

# **MATH 331: THE LITTLE QUESTIONS: SPRING 2017**

## **HOMEWORK SOLUTION KEY**

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ABSTRACT. A key part of any math course is doing the homework. This ranges from reading the material in the book so that you can do the problems to thinking about the problem statement, how you might go about solving it, and why some approaches work and others don't. Another important part, which is often forgotten, is how the problem fits into math. Is this a cookbook problem with made up numbers and functions to test whether or not you've mastered the basic material, or does it have important applications throughout math and industry? Below I'll try and provide some comments to place the problems and their solutions in context.

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A000124 Central polygonal numbers (the Lazy Caterer's sequence):  $n(n+1)/2 + 1$ ; or, maximal number of pieces formed when slicing a pancake with  $n$  cuts. <sup>297</sup>  
(Formerly M1041 N0391)

1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, 137, 154, 172, 191, 211, 232, 254, 277, 301, 326, 352, 379, 407, 436, 466, 497, 529, 562, 596, 631, 667, 704, 742, 781, 821, 862, 904, 947, 991, 1036, 1082, 1129, 1177, 1226, 1276, 1327, 1379 ([list](#): [graph](#): [ref](#): [listen](#): [history](#): [text](#): [internal format](#))

OFFSET 0,2

COMMENTS These are Hogben's central polygonal numbers with the (two-dimensional) symbol  $\cdot$ .  
P  
1 n  
The first line cuts the pancake into 2 pieces. For  $n > 1$ , the  $n$ -th line crosses every earlier line (avoids parallelism) and also avoids every previous line intersection, thus increasing the number of pieces by  $n$ . For 16 lines, for example, the number of pieces is  $2 + 2 + 3 + 4 + 5 + \dots + 16 = 137$ . These are the triangular numbers plus 1 (cf. [A000217](#)).

Define a number of straight lines in the plane to be in general arrangement when (1) no two lines are parallel, (2) there is no point common to three lines. Then these are the maximal numbers of regions defined by  $n$  straight lines in general arrangement in the plane. - Peter C. Heinig (algorithms(AT)gmx.de), Oct 19 2006

Note that  $a(n) = a(n-1) + \text{A000027}(n-1)$ . This has the following geometrical interpretation: Suppose there are already  $n-1$  lines in general arrangement, thus defining the maximal number of regions in the plane obtainable by  $n-1$  lines and now one more line is added in general arrangement. Then it will cut each of the  $n-1$  lines and acquire intersection points which are in general arrangement. (See the comments on [A000027](#) for general arrangement with points.) These points on the new line define the maximal number of regions in 1-space definable by  $n-1$  points, hence this is  $\text{A000027}(n-1)$ , where for [A000027](#) an offset of 0 is assumed, that is,  $\text{A000027}(n-1) = (n+1)-1 = n$ . Each of these regions acts as a dividing wall, thereby creating as many new regions in addition to the  $a(n-1)$  regions already there, hence  $a(n) = a(n-1) + \text{A000027}(n-1)$ . Cf. the comments on [A000125](#) for an analogous interpretation. - Peter C. Heinig (algorithms(AT)gmx.de), Oct 19 2006

FIGURE 1. Proof from the OEIS.

## 1. HW #2: DUE FEBRUARY 10, 2017

1.1. **Problems:** (1) Go to Project Euler (<https://projecteuler.net/>) and create an account for yourself, and solve the first problem. You do not need to submit this, just email me when done. (2) Read <http://www.math.ucla.edu/~radko/circles/lib/data/Handout-142-159.pdf> and do Problem #1: If  $n$  lines are drawn in a plane, and no two lines are parallel, how many regions do they separate the plane into? (3) Prove that  $(1 - 1/4)(1 - 1/9) \cdots (1 - 1/n^2) = (n + 1)/2n$ .

1.2. **Solutions: (2):** Read <http://www.math.ucla.edu/~radko/circles/lib/data/Handout-142-159.pdf>. **Do Problem #1: If  $n$  lines are drawn in a plane, and no two lines are parallel, how many regions do they separate the plane into?**

**Solution:** The problem is not phrased well; it implies there is a unique answer, but if they all intersect in a common point the answer is different than if they do not. One student said a good way to rephrase is that no three lines may intersect in a common point. Doing a little work, if we have  $n$  lines (starting at 0) the number of regions is 1, 2, 4, 7, 11. Plugging this into the OEIS yields <http://oeis.org/A000124> (there were other suggestions but reading them it is clear this is the one we want!).

