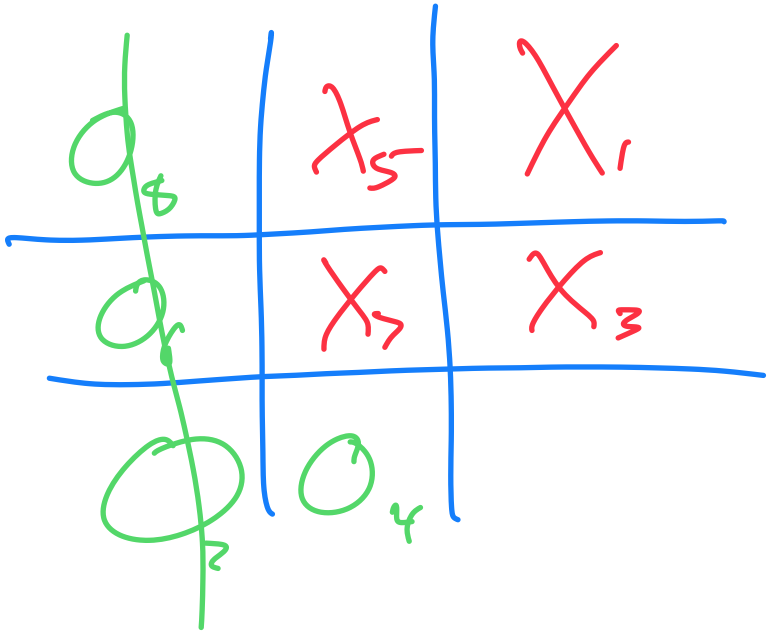
$$\Rightarrow \quad 2 \cos'(0) = 0 \Rightarrow \cos'(0) = 0$$

Radians make $\sin'(0) = 1$

Reverse Tic-Tac-Toe (I prefer Toe-Tac-Tic): Cameron, Elizabeth and Steven Miller:

In standard Tic-Tac-Toe, whomever gets three in a row (vertically, horizontally or diagonally) wins. A simple calculation (though tedious if you don't use symmetry to combine cases) shows that if both play optimally, the game will always end in a tie. Consider now *Reverse Tic-Tac-Toe*, where the object is to force the other person to get three in a row. Does one of the players have a winning strategy (and if so who and what is it), or will the game always end in a tie if each play optimally?





MSTDs : More Sums Than Differences Sets

S finite set of non-neg integers

sumset $S+S = \{x+y : x, y \in S\}$

difference set $S-S = \{x-y : x, y \in S\}$

$|S+S|$ vs $|S-S|$

↑ maybe larger as (x,x) gives $0 = x-x$ but $2x$ for sum

← maybe larger as (x,y) gives $x-y$, $y-x$ vs $x+y$

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https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 30: November 15: Putnam A1 Problems: https://youtu.be/GPX8m8KJBU0

https://www.math.uh.edu/~torok/Putnam/problems_A1.pdf

PRIMEary Colors

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We define the coloring number of a graph of V vertices and E edges to be the smallest number of colors such that no two vertices that share an edge are colored the same. Consider the following graph:

1025

1. The vertices are the integers 2, 3, 4, 5, 6, ..., 9,998, 9,999, and 10,000
2. Two vertices are connected by an edge if they share a common factor. So there isn't an edge between 2 and 3, but there is between 2 and 6, another between 3 and 6, but none between 2 and 5, between 3 and 5, ...

Prove that the coloring number of this graph is at least 13. 10

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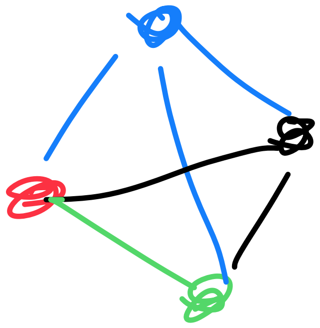
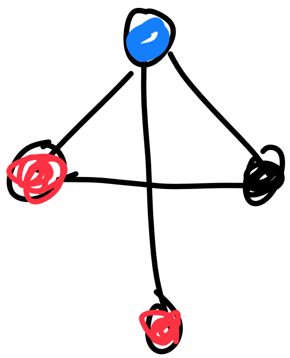


K_4



all evens: at least 5000

↳ at most 5000: color $2n$, $2n+1$ the same!

$\{2, 2^2, 2^3, \ldots, 2^{13}\}$

https://math.mit.edu/~rstan/myputnam.pdf

(A1 or B1 problem) If n is a positive integer, then define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(n)$ and $Q(n)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

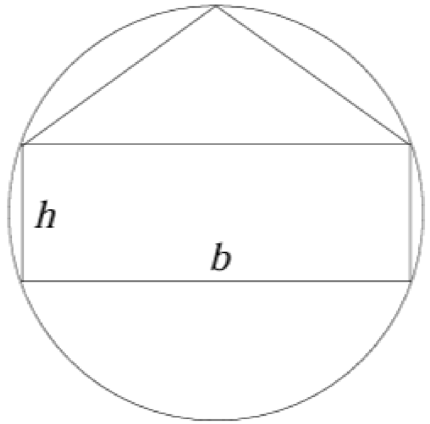
for all $n \geq 1$.

Let $x(t)$ and $y(t)$ be real-valued functions of the real variable t satisfying the differential equations

$$\frac{dx}{dt} = -xt + 3yt - 2t^2 + 1$$
$$\frac{dy}{dt} = xt + yt + 2t^2 - 1,$$

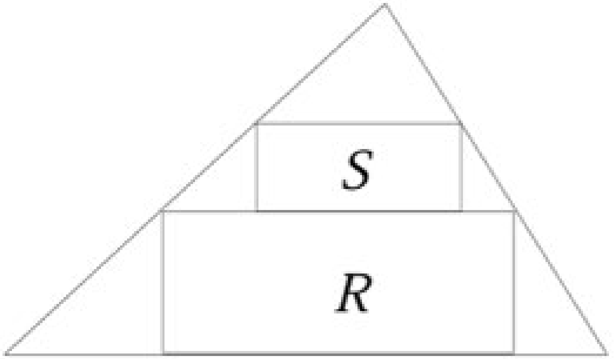
with the initial conditions $x(0) = y(0) = 1$. Find $x(1) + 3y(1)$. (This problem was later withdrawn for having an easier than intended solution.)

(A1 or B1 problem) Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown.



For what value of h do the rectangle and triangle have the same area?

Let T be a triangle and R, S rectangles inscribed in T as shown:



Find the maximum value, or show that no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)},$$

where T ranges over all triangles and R, S over all rectangles as above, and where A denotes area.

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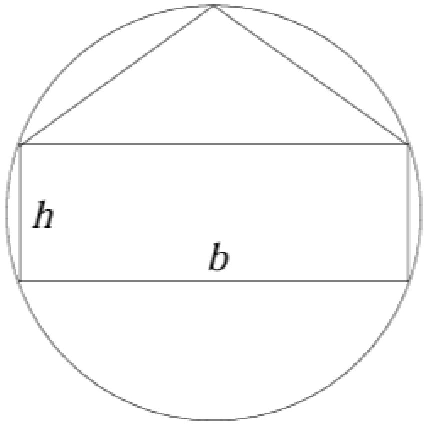

what h, b give the same area?

Let $x(t)$ and $y(t)$ be real-valued functions of the real variable t satisfying the differential equations

$$\frac{d}{dt}(x+3y)\left\{\begin{aligned} 1\,\frac{dx}{dt} &= -xt + 3yt - 2t^2 + 1 \\ 3\,\frac{dy}{dt} &= xt + yt + 2t^2 - 1, \end{aligned}\right\}$$

$2xt + 6yt + \cdots$

$2t(x+3y) + \cdots$

with the initial conditions $x(0) = y(0) = 1$. Find $x(1) + 3y(1)$. (This problem was later withdrawn for having an easier than intended solution.)

$X(t) + 3Y(t)$

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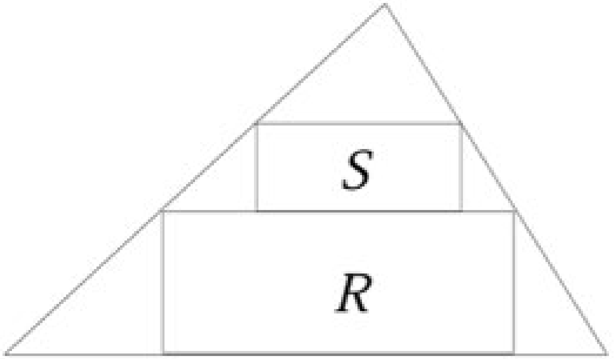

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$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(n)$ and $Q(n)$ such that

$$\underbrace{f(n+2)}_{(n+2)!} = P(n)\underbrace{f(n+1)}_{\approx(n+1)!} + Q(n)\underbrace{f(n)}_{\approx n!},$$

for all $n \geq 1$.

Maybe $P(n)$ is deg 1 : 2 terms
$Q(n)$ is deg 2 : 3 terms

GATHER DATA!

n	$f(n)$	$f(n+1)$	$f(n+2)$	$P(n)$	$Q(n)$
1	1	3	9		
2	3	9	23		
3	9	23	143		
4	23	143	863		

Aside: $f(n) = 1! + 2! + \dots + n!$: How big is $f(n)$?

$$n! \le f(n) \le n! + (n-1)! + (n-2)! + (n-3)! + \dots + 1!$$

$$\le n! + (n-1)! + (n-2)! + (n-3)!\,(n-3)$$

$$\le n! + (n-1)! + (n-2)! + (n-2)!$$

$$= n! + (n-1)! + 2 \cdot (n-2)!$$

$$= n! + (n-1+2)\,(n-2)!$$

$$= n! + (n+1)(n-2)!$$

$$= n! + (n-1+2)\,(n-2)! = n! + (n-1)! + 2\,(n-2)!$$

$$\le n! + 2\,(n-1)!$$

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for all $n \geq 1$.

$P(n) = n + 2 + p(n)$

$(n+2)! + \cdots + 1! = P(n)\left((n+1)! + \cdots + 1!\right) + Q(n)\left(n! + \cdots + 1!\right)$

$(n+2)! + \cdots + 1! = (n+2)! + (n+2)(n+1)! + (n+2)\,n! + \cdots + Q(n)\left(n! + \cdots + 1!\right)$

$+\; p(n)\left((n+1)! + \cdots + 1!\right)$

Can we make $P(n)f(n+1) = (n+2)! + (n+1)!$

or can we make $Q(n)f(n) = (n+2)!$

$$(n^2 + an + b)(n! + (n-1)! + \cdots + 1!) = (n+2)!$$

(A1 or B1 problem) If n is a positive integer, then define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(n)$ and $Q(n)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

for all $n \geq 1$.

$$f(n+3) = P(n+1)f(n+2) + Q(n+2)f(n+1)$$

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

$$(n+3)! = P(n+1)f(n+2) + (P(n) + Q(n+2))f(n+1) + Q(n)f(n)$$

$$f(n+3) = P(n+1)f(n+2) + Q(n+2)f(n+1)$$

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

$$f(n+1) = P(n-1)f(n) + Q(n-1)f(n-1)$$

$$f(n+2) = [P(n)P(n-1) + Q(n)]f(n) + Q(n-1)f(n-1)$$

$$f(n+1) = f(n) + (n+1)!$$

$$f(n+2) = f(n) + (n+1)! + (n+2)!$$

Math 331: The little Questions (Fall 2024)

Steven J Miller - Williams College - sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 31: November 18: Putnam A1 Problems: https://youtu.be/HEdt7OjSqWA

https://www.math.uh.edu/~torok/Putnam/problems_A1.pdf

https://math.mit.edu/~rstan/myputnam.pdf

(A1 or B1 problem) If n is a positive integer, then define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(n)$ and $Q(n)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

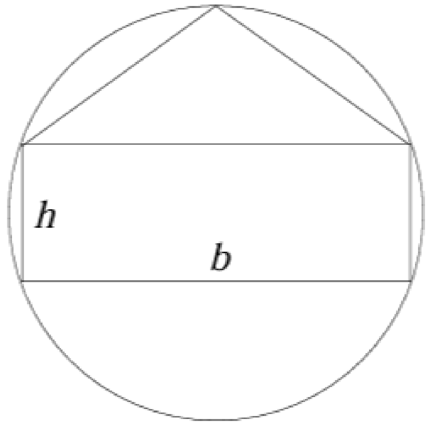
for all $n \geq 1$.