**(3): (3) Prove that**  $(1 - 1/4)(1 - 1/9) \cdots (1 - 1/n^2) = (n + 1)/2n$ .

**Solution:** This follows by induction. Let  $P(n)$  be the statement  $(1 - 1/4)(1 - 1/9) \cdots (1 - 1/n^2) = (n + 1)/2n$ . The base case is immediate. We now assume  $P(n)$  holds and must show  $P(n + 1)$  is true. We have

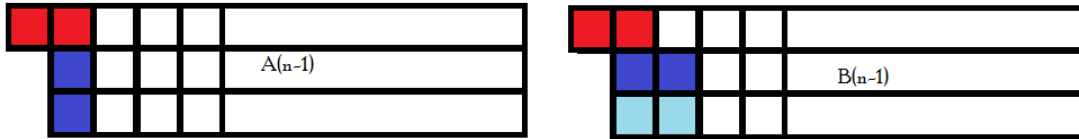
$$\begin{aligned} \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right) \left(1 - \frac{1}{(n+1)^2}\right) &= \frac{n+1}{2n} \frac{(n+1)^2 - 1}{(n+1)^2} \\ &= \frac{n^2 + 2n}{2n(n+1)} = \frac{n+2}{2n+2} = \frac{(n+1)+1}{2(n+1)}, \end{aligned}$$

as claimed, completing the proof.

Another proof (from John Fan):

$$\prod_{k=2}^n \left(1 - \frac{1}{k^2}\right) = \prod_{k=2}^n \frac{(k-1)(k+1)}{k \cdot k} = \prod_{k=2}^n \frac{1}{k_1} \prod_{k_2=2}^n \frac{1}{k_2} \prod_{k_3=2}^n (k_3 - 1) \prod_{k_4=2}^n (k_4 + 1) = \frac{1 \cdot (n+1)}{2 \cdot n}.$$

**HW #3: Due February 17, 2017: (1) Make sure you have done the first 10 problems on Project Euler. (2) How many ways are there to cover a  $3 \times n$  board using just  $1 \times 2$  tiles? (3) What if now we have a  $2 \times 2 \times n$  box and just  $1 \times 2$  tiles?**

FIGURE 2. The different configurations needed to study to find the recurrence for  $A_n$ .FIGURE 3. The different configurations needed to study to find the recurrence for  $B_n$ .

## 2. HW #3: DUE FEBRUARY 17, 2017

2.1. **Problems.** (1) Make sure you have done the first 10 problems on Project Euler. (2) How many ways are there to cover a  $3 \times n$  board using just  $1 \times 2$  tiles? (3) What if now we have a  $2 \times 2 \times n$  box and just  $1 \times 2$  tiles?

2.2. **Solutions. #2: How many ways are there to cover a  $3 \times n$  board using just  $1 \times 2$  tiles?**

**Solution:** We need to find a recurrence. Note that  $n$  has to be even, as otherwise we cannot cover as any number of  $1 \times 2$  tiles covers an even number of squares. Thus, let us assume we have a  $3 \times 2n$  board. Let  $A_n$  be the number of ways to cover a  $3 \times 2n$  board with  $1 \times 2$  tiles, and let  $B_n$  be the number of ways to cover a  $3 \times 2n$  board where in the first column (in the far left) we only have the upper left corner entry (and not the middle or bottom left corner). We find a system of recurrences.

We have (note the re-grouping is to simplify some algebra later)

$$A_n = 2A_{n-1} + A_{n-2} + B_n + B_{n-2} = 2A_{n-1} + B_n + (A_{n-2} + B_{n-2}).$$

Why? Consider the bottom left square: either it is covered with a vertical or a horizontal tile. See Figure 2.

- If it is covered by a vertical tile then we are left with just one square in the upper left corner in the first column. By definition the number of ways to cover what remains is  $B_n$ .
- If it is covered by a horizontal tile we have several options. We could have two horizontal tiles above it, which would completely cover the first two columns and leave us with a  $3 \times (2n - 2)$  board to cover; there are  $A_{n-1}$  ways to do this. We could have two vertical tiles, again completely covering the first two columns and leaving us with a  $3 \times (2n - 2)$  board to cover, which again can be done  $A_{n-1}$  ways. Finally we could have a vertical tile for the last column, and then two horizontal tiles. In that case we would need a horizontal in the bottom row (so now the bottom four squares are covered). We either now have a vertical tile completing the covering of column four (which leaves us with a  $3 \times (2n - 4)$  board, which can be covered in  $A_{n-2}$  ways), or we have two horizontal tiles and thus only the bottom element in column 5 is left in the first five columns (and by definition there are  $B_{n-2}$  ways to tile what remains).

We now need a recurrence for  $B_n$ ; see Figure 3. Clearly we must have the first tile coming in from the upper left corner. If the next tile is vertical underneath its overhang, we now have a  $3 \times 2(n - 1)$  board and the number of ways to cover that is  $A_{n-1}$ . If instead our tile is horizontal then the one below is also horizontal, and we have a region that looks like our original but is two shorter, and thus the number of ways to cover it is  $B_{n-1}$ . Thus

$$B_n = A_{n-1} + B_{n-1},$$

which implies  $A_{n-1} = B_n - B_{n-1}$ , or shifting indices  $A_n = B_{n+1} - B_n$ .

We can now find a recurrence for just  $B$ 's by substituting for the  $A$ 's in the first relation (note we can replace the grouping we did there with  $B_{n-1}$ ), which yields

$$B_{n+1} - B_n = 2B_n - 2B_{n-1} + B_n + B_{n-1} \quad \text{or} \quad B_{n+1} = 4B_n - B_{n-1}.$$

The initial conditions are easy:  $B_1 = 1$  and  $B_2 = 4$ . We can now use this to solve for  $B_n$  (try  $B_n = r^n$ , get the characteristic equation), and then get  $A_n$ . The first few values for  $B_n$  are 1, 4, 15, 56 and 209, and the general formula is

$$B_n = -\frac{(2\sqrt{3}+3)\left((2-\sqrt{3})^n - (\sqrt{3}+2)^n\right)}{6(\sqrt{3}+2)}.$$

This can be found by using the Method of Divine Inspiration, or using RSolve in Mathematica:

`RSolve[{B[n + 1] == 4 B[n] - B[n - 1], B[1] == 1, B[2] == 4}, B[n], n]`

Of course, since we only care about  $A_n$  we could instead note that if  $A_n = 2A_{n-1} + B_n + B_{n-1}$  then  $A_{n-1} = 2A_{n-2} + B_{n-1} + B_{n-2}$ . Subtracting the two yields

$$A_n - A_{n-1} = 2A_{n-1} - 2A_{n-2} + (B_n - B_{n-1}) + (B_{n-1} - B_{n-2});$$

however, from the recurrence for  $B$  we know  $B_n - B_{n-1} = A_{n-1}$  and thus

$$A_n - A_{n-1} = 2A_{n-1} - 2A_{n-2} + A_{n-1} + A_{n-2} \quad \text{or} \quad A_n = 4A_{n-1} - A_{n-2},$$

which is the same recurrence!

### #3: What if now we have a $2 \times 2 \times n$ box and just $1 \times 1 \times 2$ tiles?

**Solution:** Let  $A_n$  be the number of ways to tile a  $2 \times 2 \times n$  box. We again find a recurrence. There are 2 ways to tile the bottom level completely (both parallel to the  $x$ -axis, or both parallel to the  $y$ -axis), and thus our recurrence begins  $A_n = 2A_{n-1} + \dots$ ; we now figure out the remainder. There are two possibilities. The first is all tiles in the bottom level point up; there is one way to do this, and it leaves us with a  $2 \times 2 \times (n-2)$  box, which has  $A_{n-2}$  ways to tile. Thus  $A_n = 2A_{n-1} + A_{n-2} + \dots$ , and the lone case remaining is that we have two vertical tiles in the bottom row and one horizontal (note the two vertical tiles must be next to each other). There are four ways to choose where to place the one horizontal tile. Thus let  $B_n$  be the number of ways to tile a  $2 \times 2 \times n$  box where there is a horizontal tile removed from the bottom row. We have