Let $x(t)$ and $y(t)$ be real-valued functions of the real variable t satisfying the differential equations

$$\begin{aligned} \frac{dx}{dt} &= -xt + 3yt - 2t^2 + 1 \\ \frac{dy}{dt} &= xt + yt + 2t^2 - 1, \end{aligned}$$

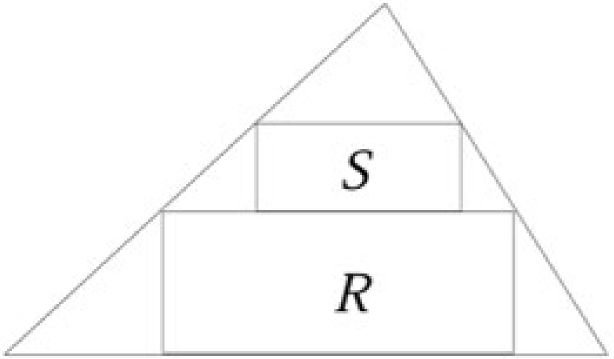
with the initial conditions $x(0) = y(0) = 1$. Find $x(1) + 3y(1)$. (This problem was later withdrawn for having an easier than intended solution.)

(A1 or B1 problem) Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown.



For what value of h do the rectangle and triangle have the same area?

Let T be a triangle and R, S rectangles inscribed in T as shown:



Find the maximum value, or show that no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)},$$

where T ranges over all triangles and R, S over all rectangles as above, and where A denotes area.

$f(n) = 1! + \cdots + n!$ $\quad f(n+2) = P(n) f(n+1) + Q(n) f(n)$

Idea: $f(n+1) = (n+1)! + f(n)$ and $f(n+2) = (n+2)! + (n+1)! + f(n)$

$\hookrightarrow (n+2)! + (n+1)! + f(n) = P(n)\left[(n+1)! + f(n)\right] + Q(n) f(n)$

$$\underbrace{(n+2)! + (n+1)! - P(n)(n+1)!}_{\text{choose } P(n) \text{ to be zero}} = \underbrace{\left[P(n) + Q(n) - 1\right]}_{\text{choose to be zero}} f(n)$$

LHS is $\left[(n+2)+1\right](n+1)! - P(n)(n+1)!$

$= \left[n+3 - P(n)\right](n+1)!$

If $\exists P(n)$ st LHS is 0, take $Q(n) = -P(n) + 1$

if $P(n)$ poly, so is $Q(n)$

So $P(n) = n+3$ $\implies Q(n) = -n-2$

- Does a max exist?
- Gather Data!

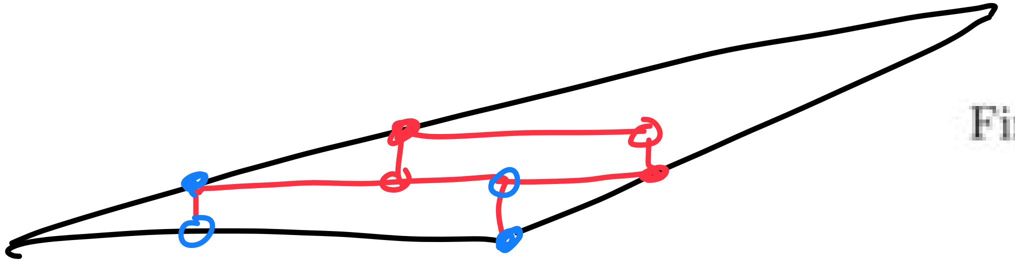
Candidates:

↳ Equilateral

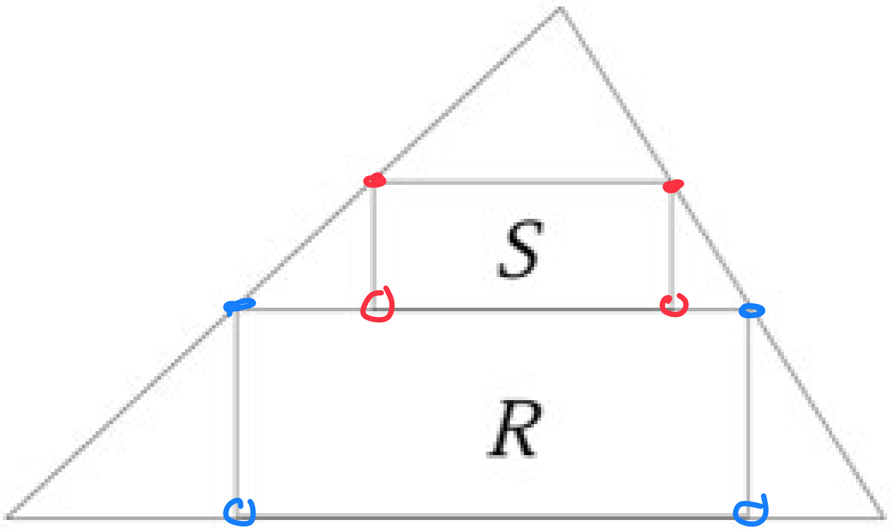
↳ Isocelese Right

↳ Right Triangle

↳ obtuse?



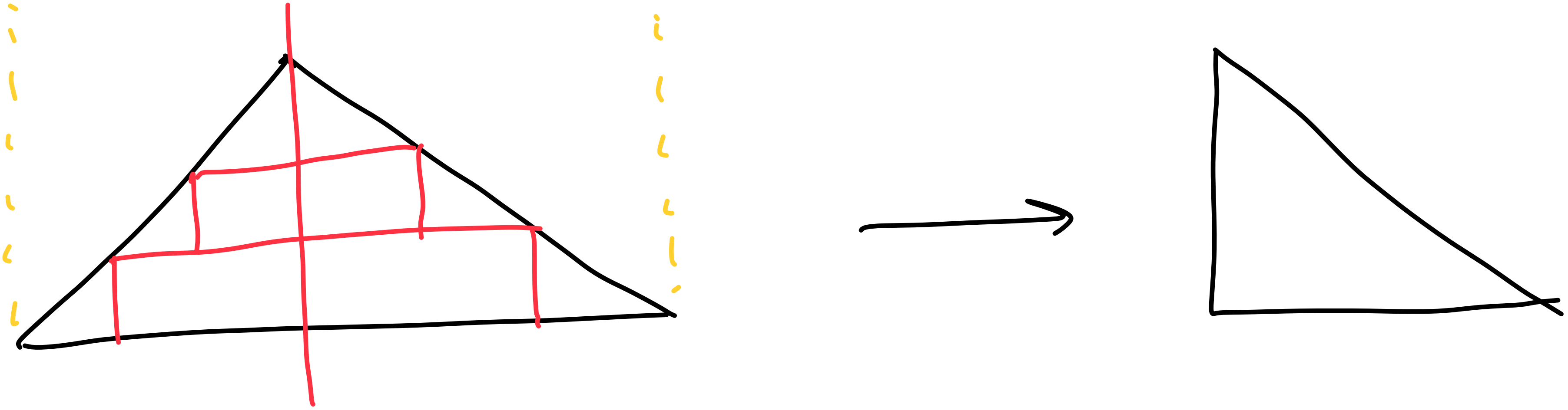
Let T be a triangle and R, S rectangles inscribed in T as shown:

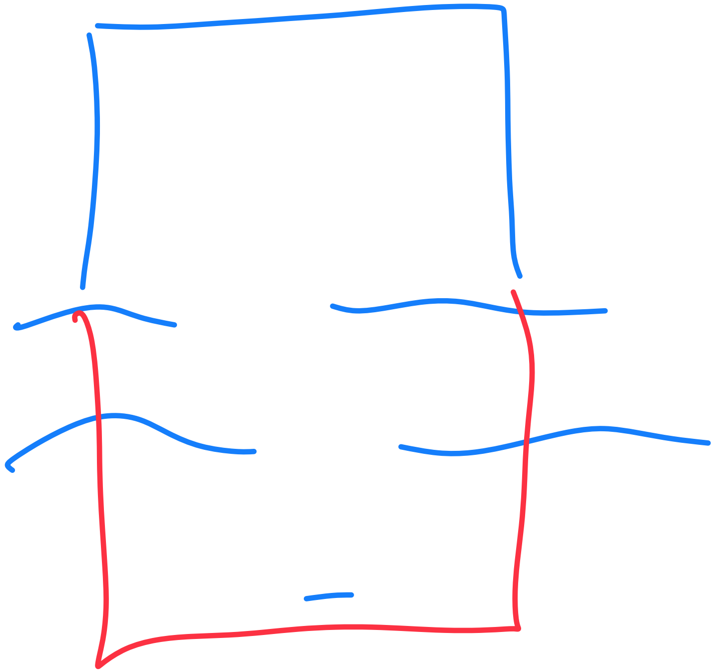


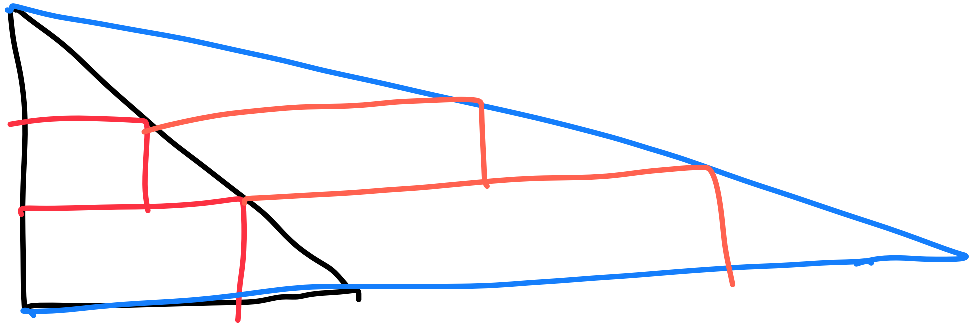
Find the maximum value, or show that no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)},$$

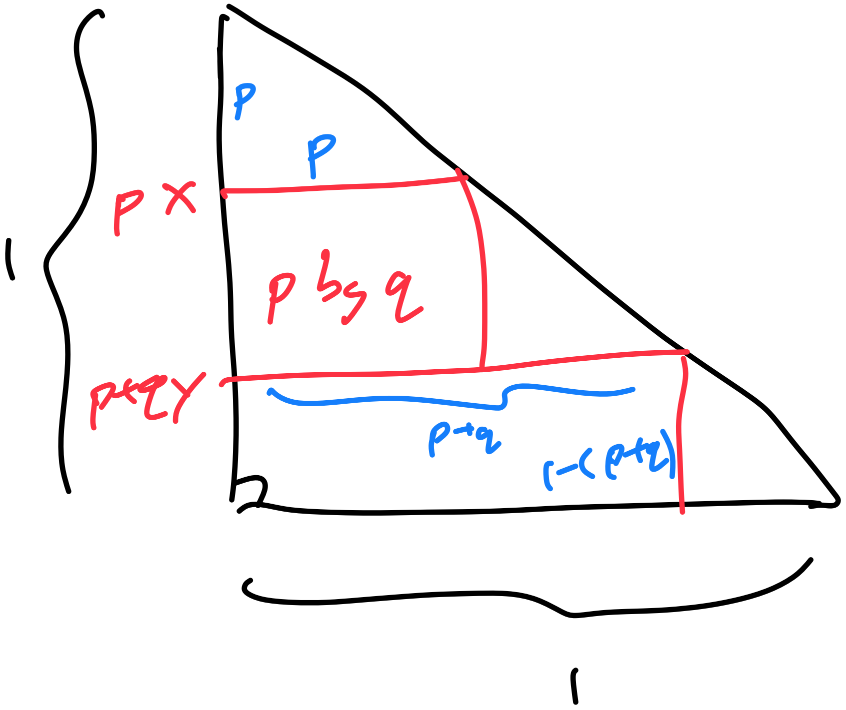
where T ranges over all triangles and R, S over all rectangles as above, and where A denotes area.







wlog



$0 \le x \le y \le 1$

$0 \le p, q, p+q \le 1$

$A(p,q) = pq + (p+q)(1-(p+q))$

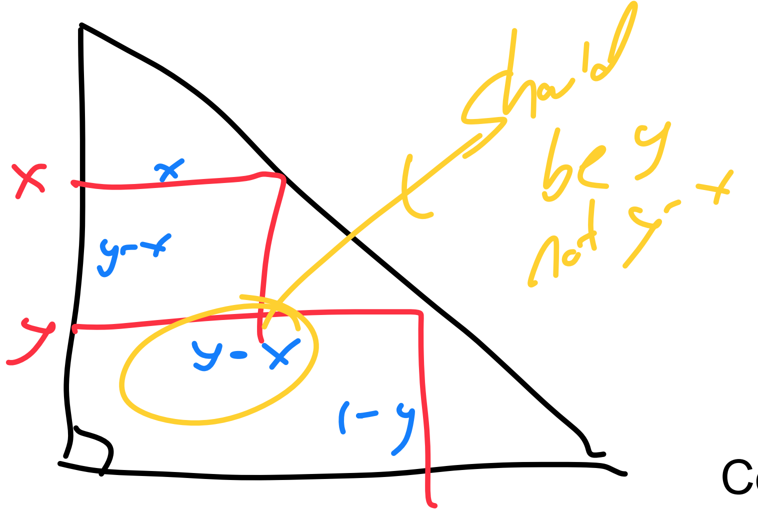
$A(x,y) = x(y-x) + (y-x)(1-y)$

$A(x,y) = (x+1-y)(y-x)$.

wlog either $x \geq y$ or $y \geq x$

Two calculus-I problems

Assume $y \le x$



Comment 11-19-24: The mistake is that the lower box is not y-x by 1-y; it is y by 1-y.