$$A_n = 2A_{n-1} + A_{n-2} + 4B_{n-1}.$$

We need a recurrence for  $B_n$ . If we add a horizontal tile in the last level, that gives us a completed level and now we have a  $2 \times 2 \times (n-1)$  box, and there are  $A_{n-1}$  ways to tile. If we add two vertical tiles then we have the same region as we started but smaller, and the number of ways to fill that is  $B_{n-1}$ . Thus our second recurrence is

$$B_n = A_{n-1} + B_{n-1} \quad \text{or} \quad A_{n-1} = B_n - B_{n-1}.$$

The initial conditions can be found by straightforward computation.

We can now get a recurrence just involving  $B$ , solve by looking at the characteristic polynomial and doing linear combinations, and then deduce the one for  $A$ . Using  $2B_n = 2A_{n-1} + 2B_{n-1}$  we find

$$A_n = A_{n-2} + 2B_n + 2B_{n-1} \quad \text{or} \quad B_{n+1} = 3B_n + 3B_{n-1} - B_{n-2},$$

with initial conditions  $B_1 = 1$ ,  $B_2 = 3$  and  $B_3 = 11$  (this last takes a bit of counting). Typing

`RSolve[{B[n+1] == 3 B[n] + 3 B[n-1] - B[n-2], B[1] == 1, B[2] == 3, B[3] == 11}, B[n], n]` yields

$$B_n = \frac{(4\sqrt{3}+7)(2-\sqrt{3})^n + (\sqrt{3}+2)^{n+1}}{5\sqrt{3}+9},$$

and hence the first few values of  $A_n$  are 0, 2, 8, 30 and 112, or

$$A_n = \frac{(\sqrt{3}-1)(\sqrt{3}+2)^n - (3\sqrt{3}+5)(2-\sqrt{3})^n}{\sqrt{3}+3}.$$

**Homework #4: Due Feb 24, 2017: #1: Prove for  $a_i > 0$  that  $(1+a_1)\cdots(1+a_n) \geq 2^n\sqrt{a_1\cdots a_n}$ . #2: Prove for  $a, b > 0$  that  $a/b + b/a \geq 2$ , both by using an inequality approach and without using an inequality! #3: Solve the double recurrence  $f_n = f_{n-1} + 3g_{n-1}$ ,  $g_n = -3f_{n-1} + 9g_{n-1}$ . #4: Define a set to be selfish if it contains its cardinality (i.e., its number of elements) as an element; thus  $\{1, 3, 5\}$  is selfish, while  $\{1, 2, 3, 5\}$  is not. Find, with proof, the number of subsets of  $\{1, 2, \dots, n\}$  that are minimal selfish sets (that is, selfish sets none of whose subsets are selfish; thus  $\{1, 3, 5\}$  is not minimal selfish as  $\{1\}$  is a subset). This is a Putnam problem..... Also, make sure you have done the first 15 Project Euler Problems.**

## 3. HW #4: DUE FEBRUARY 24, 2017

**#1: Prove for  $a_i > 0$  that  $(1 + a_1) \cdots (1 + a_n) \geq 2^n \sqrt{a_1 \cdots a_n}$ . #2: Prove for  $a, b > 0$  that  $a/b + b/a \geq 2$ , both by using an inequality approach *and* without using an inequality! #3: Solve the double recurrence  $f_n = f_{n-1} + 3g_{n-1}$ ,  $g_n = -3f_{n-1} + 9g_{n-1}$ . #4: Define a set to be selfish if it contains its cardinality (i.e., its number of elements) as an element; thus  $\{1, 3, 5\}$  is selfish, while  $\{1, 2, 3, 5\}$  is not. Find, with proof, the number of subsets of  $\{1, 2, \dots, n\}$  that are minimal selfish sets (that is, selfish sets none of whose subsets are selfish; thus  $\{1, 3, 5\}$  is not minimal selfish as  $\{1\}$  is a subset). This is a Putnam problem..... Also, make sure you have done the first 15 Project Euler Problems.**

**#1: Prove for  $a_i > 0$  that  $(1 + a_1) \cdots (1 + a_n) \geq 2^n \sqrt{a_1 \cdots a_n}$ .**

**Solution:** This follows by the AM-GM inequality applied to each factor:  $(1 + a_i)/2 \geq \sqrt{1 \cdot a_i}$ . The claim now follows by multiplication. Note there is an asymmetry here in that, for the first time, our quantities are not all of the same dimension. We can't just rescale the  $a_i$ 's without changing things. The solution is to introduce  $b_1, \dots, b_n$  and see this is the same as  $(b_1 + a_1) \cdots (b_n + a_n) \geq 2^n \sqrt{a_1 b_1 \cdots a_n b_n}$ , and now if each variable is in meters, both sides are in meters<sup>n</sup>. See [http://www.aam.org.in/site/st\\_material/14.pdf](http://www.aam.org.in/site/st_material/14.pdf) for more.

**#2: Prove for  $a, b > 0$  that  $a/b + b/a \geq 2$ , both by using an inequality approach *and* without using an inequality!**

**Solution:** We can do this via the AM-GM: we would get

$$\frac{a/b + b/a}{2} \geq \sqrt{\frac{a}{b} \frac{b}{a}} = 1.$$

We can also do this with one variable calculus: let  $x = a/b$ . Then we must show, for  $x > 0$ , that  $x + 1/x \geq 2$ . It's nice to have a compact set (closed and bounded) so we can use the wonderful result from real analysis that a continuous function on a closed and bounded set attains its maximum and minimum. Without loss of generality we may assume  $x \in [1, 2]$ ; the claim is clearly true for  $x \geq 2$ , and for  $x < 1$  we just consider  $1/x$  instead. We now have the function  $f(x) = x + 1/x$  on  $[1, 2]$  and we want to find its minimum.

Since  $f'(x) = 1 - 1/x^2$ , we see the critical points (where it equals zero) in our set is just  $x = 1$ , which also happens to be an endpoint!. We find  $f(1) = 2$ ,  $f(2) = 2.5$ , and thus the minimum is 2.

Notice if we tried to do this by scaling, we can say without loss of generality  $ab = 1$ ; to see this, replace  $a$  by  $a' = a\sqrt{t}$  and  $b$  by  $b' = b\sqrt{t}$ , which doesn't change the sum of the fractions but now  $a'b' = t$ . So, using  $ab = 1$  leads to showing  $a^2 + 1/a^2 \geq 2$ . While we could differentiate or apply results to this expression, we can of course just replace  $a^2$  with  $x$  and use the previous argument.

**#3: Solve the double recurrence  $f_n = f_{n-1} + 3g_{n-1}$ ,  $g_n = -3f_{n-1} + 9g_{n-1}$ .**

**Solution:** We solve for one in terms of the other. Using the second relation we get  $f_{n-1} = -\frac{1}{3}g_n + 3g_{n-1}$ ; as this holds for all indices we can increment  $n$  by 1 and find  $f_n = -\frac{1}{3}g_{n+1} + 3g_n$ . We now substitute these values into the first recurrence, and find

$$\left(-\frac{1}{3}g_{n+1} + 3g_n\right) = \left(-\frac{1}{3}g_n + 3g_{n-1}\right) + 3g_{n-1} \quad \text{or} \quad g_{n+1} = 10g_n - 18g_{n-1}.$$

We try  $g_n = r^n$  and find a characteristic polynomial of

$$r^2 - 10r + 18 = 0.$$

If instead we tried to write the  $g$ 's in terms of the  $f$ 's we would again obtain this recurrence relation. We now solve in the usual way.

Explicitly, we assume that  $f_n = r^n$  to see if we can satisfy the equation. This gives

$$r^{n+1} - 10r^n + 18r^{n-1} = 0,$$

which means that non-trivial solutions are the roots of the quadratic equation

$$r^2 - 10r + 18 = 0,$$

which are given by

$$\begin{aligned} r_1 &= 5 + \sqrt{7} \\ r_2 &= 5 - \sqrt{7}. \end{aligned}$$

So, a general solution is of the form

$$\alpha_1 r_1^n + \alpha_2 r_2^n.$$

The problem doesn't give any initial values, so we cannot simplify further. To highlight the method, let's assume  $f_0 = 0$  and  $f_1 = 1$ . This gives  $\alpha_1 = -\alpha_2$ . Consequently,

$$1 = -\alpha_2(5 + \sqrt{7}) + \alpha_2(5 - \sqrt{7})$$

and we have that

$$\alpha_2 = -\frac{1}{2\sqrt{7}}$$

$$\alpha_1 = \frac{1}{2\sqrt{7}},$$

which concludes the proof. A solution is of the form

$$f_n = \left(\frac{1}{2\sqrt{7}}\right)(5 + \sqrt{7})^n - \left(\frac{1}{2\sqrt{7}}\right)(5 - \sqrt{7})^n,$$

and we can use similar methods to find a recursion for the  $g_n$ .