Assume: $0 \le x, y \le 1$ maximize: $A(x,y) = (x+1-y)(y-x)$

Wlog can have $x \le y \le 1$

check extreme cases, get 0 added area

Fix x, find best y for that x, then vary x.

Extremes bad: area is 0 if $y = x$ or $y = 1$

Thoreau: Simplify, Simplify.

$$A(x,y) = (x+1)(-x) - y^2 + xy + (x+1)y$$

$$\frac{\partial A}{\partial y} = -2y + x + x + 1 \quad \text{So} \quad \frac{\partial A}{\partial y} = 0 \to -2y + 2x + 1 = 0$$

$$\text{or} \quad y = \frac{2x+1}{2} = x + \frac{1}{2}$$

$$\text{Area}(x) = A(x, x+\tfrac{1}{2}) = \frac{1}{2}\,\frac{1}{2}$$

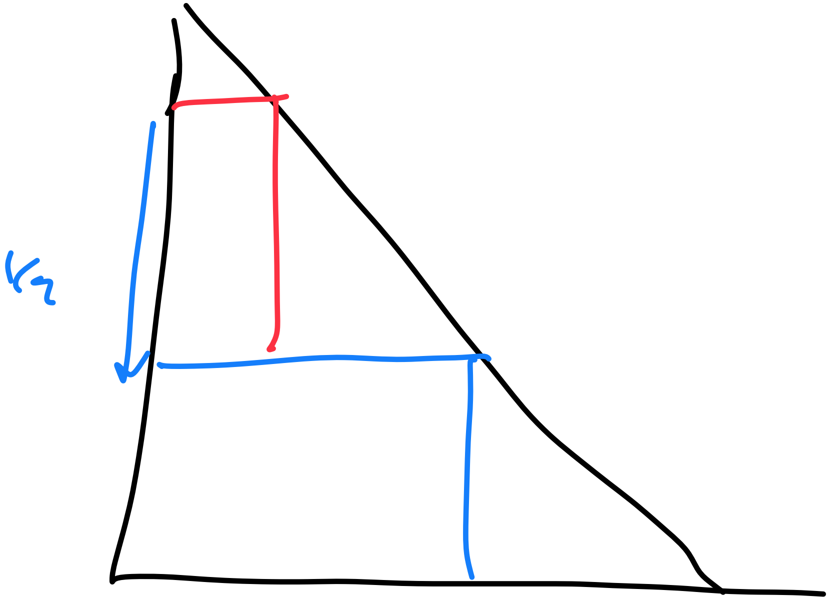
Comment 11-19-24: The mistake is that the lower box is not y-x by 1-y; it is y by 1-y.
Thus A(x,y) = x(y-x) + y(1-y) = xy – x^2 + y – y^2, so partial A/partial y = x + 1 – 2y so y = (x+1)/2 (b/w x and 1).
Then Area(x) = A(x, (x+1)/2) = x(1/2 – x/2) + (x+1)/2 – (x+1)^2/4 so A’(x) = ½ - x + ½ – ½ (x+1), is 0 when x = 1/3 and thus y = 2/3.

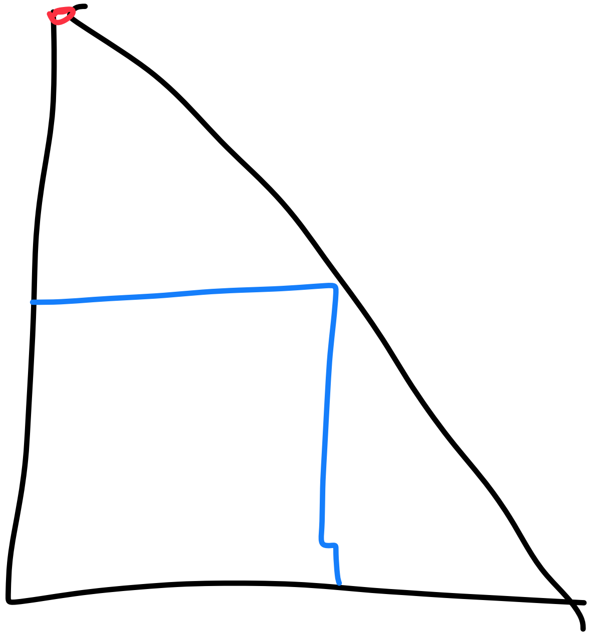
$$A(x,y) = x(y-x) + (y-x)(1-y) = xy - x^2 + y - y^2 - x + xy$$

$$\frac{\partial A}{\partial y} = x + 1 - 2y + x \qquad \frac{\partial A}{\partial y} = 0 \Rightarrow 2y = 2x+1$$

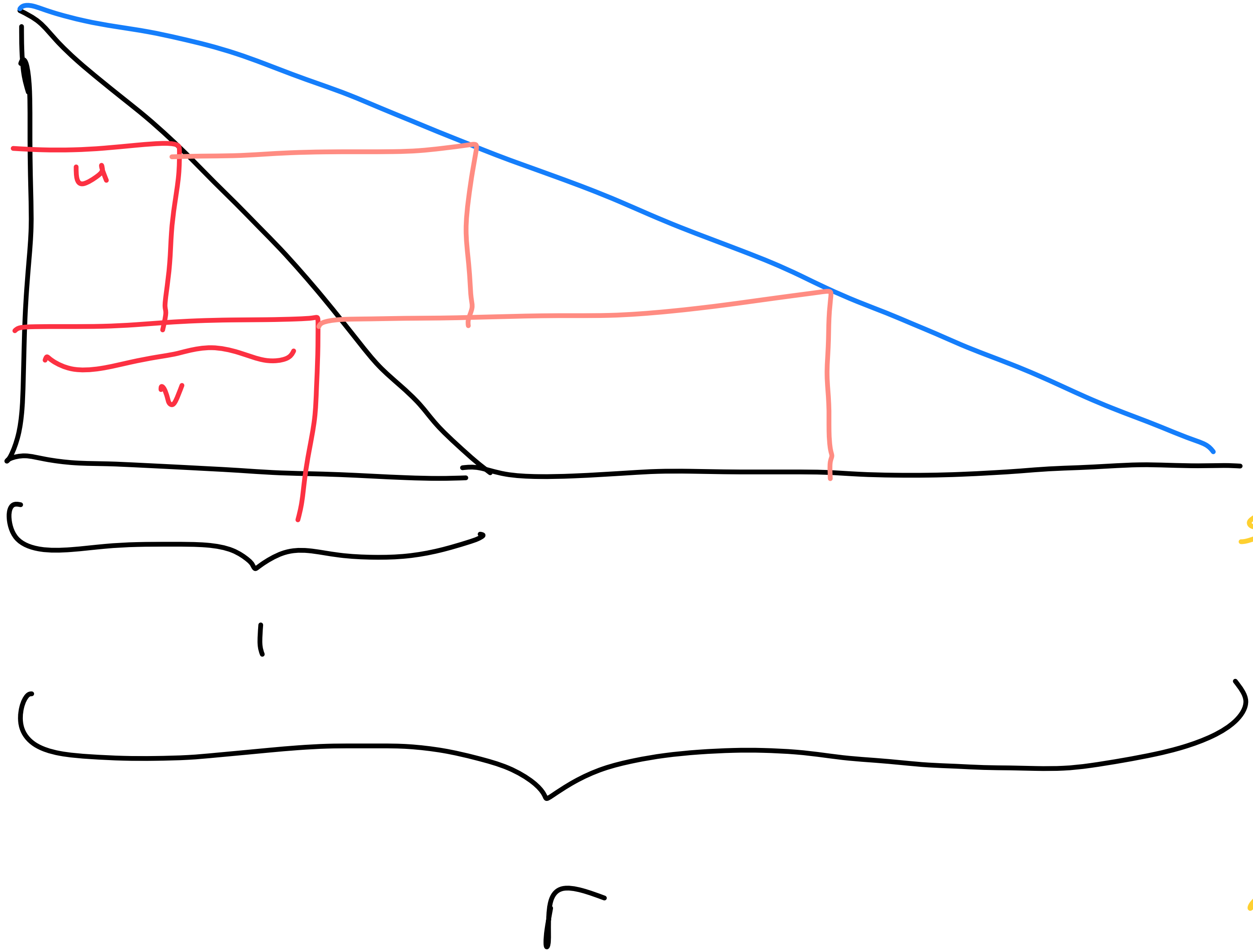
$$\text{or } y = x + 1/2$$

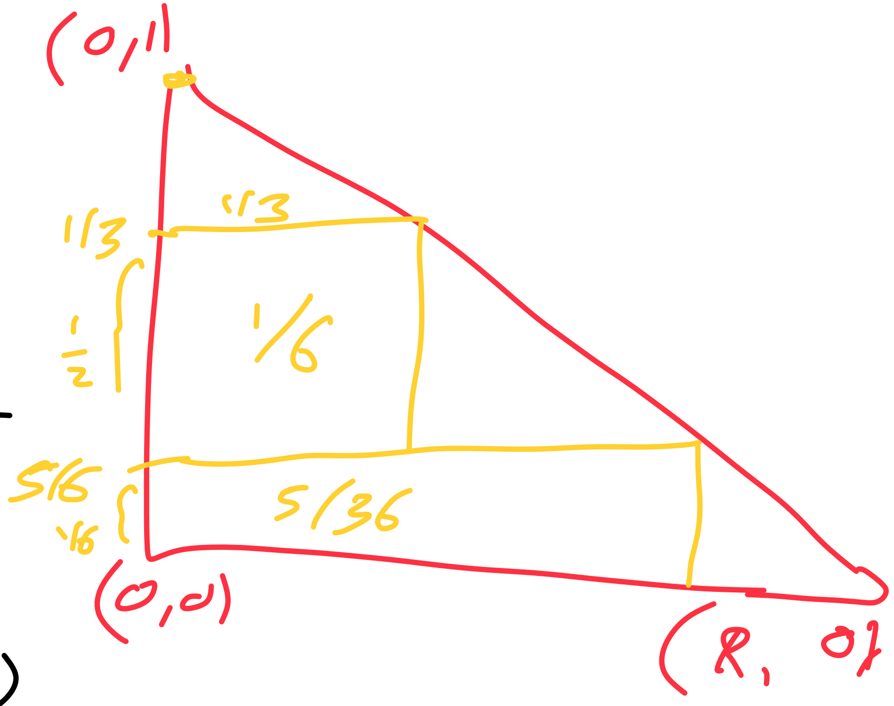
$$A(x, x+\tfrac{1}{2}) = x(\tfrac{1}{2}) + \tfrac{1}{2}(\tfrac{1}{2} - x) = \tfrac{1}{4}$$





Algebra mistake.
Wrong function.
See previous page.





Note $\frac{1}{6}+\frac{5}{36}=\frac{11}{36}\neq\frac{1}{4}$

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Steven J Miller - Williams College - sjm1@williams.edu

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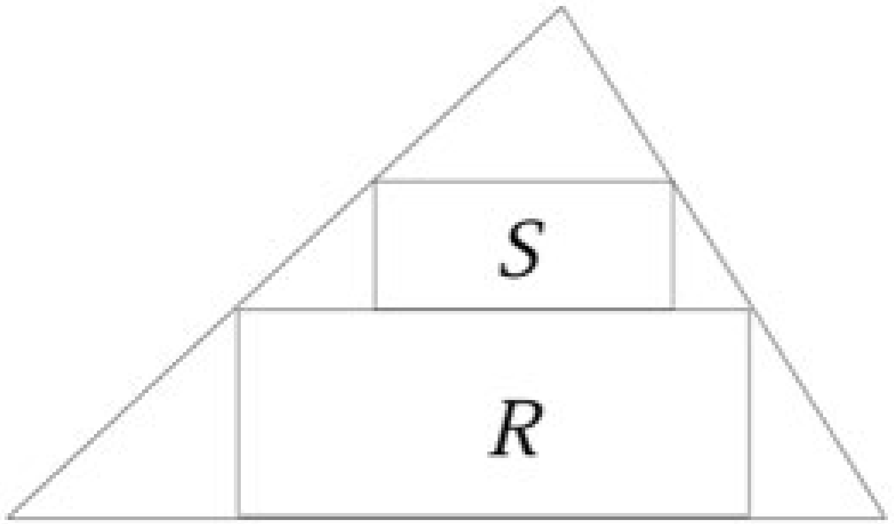
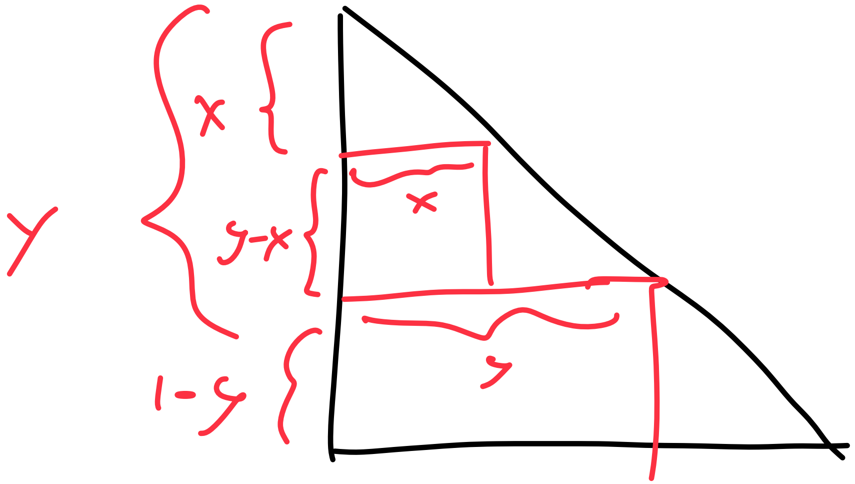
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 32: November 20: Putnam A1 Problems: https://youtu.be/1SlsUek0GqU

https://www.math.uh.edu/~torok/Putnam/problems_A1.pdf

Let T be a triangle and R, S rectangles inscribed in T as shown:





Find the maximum value, or show that no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)},$$

where T ranges over all triangles and R, S over all rectangles as above, and where A denotes area.

$$A(x,y) = x(y-x) + y(1-y)$$
$$= xy - x^2 + y - y^2$$

$$\frac{\partial A}{\partial y} = x + 1 - 2y$$

so $\frac{\partial A}{\partial y} = 0 \Rightarrow y = \frac{x+1}{2}$

Passes small test: $x \le y \le 1$

note $\frac{x+1}{2} \geq x$

$$\left.\begin{aligned} A(x,y) &= x(y-x) + y(1-y) \\ &= \underbrace{xy - x^2} + y - y^2 \end{aligned}\right\} \text{largest when } y = \frac{x+1}{2}$$

$$y - x = \frac{x+1}{2} - x = \frac{1-x}{2}$$

$$\text{Area}(x) = A\left(x, \frac{x+1}{2}\right) = \underbrace{x\,\overbrace{\frac{1-x}{2}}^{y-x}}_{\frac{x}{2} - \frac{x^2}{2}} - \frac{x^2}{2} + \frac{x+1}{2} - \frac{(x+1)^2}{4}$$

$$\text{so } \text{Area}'(x) = \frac{1}{2} - \frac{1}{2}x - x + \frac{1}{2} - \frac{1}{2}(x+1)$$

$$\text{Area}'(x) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2}x - x - \frac{1}{2}x$$

$$\text{Area}'(x) = \frac{1}{2} -$$

$A(x,y) = x(y-x) + y(1-y)$

$= xy - x^2 + y - y^2$ and $y = \frac{x+1}{2}$

$\text{Area}(x) = A\left(x, \frac{x+1}{2}\right) = x\left(\frac{x+1}{2}\right) - x^2 + \frac{x+1}{2} - \left(\frac{x+1}{2}\right)^2$

$\text{Area}(x) = \frac{x^2}{2} + \frac{x}{2} - x^2 + \frac{x}{2} + \frac{1}{2} - \frac{x^2+2x+1}{4}$

$= -\frac{x^2}{2} + x + \frac{1}{2} - \frac{x^2}{4} - \frac{x}{2} - \frac{1}{4}$

$= -\frac{3}{4}x^2 + \frac{x}{2} + \frac{1}{4}$

$\text{Area}'(x) = -\frac{3}{2}x + \frac{1}{2}$ So $\text{Area}'(x) = 0 \to \frac{3}{2}x = \frac{1}{2}$

or $x = 1/3$

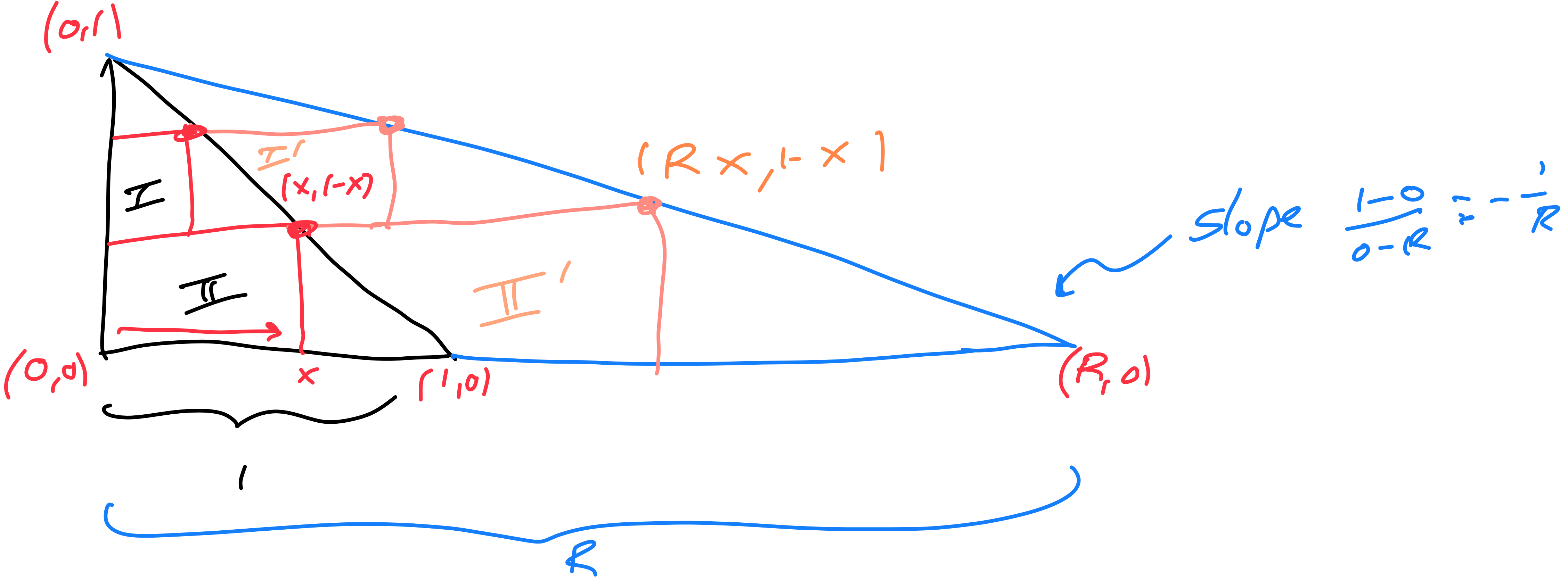
As $y = \frac{x+1}{2}$ Get $y = \frac{\frac{1}{3}+1}{2} = \frac{4/3}{2} = 2/3$

Go down $1/3$rd, Go down another $1/3$rd

$A\left(\frac{1}{3}, \frac{2}{3}\right)$

$= \frac{1}{3}\frac{1}{3} + \frac{2}{3}\frac{1}{3}$

$= \frac{1}{3}$

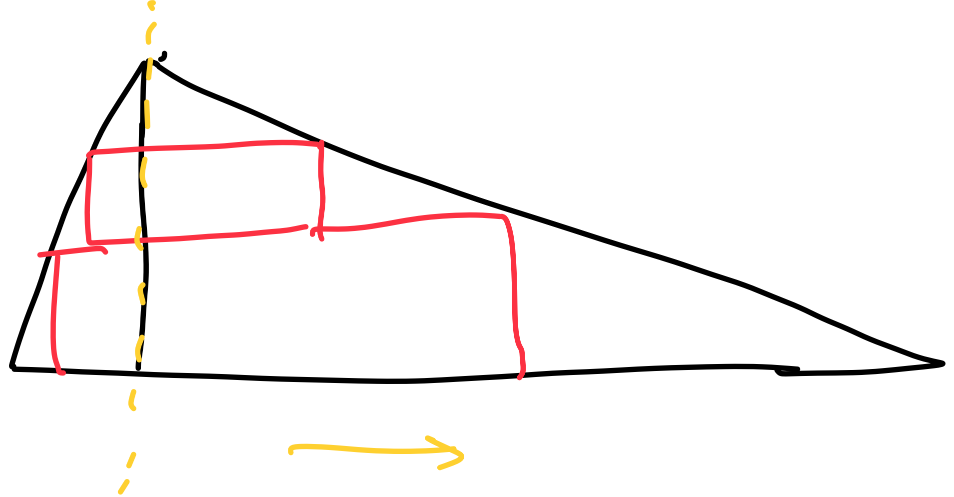


Eq black line: $y-1 = -1(x-0)$ or $y = -x+1$ or $x = 1-y$

Eq blue line: $y-1 = -\frac{1}{R}(x-0)$ or $y = -\frac{1}{R}x+1$ or $x = R-Ry$

"x" $= R - R(1-x)$
$= R - R + Rx$

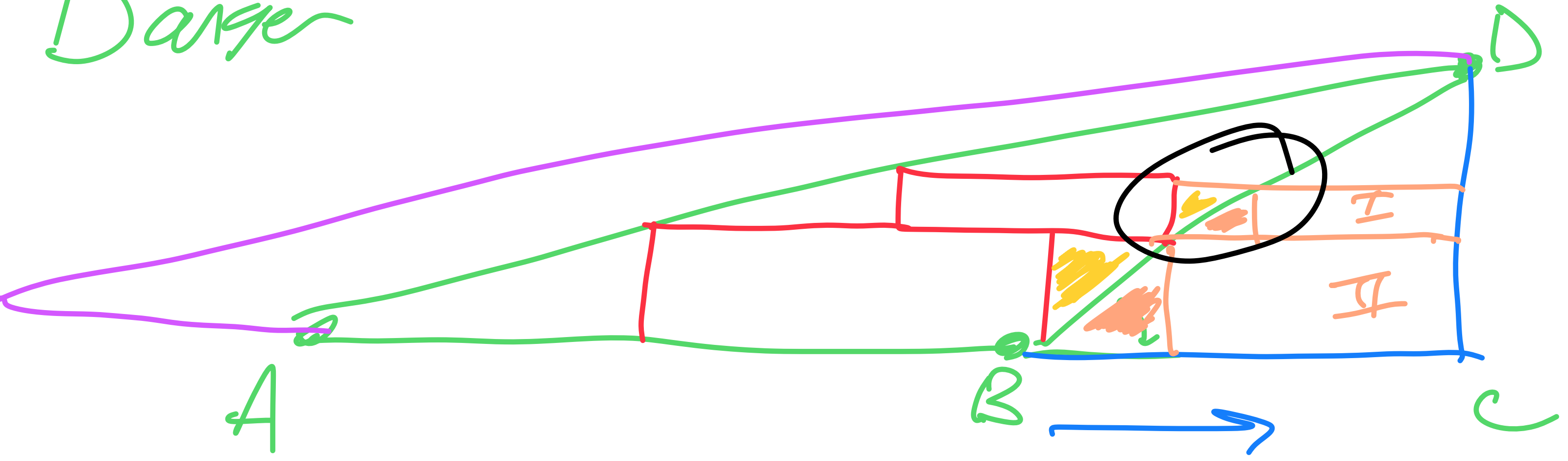
$$\frac{A(I)+A(II)}{A(T)} = \frac{A(I')+A(II')}{A(T')}$$



look at right half
can "rescale" to isosceles at
so 1/3, 2/3 for rectangles

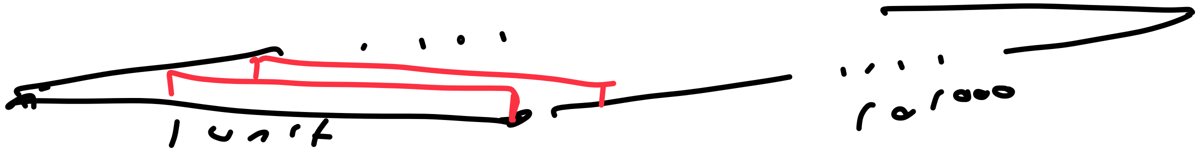
Same for left!