**#4: Define a set to be selfish if it contains its cardinality (i.e., its number of elements) as an element; thus  $\{1, 3, 5\}$  is selfish, while  $\{1, 2, 3, 5\}$  is not. Find, with proof, the number of subsets of  $\{1, 2, \dots, n\}$  that are minimal selfish sets (that is, selfish sets none of whose subsets are selfish; thus  $\{1, 3, 5\}$  is not minimal selfish as  $\{1\}$  is a subset). This is a Putnam problem....**

**Solution:** For problems like this, it's best to do a few cases and get a feel. Doing this we find the number of minimal selfish sets, for the first few  $n$ , to be the Fibonacci numbers!

Let  $S_n$  denote the number of subsets of  $\{1, \dots, n\}$  that are minimal selfish. Consider one of the minimal selfish sets; it either contains  $n$ , or it doesn't. By definition the number of minimal selfish sets of  $\{1, \dots, n\}$  not containing  $n$  is  $S_{n-1}$ . Imagine now we have a minimal selfish set containing  $n$ . Note it's cardinality is its size, and it has no selfish subset. Its cardinality cannot be  $n$  if  $n > 1$  (as that would mean we have all numbers, and thus selfish subsets). If we subtract 1 from each element we now have a subset of  $\{1, \dots, n-1\}$  (note we could not have had 1 and  $n$  both in our original set, and thus since we assumed  $n$  was in, 1 was not). We remove  $n-1$  now, and notice we've decreased all the elements by 1 and removed one element from the original set which had  $n$  and was minimal selfish; we now have a minimal selfish subset of  $\{1, \dots, n-2\}$  (its cardinality must be in here). Thus the number of minimal selfish sets containing  $n$  here is  $S_{n-2}$ , and we get the recurrence  $S_n = S_{n-1} + S_{n-2}$ . We just need the initial conditions, which are  $S_1 = 1$  and  $S_2 = 1$ , to see that it's the Fibonacci.

**Homework #5: Due Monday, February 27 (we will discuss in class): Show that no matter what 5 points are chosen on the surface of a unit sphere, there is at least one closed hemisphere containing at least 4 of the points. Homework: Due Friday, March 3rd: (1) Prove the law of cosines: if  $a$ ,  $b$  and  $c$  are the sides of a triangle and  $\theta$  is the angle between  $a$  and  $b$ , then  $c^2 = a^2 + b^2 - 2ab \cos(\theta)$ . (2-21) Complete the first 20 Project Euler Problems, and include in your HW a screenshot showing that you have completed all of these. Note this problem is worth 200 points (20 questions), and is thus giving you credit for all the work you have been doing. We will spend Friday discussing the coding and these problems, so let me know in advance ones you find particularly interesting. Homework (optional): Geometry problems typically invoke extreme reactions: some love, and some hate. If you like geometry problems look at the resources above, and choose 1-2 problems to do and submit. You may use these as HW exemptions for problems in future weeks (i.e., if you get full credit on either of these, you can skip a future problem and receive full credit).**

## 4. HW #5: DUE MARCH 3, 2017

(1) Prove the law of cosines: if  $a$ ,  $b$  and  $c$  are the sides of a triangle and  $\theta$  is the angle between  $a$  and  $b$ , then  $c^2 = a^2 + b^2 - 2ab \cos(\theta)$ . (2-21) Complete the first 20 Project Euler Problems, and include in your HW a screenshot showing that you have completed all of these. Note this problem is worth 200 points (20 questions), and is thus giving you credit for all the work you have been doing. We will spend Friday discussing the coding and these problems, so let me know in advance ones you find particularly interesting. Homework (optional): Geometry problems typically invoke extreme reactions: some love, and some hate. If you like geometry problems look at the resources above, and choose 1-2 problems to do and submit. You may use these as HW exemptions for problems in future weeks (i.e., if you get full credit on either of these, you can skip a future problem and receive full credit).

Drop the perpendicular onto the side  $c$  to get (see Fig. 5)

$$c = a \cos \beta + b \cos \alpha.$$

(This is still true if  $\alpha$  or  $\beta$  is obtuse, in which case the perpendicular falls outside the triangle.) Multiply through by  $c$  to get

$$c^2 = ac \cos \beta + bc \cos \alpha.$$

By considering the other perpendiculars obtain

$$a^2 = ac \cos \beta + ab \cos \gamma,$$

$$b^2 = bc \cos \alpha + ab \cos \gamma.$$

Adding the latter two equations gives

$$a^2 + b^2 = ac \cos \beta + bc \cos \alpha + 2ab \cos \gamma.$$

Subtracting the first equation from the last one we have

$$a^2 + b^2 - c^2 = -ac \cos \beta - bc \cos \alpha + ac \cos \beta + bc \cos \alpha + 2ab \cos \gamma$$

which simplifies to

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

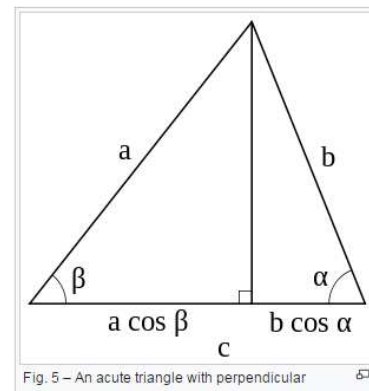


FIGURE 4. Proof from [https://en.wikipedia.org/wiki/Law\\_of\\_cosines](https://en.wikipedia.org/wiki/Law_of_cosines).

**HW #6: Due Friday, March 10: #1:** Let  $a_1, a_2, \dots, a_n$  be positive integers. Show a subset sums to a multiple of  $n$ . **#2:** Given any  $n$ , show there is a number  $x_n$  whose digits are only 0's and 7's such that  $n$  divides  $x_n$ . **#3:** Consider the previous problem. Find such a number for  $n = 2017$ ; what is the smallest such number? **#4:** Show that if  $n$  divides a Fibonacci number that it divides infinitely many Fibonacci numbers. **#5:** For all positive real numbers  $a, b, c$  show that  $a^a b^b c^c \geq a^b b^c c^a$ .



## 5. HW #6: DUE FRIDAY, MARCH 10, 2017

**HW #6: Due Friday, March 10: #1: Let  $a_1, a_2, \dots, a_n$  be positive integers. Show a subset sums to a multiple of  $n$ . #2: Given any  $n$ , show there is a number  $x_n$  whose digits are only 0's and 7's such that  $n$  divides  $x_n$ . #3: Consider the previous problem. Find such a number for  $n = 2017$ ; what is the smallest such number? #4: Show that if  $n$  divides a Fibonacci number that it divides infinitely many Fibonacci numbers. #5: For all positive real numbers  $a, b, c$  show that  $a^a b^b c^c \geq a^b b^c c^a$ .**

**#1: Let  $a_1, a_2, \dots, a_n$  be positive integers. Show a subset sums to a multiple of  $n$ .**

**Solution:** As we are trying to prove a sum is a multiple of  $n$ , it is natural to look at sums modulo  $n$ . For pigeon hole problems we need to identify the boxes and the pigeons. The boxes should be the  $n$  residue classes modulo  $n$ . What about the pigeons? There are  $2^n$  possible sums, but that's a lot more than  $n$ . Let's try looking at something smaller such as  $a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + \dots + a_n$ . Either one of these  $n$  sums is zero modulo  $n$  and we win, or else two by the Pigeon Hole Principle have the same remainder, say  $i > j - 1$ . If we subtract, we get  $a_i + \dots + a_j = 0 \pmod n$ , completing the claim.