Danger



Can do $\triangle ACD$, ABCD

$\text{Area}(I) + \text{Area}(II) = \frac{1}{3} \text{Area } ABCD$



Math 331: The little Questions (Fall 2024)

Steven J Miller - Williams College - sjm1@williams.edu

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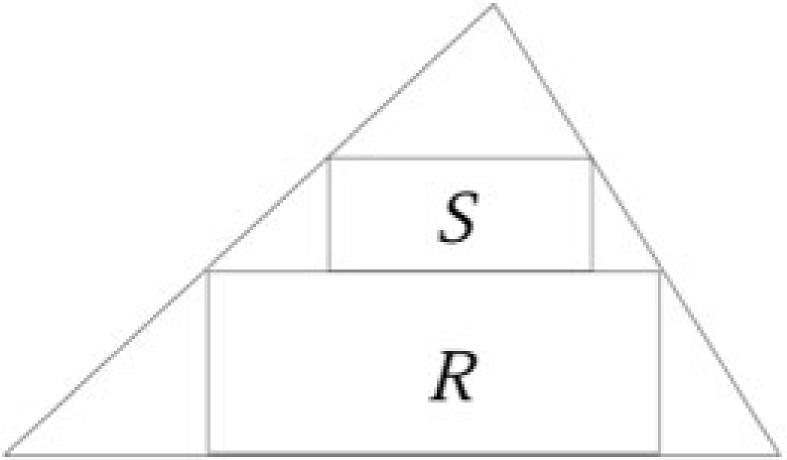
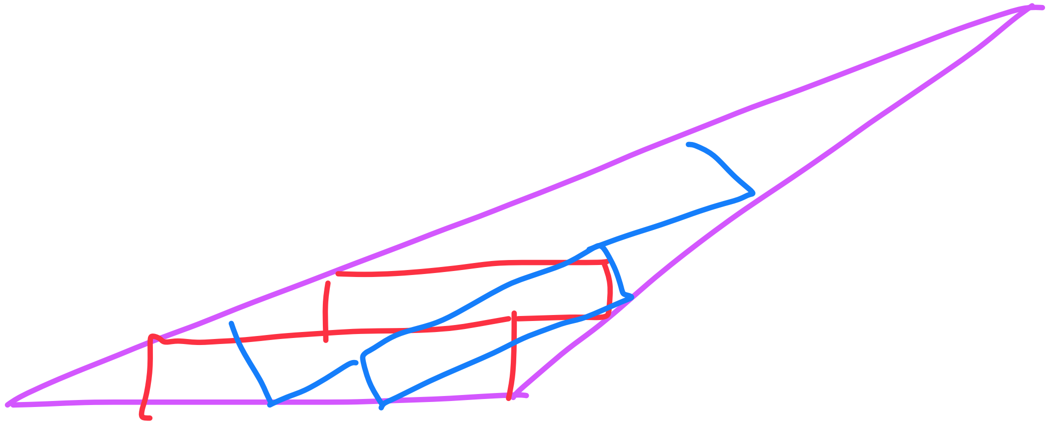
Lecture 33: November 22: Putnam A1 Problems: https://youtu.be/aL2oOxoKZPY

https://www.math.uh.edu/~torok/Putnam/problems_A1.pdf

Let T be a triangle and R, S rectangles inscribed in T as shown:



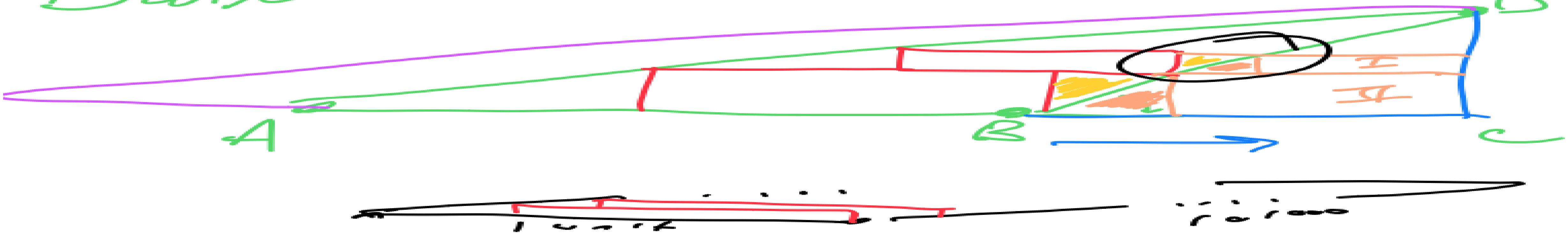


Find the maximum value, or show that no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)},$$

where T ranges over all triangles and R, S over all rectangles as above, and where A denotes area.

Danger



(R,h)

(R,1)

Line from (0,0) to (R,1) is y - 0 = 1/R (x - 0) or y = x/R.

Line from (1,0) to (R,1) is y - 0 = 1/(R-1) (x -1) or y = (x - 1) / (R-1) or x = (R-1)y + 1.

Lower rectangle: vertices (a, 0), (a, a/R), (1,0), (1, a/R) for an area of (1-a) a / R.

Upper rectangle: vertices (b, a/R), (b, b/R), (1 + (R-1)a/R), (1 + (R-1)a/R, b/R) for an area of (b-a)/R * (1 + (a-b) - a/R).

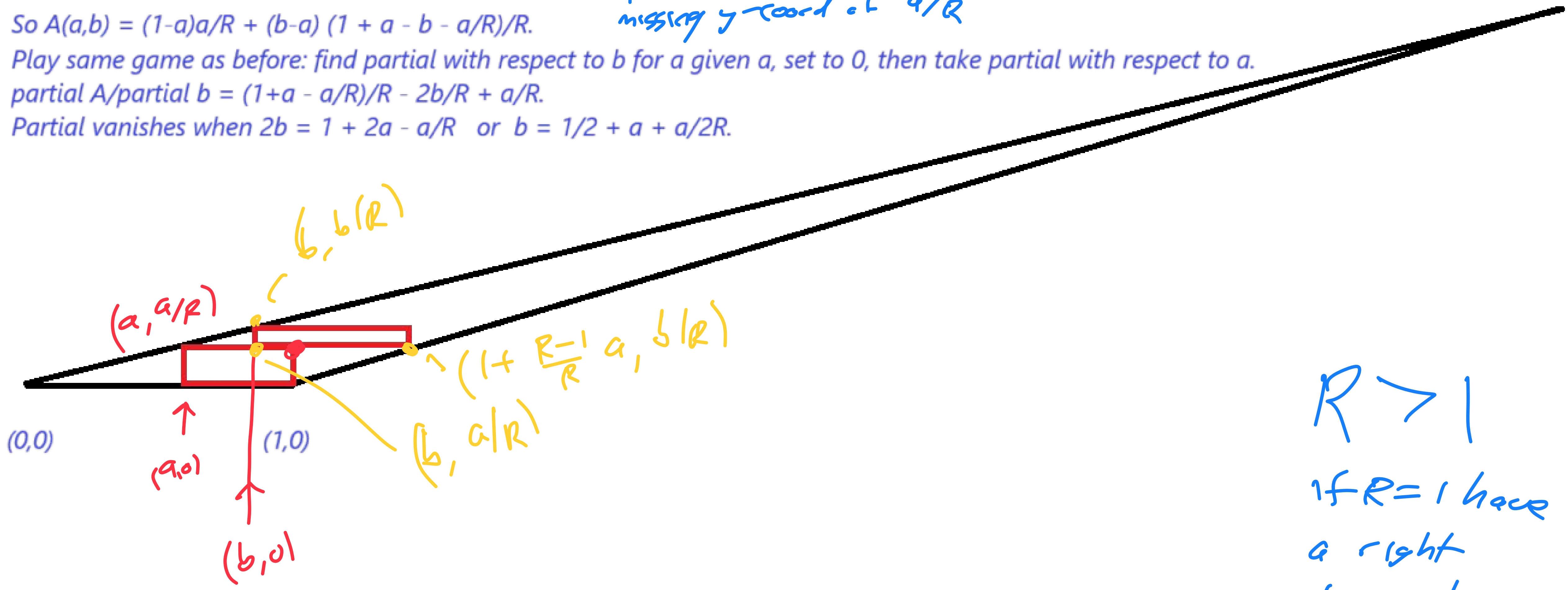
missing y-coord of a/R

So A(a,b) = (1-a)a/R + (b-a) (1 + a - b - a/R)/R.

Play same game as before: find partial with respect to b for a given a, set to 0, then take partial with respect to a.

partial A/partial b = (1+a - a/R)/R - 2b/R + a/R.

Partial vanishes when 2b = 1 + 2a - a/R   or b = 1/2 + a + a/2R.



R > 1

if R = 1 have a right triangle

```
A[a_, b_] := (1 - a) a / R + (b - a) (1 + a - b - a / R) / R
pApb[a_, b_] := D[A[a, b], b]
pApb[a, b]
Solve[pApb[a, b] == 0, b]
AREA[a_] := A[a, (-a + R + 2 a R)/(2 R)]
Simplify[AREA[a]]
dAREAa[a_] := Simplify[D[AREA[a], a]]
Solve[dAREAa[a] == 0, a]
Simplify[AREA[R/(1 + 2 R)]]
```

$$-\frac{-a+b}{R}+\frac{1+a-b-\frac{a}{R}}{R}$$

$$\left\{\left\{b\to\frac{-a+R+2aR}{2R}\right\}\right\}$$

$$\frac{R^2+2aR(-1+2R)+a^2(1-4R^2)}{4R^3}$$

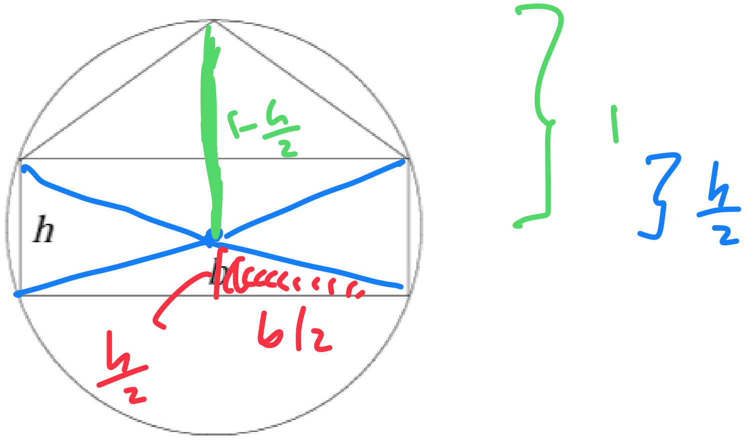
$$\left\{\left\{a\to\frac{R}{1+2R}\right\}\right\}$$

$$\frac{1}{1+2R}$$

if $R>1$, $\frac{1}{1+2R}<\frac{1}{3}$

if $R=1$, $\frac{1}{1+2R}=\frac{1}{3}$

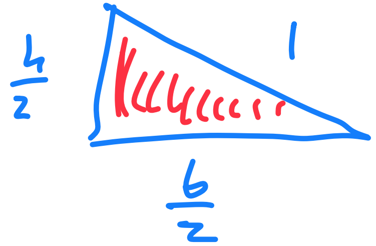
(A1 or B1 problem) Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown.



For what value of h do the rectangle and triangle have the same area?

wlog, radius $= 1$

why is b ignored?



so $\left(\frac{h}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = 1$

or $h^2 = 4 - b^2$

or $b^2 = 4 - h^2$

Area Rectangle: $bh = h\sqrt{4-h^2}$

(Maybe Area-squared is better)

Area Triangle: $\frac{1}{2} b \cdot \left(1 - \frac{h}{2}\right) \Rightarrow (\text{Area Triangle})^2 = \frac{1}{4} b^2 \left(1 - \frac{h}{2}\right)^2$

$(\text{Area Rectangle})^2 = h^2(4-h^2)$ $\quad$ $(\text{Area Triangle})^2 = \frac{1}{4}(4-h^2)\frac{(2-h)^2}{4}$

$(\text{Area Rectangle})^2 = h^2(4-h^2)$ $\quad (\text{Area Triangle})^2 = \frac{1}{4}(4-h^2)\frac{(2-h)^2}{4}$

Get $h^2(4-\cancel{h^2}) = \frac{1}{16}(4-\cancel{h^2})(2-h)^2$

Take square-root: $h = \frac{1}{4}(2-h)$ ignore $\pm$ as $0<h<2$

so $4h = 2-h$

or $5h = 2 \rightarrow h = 2/5$

CHECK! IS Area(Δ) = Area(Rectangle)?

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Steven J Miller - Williams College - sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 34: November 25: Putnam A1 Problems: https://youtu.be/FV2eWA71nMA

https://www.math.uh.edu/~torok/Putnam/problems_A1.pdf

(A1 or B1 problem) Let B be an $a \times b \times c$ brick. Let C_1 be the set of all points p in $\mathbb{R}^3$ such that the distance from p to C (i.e., the minimum distance between p and a point of C) is at most one. Find the volume of C_1.

Let R be a ring (not necessarily with identity). Suppose that there exists a nonzero element x of R satisfying

$$x^4 + x = 2x^3.$$

Prove or disprove: There exists a nonzero element y of R satisfying $y^2 = y$.

Let

$$S = \sum \frac{1}{m^2n^2},$$

where the sum ranges over all pairs (m, n) of positive integers such that the largest power of 2 dividing m is different from the largest power of 2 dividing n. Express S in the form $\alpha\pi^k$, where k is an integer and α is rational. You may assume the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

https://www.youtube.com/watch?v=d-o3eB9sfls

Define $\int f$ by $\int f$ is $\int_0^1 f(t)\,dt$

Find f st $\int f = f(x) - 1$

Intuition: $f(x)$ is related to $\exp(x)$

Note if $f(x) = e^x$ then $\int f$ is $\int_0^x f(t)\,dt = \int_0^x e^t\,dt = e^x - e^0$

YES!

Questions: Anything else?

Prove it!

"Proof" Solve $\int f = f(x) - 1$ or $\int f = f - 1$

Same as $f - \int f = 1$

Same as $(1 - \int) f = 1$

Same as $f = (1 - \int)^{-1} 1$

Aside: $(1-\int)^{-1}$ Think of as $\frac{1}{1-\int} = 1 + \int + \int\int + \int\int\int + \dots$

Recall $\frac{1}{1-x} = 1 + x + x^2 + \dots$

Thus $f(x) = (1 + \int + \int\int + \int\int\int + \dots) 1$

$\int_0^x t\,dt$ $\int_0^x \frac{t^2}{2}\,dt$

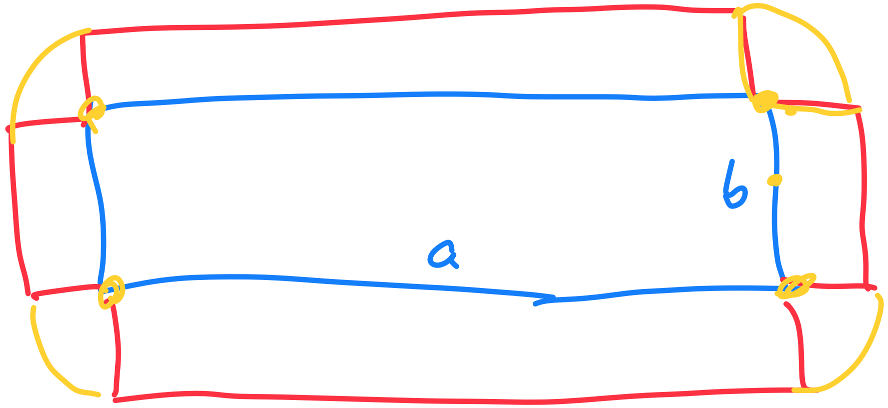
or $f(x) = 1 + \int 1 + \int[\int 1] + \int[\int[\int 1]] + \dots$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$(1-A)(1 + A + A^2 + A^3 + A^4 + \dots) = 1$

(A1 or B1 problem) Let B be an $a \times b \times c$ brick. Let C_1 be the set of all points p in $\mathbb{R}^3$ such that the distance from p to C (i.e., the minimum distance between p and a point of C) is at most one. Find the volume of C_1.

Try 2-dim



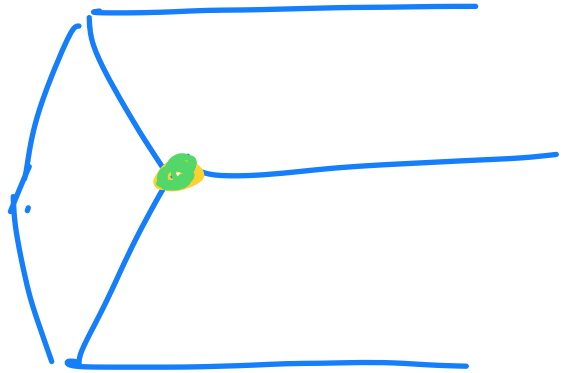
4 quarter circles of radius 1 so area $\pi 1^2 = \pi$

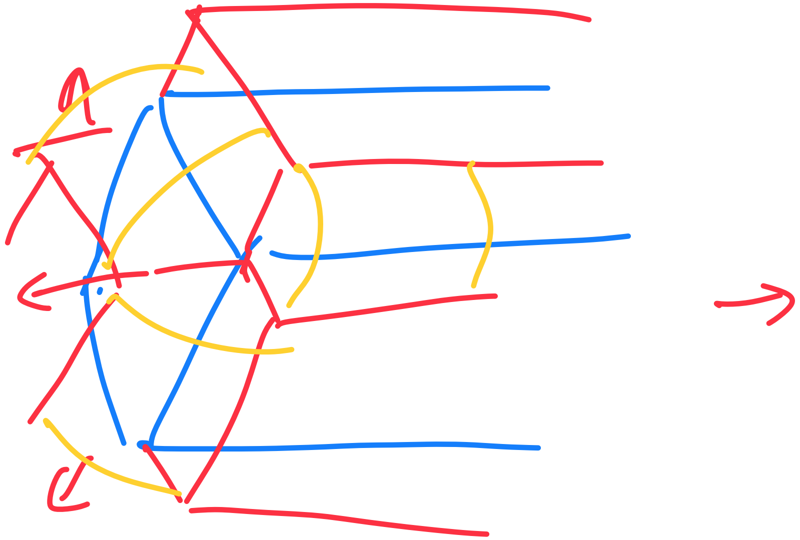
Area rectangles: $2(a*1 + b*1) = 2(a+b)$

answer MUST be symmetric in a, b, c

not in box

not in cylinders $\frac{4}{3}\pi r^3$





3 sets of 4 quarter cylinders

$\pi r^2 a + \pi r^2 b + \pi r^2 c$

Boxes get

$2(ab*1 + ac*1 + bc*1)$

Shows volume as 3 factors

$$\frac{4}{3}\pi + \pi(a+b+c) + 2(ab+ac+bc)$$

Let

$$S = \sum \frac{1}{m^2 n^2},$$

where the sum ranges over all pairs (m, n) of positive integers such that the largest power of 2 dividing m is different from the largest power of 2 dividing n. Express S in the form $\alpha\pi^k$, where k is an integer and α is rational. You may assume the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Remove different condition

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2 n^2}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \left(\frac{\pi^2}{6}\right)^2$$

sum with different = sum with no constraint − sum with same

Think inclusion exclusion

$$\sum_{\text{different}} \frac{1}{m^2 n^2} = \sum_{\text{no constraint}} \frac{1}{m^2 n^2} - \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(2k_1+1)^2 (2k_2+1)^2 2^{2l}}$$

$n = (2k_1+1)2^l \quad m = (2k_2+1)2^l$

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(2k_1+1)^2 (2k_2+1)^2 2^{2l}}$$

$$= \underbrace{\left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \right)^2}_{\text{Given } \sum \frac{1}{n^2} = \frac{\pi^2}{6}} \left(\sum_{l=0}^{\infty} \frac{1}{4^l} \right)$$

Geometric Series $\Rightarrow$ $\frac{1}{1-1/4} = \frac{4}{3}$

Reduced to $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$, know $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \underbrace{\sum_{k=1}^{\infty} \frac{1}{(2k)^2}}_{} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

If we can figure this out, we know the sum of the odd reciprocals squared

But $\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{4} \frac{\pi^2}{6}$

Sum over odds is $\frac{3}{4} \frac{\pi^2}{6}$ or $\frac{3\pi^2}{24}$

Math 331: The little Questions (Fall 2024)

Steven J Miller - Williams College - sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 35: December 2: 2024 & Putnam A1 Problems:
https://youtu.be/bG3CPgrCGv4

https://www.math.uh.edu/~torok/Putnam/problems_A1.pdf

Let R be a ring (not necessarily with identity). Suppose that there exists a nonzero element x of R satisfying

$$x^4 + x = 2x^3.$$

Prove or disprove: There exists a nonzero element y of R satisfying $y^2 = y$.

$2024 = 2^3 \cdot 11 \cdot 23$
2024 has 16 divisors (see below), whose sum is σ = 4320. Its totient is ϕ = 880.
The previous prime is 2017. The next prime is 2027. The reversal of 2024 is 4202.
$2024 = T_1 + T_2 + \ldots + T_{22}$.
$2024 = 2^3 + 3^3 + \ldots + 9^3$.
It is a Cunningham number, because it is equal to 45^2-1.
2024 is a nontrivial binomial coefficient, being equal to C(24, 3).
Together with 2295 it forms a betrothed pair.
It is a Harshad number since it is a multiple of its sum of digits (8).
It is a plaindrome in base 9 and base 15.
It is not an unprimeable number, because it can be changed into a prime (2027) by changing a digit.
2024 is an untouchable number, because it is not equal to the sum of proper divisors of any number.
It is the 22-nd tetrahedral number.
It is a pernicious number, because its binary representation contains a prime number (7) of ones.
It is a polite number, since it can be written in 3 ways as a sum of consecutive naturals, for example, 77 + ... + 99.
It is an arithmetic number, because the mean of its divisors is an integer number (270).
2^{2024} is an apocalyptic number.
It is an amenable number.
It is a practical number, because each smaller number is the sum of distinct divisors of 2024, and also a Zumkeller number, because its divisors can be partitioned in two sets with the same sum (2160).
2024 is an abundant number, since it is smaller than the sum of its proper divisors (2296).

https://www.numbersaplenty.com/2024

2024 is a wasteful number, since it uses less digits than its factorization.
2024 is an odious number, because the sum of its binary digits is odd.
The sum of its prime factors is 40 (or 36 counting only the distinct ones).
The product of its (nonzero) digits is 16, while the sum is 8.
The square root of 2024 is about 44.9888875168. The cubic root of 2024 is about 12.6494070868.
Adding to 2024 its reverse (4202), we get a palindrome (6226).

https://www.numbersaplenty.com/2024

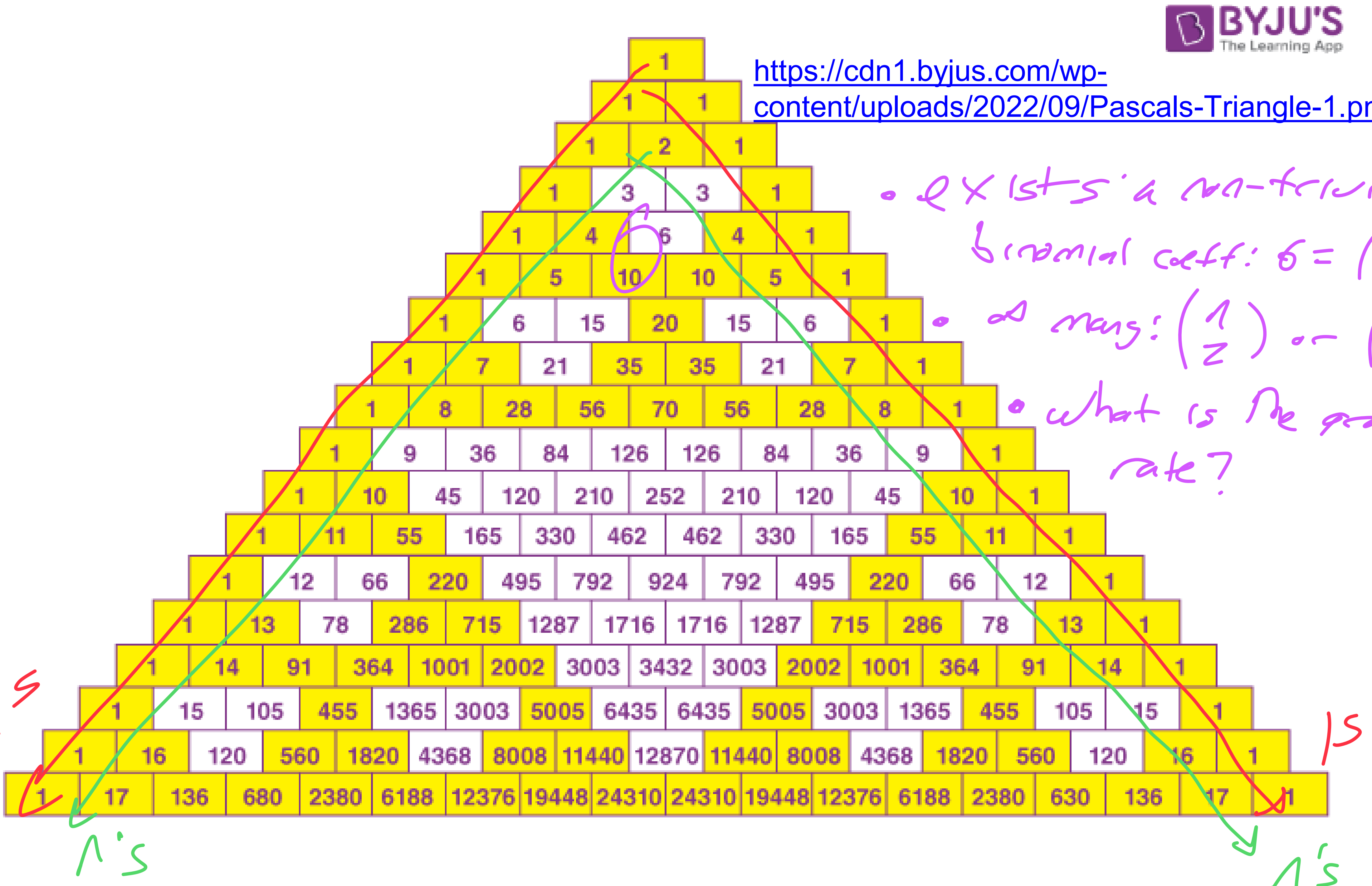
Good Questions to always ask:

(1) do examples exist?

(2) if yes, infinitely many?

(3) if yes, growth rate?

(4) can you give an explicit formula?



https://cdn1.byjus.com/wp-content/uploads/2022/09/Pascals-Triangle-1.png

- exists a non-trivial binomial coeff: $6 = \binom{4}{2}$
- as many: $\binom{n}{2}$ or $\binom{2n}{n}$
- what is the growth rate?

```
findnumbersnotnontrivialbincoeff[max_] := Module[{},
 list = {};
 For[n = 4, n <= max, n++,
  {
   number = n;
   For[k = 1,  k <=  n/2 - 1, k++,
    {
     number = number * (n-k)/(k+1);
     If[MemberQ[list, number] == False,
          list = AppendTo[list, number]];
     }
    ] (* end of k loop *)
   }]; (* end of n loop *)
 list = Sort[list];
 Print[list];
 ]
(* n!/k!(n-k)!
   n!/(k+1)!(n-k-1)!
   = n!/k! (n-k)!  * (n-k)/(k+1) *)
```

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{k+1} \frac{n-k}{k+1}$$

```
findnumbersnotnontrivialbincoeff[25]

{6, 10, 15, 20, 21, 28, 35, 36, 45, 55, 56, 66, 70,
 78, 84, 91, 105, 120, 126, 136, 153, 165, 171, 190,
 210, 220, 231, 252, 253, 276, 286, 300, 330, 364,
 455, 462, 495, 560, 680, 715, 792, 816, 924, 969,
 1001, 1140, 1287, 1330, 1365, 1540, 1716, 1771,
 1820, 2002, 2024, 2300, 2380, 3003, 3060, 3432,
 3876, 4368, 4845, 5005, 5985, 6188, 6435, 7315,
 8008, 8568, 8855, 10 626, 11 440, 11 628, 12 376,
 12 650, 12 870, 15 504, 18 564, 19 448, 20 349, 24 310,
 26 334, 27 132, 31 824, 33 649, 38 760, 42 504, 43 758,
 48 620, 50 388, 53 130, 54 264, 74 613, 75 582,
 77 520, 92 378, 100 947, 116 280, 125 970, 134 596,
 167 960, 170 544, 177 100, 184 756, 203 490, 245 157,
 293 930, 319 770, 346 104, 352 716, 480 700, 490 314,
 497 420, 646 646, 705 432, 735 471, 817 190, 1 081 575,
 1 144 066, 1 307 504, 1 352 078, 1 961 256, 2 042 975,
 2 496 144, 2 704 156, 3 268 760, 4 457 400, 5 200 300}
```

https://www.scientificamerican.com/article/mathematicians-newest-assistants-are-artificially-intelligent/?utm_source=linkedin&utm_medium=social&utm_campaign=socialflow&fbclid=IwZXh0bgNhZW0CMTEAAR1e0OjknJRjgf5uN1RWjO3mKn0XlfdRlHFJaqpB8LQB22aZNu-AiJUHoDA_aem_8XV-0ansL7wERgukGbDd6Q

Recall n is polite if $n = k + (k+1) + \cdots + k+(\ell-1)$ with $\ell \geq 2$.

What % of integers $\leq x$ are polite?

Gather data (computers)

Look at differences of Triangular Numbers:

$$T_n = 1+2+\cdots+n = \frac{n(n+1)}{2}$$

$$T_n = n+(n-1)+\cdots+1$$

$$2T_n = \underbrace{(n+1)+\cdots+(n+1)}_{n \text{ times}} = n(n+1) \to T_n = \frac{n(n+1)}{2}$$

If p is polite then $p = T_n - T_m$ with $n \geq m+2$

Looking at $T_n - T_m$ or $\frac{n(n+1)}{2} - \frac{m(m+1)}{2}$; $n \geq m+1$

How big is $T_n - T_{n-1}$?

$$\frac{n(n+1)}{2} - \frac{(n-1)\,n}{2} = \frac{n}{2}\left((n+1) - (n-1)\right) = n$$

To see if 7 is polite, check finite or ∞ many pairs?

↳ ok to check $m \leq 7$

Claim is m is "at least/most" something - . . . .

Want $T_n - T_m \leq X$ with $n = m+k,\ k \geq 1$

if $m=1,\ n: 2 \to ?$, need $\frac{n(n+1)}{2} - 1 \leq x$

Shows if $m=1$

$n: 2 \to$ appear $\sqrt{2x}$

$$\left\{\begin{array}{l}
\text{so } n(n+1) \leq 2(x+1) \\
\text{so } n^2 \leq n(n+1) \leq 2(x+1) \\
\text{Thus } n \leq \sqrt{2(x+1)} \\
\text{Could do } n(n+1) = (n+\tfrac{1}{2})^2 - \tfrac{1}{4} \\
\text{so } (n+\tfrac{1}{2})^2 \leq 2(x+1) + \tfrac{1}{4} \\
(n+\tfrac{1}{2})^2 \leq 2x + \tfrac{9}{4} \\
n \leq \sqrt{2x + 9/4} - \tfrac{1}{2}
\end{array}\right.$$

If $m=2$ then $n:3 \rightarrow ?$

General Case: if $m=m$ then $n: m+1 \rightarrow ?$

$$\frac{n(n+1)}{2} - \frac{m(m+1)}{2} \leq x$$

so $n(n+1) - m(m+1) \leq 2x$

$$\left[\left(n+\frac{1}{2}\right)^2 - \frac{1}{4}\right] - \left[\left(m+\frac{1}{2}\right)^2 - \frac{1}{4}\right] \leq 2x$$

$$\left(n+\frac{1}{2}\right)^2 \leq 2x + \left(m+\frac{1}{2}\right)^2$$

$$\text{so } n \leq \underbrace{\sqrt{2x+(m+1/2)^2}}_{\text{at least } m+1/2} - \frac{1}{2}$$

want $1+\dots+k=m$

and $k=x-1$

so $m=\frac{(x-1)x}{2}$

of n is $\sqrt{2x+(m+\frac{1}{2})^2} - \frac{1}{2} - m$ and $1 \leq m \leq \frac{(x-1)x}{2}$

Math 331: The little Questions (Fall 2024)

Steven J Miller - Williams College - sjm1@williams.edu

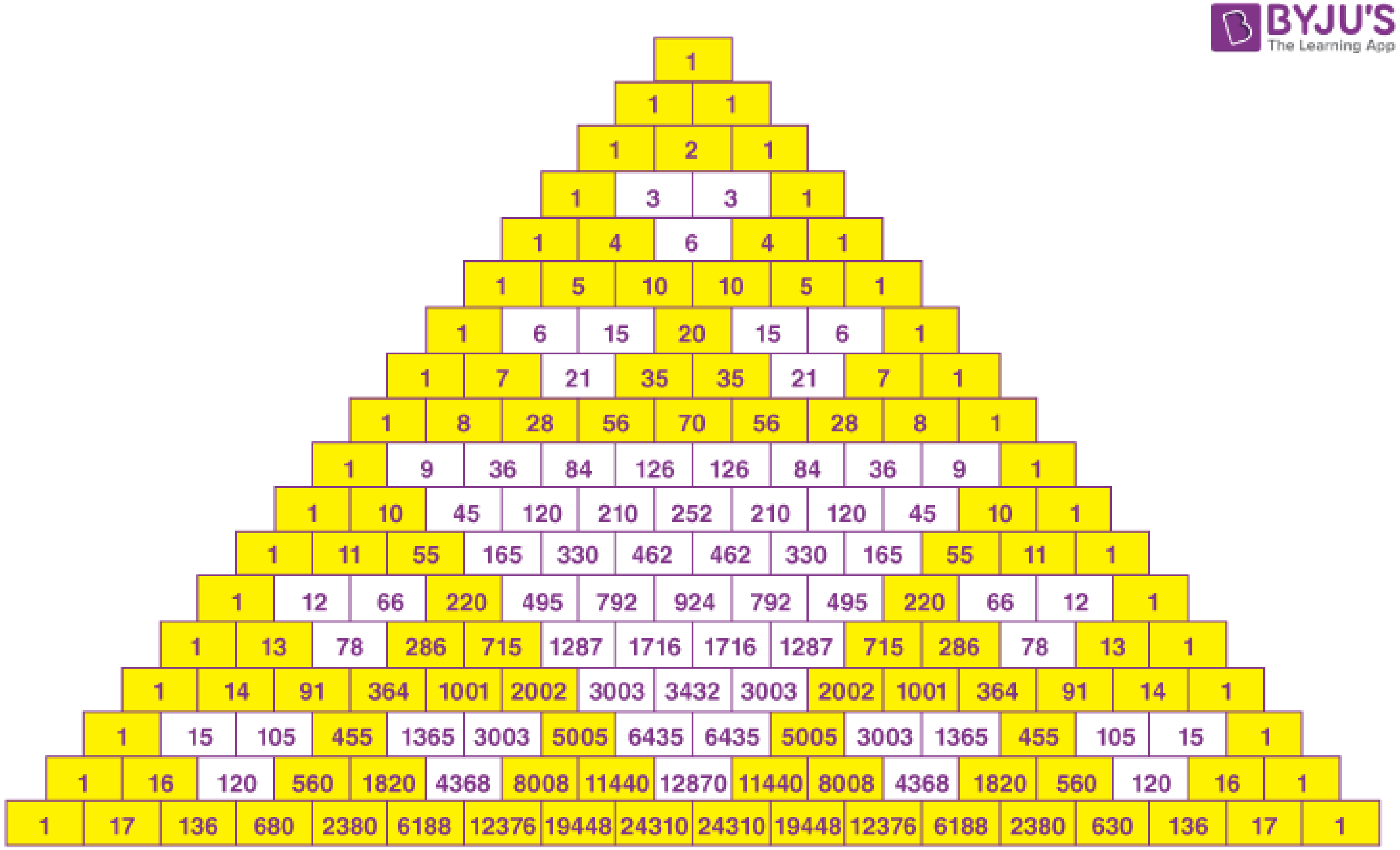
http://www.williams.edu/Mathematics/sjmiller/public_html/331Fa24
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math331fa24slides.pdf

Lecture 36: December 4: 2024 & Putnam A1 Problems:
https://youtu.be/y71SFoiPcqI

https://www.math.uh.edu/~torok/Putnam/problems_A1.pdf

2024: is a polite number, since it can be written in 3 ways as a sum of consecutive naturals, for example, 77 + ... + 99.

2024 is a nontrivial binomial coefficient, being equal to C(24, 3).



https://cdn1.byjus.com/wp-content/uploads/2022/09/Pascals-Triangle-1.png

Rather than fix n, fix # of consec summands
Want to study polite numbers at most x.

1) of 2 summands: $1+2$, $2+3$, $\ldots$, $\left(\frac{x}{2}-1\right)+\frac{x}{2}$

↳ roughly $\frac{x}{2}$ pairs

2) of 3 summands: $1+2+3$, $2+3+4$, $\ldots$, $\left(\frac{x}{3}-2\right)+\left(\frac{x}{3}-1\right)+\frac{x}{3}$

↳ roughly $\frac{x}{3}$ triples

3) of 4 summands: $1+2+3+4$, $\ldots$, $\left(\frac{x}{4}-3\right)+\left(\frac{x}{4}-2\right)+\left(\frac{x}{4}-1\right)+\frac{x}{4}$

↳ roughly $\frac{x}{4}$ quadruples

note this gives $\simeq \frac{x}{2}+\frac{x}{3}+\frac{x}{4} > x$ tuples: more than all integers at most x: clearly some collisions

Claim: Powers of 2 are never polite

$T_n - T_m$ with $n \geq m+2$ so $n = m+k$

$$\frac{n(n+1)}{2} - \frac{m(m+1)}{2} = x = \frac{(m+k)(m+k+1)}{2} - \frac{m(m+1)}{2}$$

$$\left[\left(n+\frac{1}{2}\right)^2 - \frac{1}{4}\right] - \left[\left(m+\frac{1}{2}\right)^2 - \frac{1}{4}\right] = 2x = (m+k)(m+k+1) - m(m+1)$$

$$\left(n+\frac{1}{2}\right)^2 - \left(m+\frac{1}{2}\right)^2 = 2x = \text{you do this side}$$

$$(n-m)(n+m+1) = 2x$$

parity of $n-m$ same as of $n+m$

so one even, one is odd

n	m	$n-m$	$n+m+1$
3	1	2	5
4	1	3	6
7	2	5	10
8	4	4	13
8	5	3	14

$(n+k)(n+k+1) - n(n+1)$

$= \cancel{n^2} + 2kn + k^2 + \cancel{n} + k - \cancel{n^2} - \cancel{n}$

$= 2kn + k^2 + k$

$= k(2n + k + 1)$

k and $k+1$ have opposite parities

one term/factor odd, one is even

Solve (if possible) $(n-m)(n+m+1) = 2x$

If x is a power of 2, say $x = 2^l$

Then $(n-m)(n+m+1) = 2^l$

odd * even $= 2^l$ assume $l \geq 1$

need one of these, the odd, to be 1

must be $n-m$ as $n-m$ is smaller

so $n-m = 1 \rightarrow n = m+1$ contradiction

(remember $n \geq m+2$): Powers of 2 are impolite

$10 = 4+3+2+1$ $11 = 6+5$ $12 = 5+4+3$ $9 = 5+4$

$13 = 7+6$ $14 = 5+4+3+2$ $15 = 5+4+3+2+1$

Lemma: All odds are polite

Proof: $x = 2r+1 = (r+1)+r$ $\blacksquare$

Only need to study even #s not powers of 2

$(n-m)(n+m+1)$ or $k(2m+k+1) = 2 \cdot x$

$x = 2^l(2r+1)$ with $l \geq 1$, $r \geq 1$

↳ already did $l=0$ (odds) and $r=0$ (powers of 2)

Solve / find n, m or m, k such that

$(n-m)(n+m+1) = k(2m+k+1) = 2^l(2r+1)$

Solve $k(2m+k+1) = 2^l(2r+1)$

↳ Can we do $k = 2^l$ and $2m+k+1 = 2r+1$?

↳ Can we do $k = 2r+1$ and $2m+k+1 = 2^l$

① $k = 2^\ell$ and $2m+k+1 = 2r+1$?

Then $2m+k+1 = 2m+2^\ell+1$ must equal $2r+1$

To have a chance ℓ and r have what relationship?

↳ need $2^\ell \leq 2r$

if true then $m = (2r-2^\ell)/2$ ok as $\ell \geq 1$

② $k = 2r+1$ and $2m+k+1 = 2^\ell$

Then $2m+2r+1+1 = 2^\ell$

So $2m = 2^\ell - 2r - 2$

So $m = 2^{\ell-1} - r - 1$

need $m \geq 0$

$2^{\ell-1} \geq r+1$

$2^\ell \geq 2r+2$

From ① ok if $2^\ell \leq 2r$ and from ② ok if $2^\ell \geq 2r+2$

note $2^\ell \neq 2r+1$ so done

Proof by Induction that all non-powers of 2 are polite

Gather data

3 = 2+1

5 = 3+2

6 = 3+2+1

7 = 4+3

9 = 5+4 ← 4+3+2

10 = 4+3+2+1

11 = 6+5

12 = 5+4+3

13 = 7+6

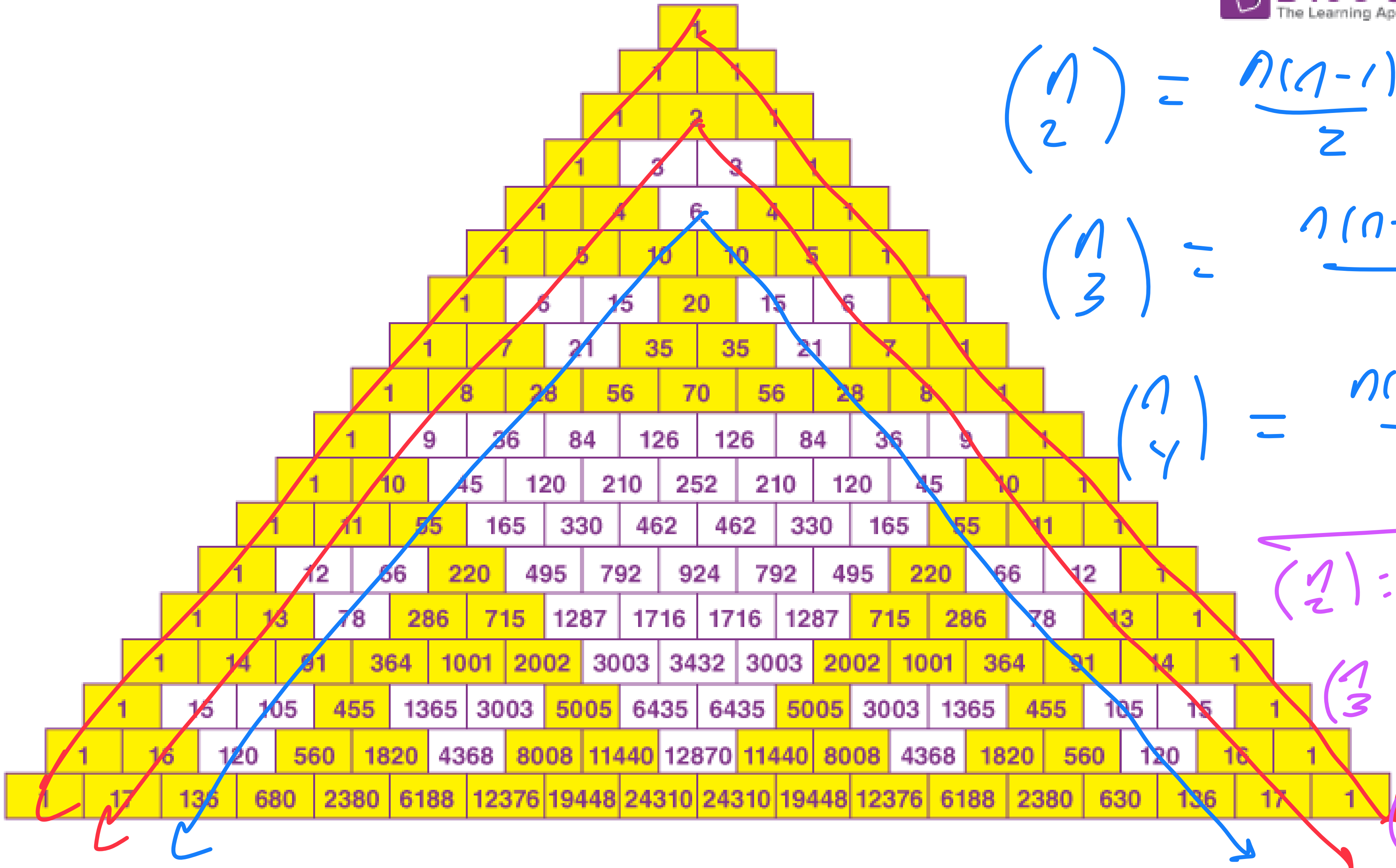
14 = 5+4+3+2

15 = 6+5+4

15 = 5+4+3+2+1

Conj: If $n = 2^l - 1$ then n is triangular

$l = 3$: $n = 7 \neq$ triangle number



$$\binom{n}{2} = \frac{n(n-1)}{2} \le x$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6} \le x$$

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24} \le x$$

$\binom{n}{2}$: $n \le$ "$\sqrt{2x}$"

$\binom{n}{3}$: $n \le$ "$\sqrt[3]{6x}$"

$\binom{n}{4}$: $n \le$ "$\sqrt[4]{24x}$"

non-trivial at most X is at most $\approx$

$$(2X)^{\frac{1}{2}} + \underbrace{(3!X)^{\frac{1}{3}} + (4!X)^{\frac{1}{4}} + (5!X)^{\frac{1}{5}} + \cdots}_{\text{size } x^{1/3}}$$

have X integers at most X

This sum $\simeq \sqrt{2x}$ + size $x^{1/3}$

Bound $\displaystyle\sum_{\substack{k \le f(x) \\ k \geq 3}} (k!\,x)^{1/k} \ll \sum_{\substack{k \geq 3 \\ k \le f(x)}} \frac{k}{e} x^{\frac{1}{k}}$

$k! \simeq k^k e^{-k} \sqrt{2\pi k}$

$(k!)^{1/k} \simeq \left(\frac{k}{e}\right) (2\pi k)^{\frac{1}{k}}$

How large can k be? Would be a "middle" term so $\binom{2n}{n} \leq X$

now $\binom{2n}{n} \leq (1+1)^{2n} = 2^{2n}$ and also its the largest of the $2n+1$ coefficients, so $\binom{2n}{n} \geq \frac{2^{2n}}{2n+1}$

Thus $\frac{1}{2n+1} 2^{2n} \leq \binom{2n}{n} \leq 2^{2n}$

If take n^{th} root get $\left(\frac{1}{2n+1}\right)^{\frac{1}{n}} 2^2 \leq \binom{2n}{n}^{\frac{1}{n}} \leq 2^2$

So $\binom{2n}{n} \sim X$ means $\frac{1}{2n+1} 2^{2n} \sim X \sim 2^{2n}$ so $n \sim \frac{1}{2} \frac{\log X}{\log 2}$

So largest index k at most $\frac{1}{2 \log 2} \log X$

So sum is at most $\approx$

$$(2!x)^{1/2} + (3!\,x)^{1/3} + \cdots + (k_{max}!\,x)^{1/k_{max}}$$

$$= \sqrt{2x} + \sum_{\substack{k \geq 3 \\ k \leq k_{max}}} (k!)^{\frac{1}{k}} x^{1/k} \quad \text{and} \quad (k!)^{k} \leq k$$

$$\leq \sqrt{2x} + \sum_{\substack{k \geq 3 \\ k \leq k_{max}}} k\,x^{1/k}$$

$$\leq \sqrt{2x} + \sqrt[3]{3}\,x^{1/3} + x^{\frac{1}{4}} \sum_{\substack{k \geq 4 \\ k \leq \frac{1}{2\log_2} \log x}} k \leq \sqrt{2}\,x^{\frac{1}{2}} + \sqrt[3]{3}\,x^{\frac{1}{3}} + x^{\frac{1}{4}} \frac{k_{max}(k_{max}+1)}{2}$$

$$\leq \sqrt{2}\,x^{\frac{1}{3}} + \sqrt[3]{3}\,x^{\frac{1}{3}} + x^{1/4} \log^2 x$$