**#2: Given any  $n$ , show there is a number  $x_n$  whose digits are only 0's and 7's such that  $n$  divides  $x_n$ .**

**Solution:** This problem is very similar to the previous. We start looking at the numbers 7, 77, 777 and so on; if we look at  $n + 1$  then two must have the same remainder modulo  $n$ . Subtracting the smaller from the larger leaves  $777 \dots 77000 \dots 0$ , which is congruent to zero modulo  $n$  and thus is our desired solution.

**#3: Consider the previous problem. Find such a number for  $n = 2017$ ; what is the smallest such number?**

We could of course apply the method from above; as 10 is a multiple of just 2 and 5 we see there can be no trailing 7's in our answer. Instead, as we are asked to find the smallest such number, we just write a simple code to do that.

```
findsmallestsevenzero[max_, target_] := Module[{ },
  smallest = Infinity;
  For[n = 1, n <= max, n++,
    {
      (* next line converts n to binary, multiples all digits by 7 *)
      digits = 7 IntegerDigits[n, 2];
      numdigits = Length[digits];
      number = Sum[digits[[d]] 10^(numdigits - d), {d, 1, numdigits}];
      If[Mod[number, target] == 0,
        {
          If[number < smallest, smallest = number];
          n = max + 10; (* exit for loop if found soln *)
        }];
    }]; (* end of n for loop *)
  If[smallest < Infinity, Print[smallest]];
];
```

This generates the answer 70077077707007; i.e., this is the first non-zero number whose digits are just 0's and 7's which is congruent to 0 modulo 2017. It might be interesting to see how the length of the smallest number varies as a function of the target. If we did 2016 we would find 77777777700000 (same number of digits), while 2018 is 700700077777770 and 2019 is the significantly shorter 700700007.

**#4: Show that if  $n$  divides a Fibonacci number that it divides infinitely many Fibonacci numbers.**

**Solution:** Note that the Fibonacci numbers are periodic modulo  $m$  for any  $m$ . The reason is the pigeonhole principle. Modulo  $m$  there are only  $m$  possible residues, and thus only  $m^2$  possible pairs of two numbers modulo  $m$ . Once we look at  $m^2 + 2$  consecutive Fibonacci numbers we have  $m^2 + 1$  pairs, and thus at least two pairs are the same.

For our problem, let's look at the Fibonacci numbers modulo  $n$ . By assumption we know  $n$  divides one of them; we now prove it divides infinitely many as the pattern repeats. To see this, imagine we have repeating pairs at indices  $(i_1, i_1 + 1)$  and  $(i_2, i_2 + 1)$ , and let's assume  $F_k$  is our given multiple of  $n$ . If  $k$  is one of these indices, or between them, it's clear. What if  $k$  isn't? Well, we had to hit  $k$  as we walked from indices  $(0, 1)$  to  $(i_1, i_1 + 1)$ ; thus if we run backwards from  $(i_1, i_1 + 1)$  we must hit  $k$ ; however, this will give us the same residues as we would get walking backwards from  $(i_2, i_2 + 1)$ , and so we must have something between our two pairs that's a multiple of  $n$ .



**#5: For all positive real numbers  $a, b, c$  show that  $a^a b^b c^c \geq a^b b^c c^a$ .**

**Solution:** If we wanted, we could rescale and assume  $abc = 1$ . Why? If we multiply each by  $r$  we get each side increases by  $r^{r(a+b+c)}$ , and thus the relation still holds or doesn't hold. It doesn't help us, but for awhile I thought about making their product 1, or setting  $b$  equal to 1.... What is more useful is there is a cyclic symmetry, and without loss of generality we may assume  $a \leq b \leq c$ . Some ordering exists, the left hand side is independent of the ordering, and seeing the cyclicity (the right hand side is also  $b^c c^a a^b$  or  $c^a a^b b^c$ ) there is no harm in assuming an ordering.

In some sense, if you look at this problem the right way it's "obvious". Why? Imagine our numbers are integers. We're talking about having some number of powers of  $a, b$  and  $c$ . We can choose  $a + b + c$  numbers. Clearly you want to have as many powers of  $c$  as possible, so give it the exponent  $c$ . Then let's take as many  $b$ 's as we can, namely  $b$  of them, and finally let's take the rest to be  $a$ .

More formally, we have the following chain (which holds for positive real numbers  $a \leq b \leq c$ ):

$$\begin{aligned} a^a b^b c^c &= a^a b^b c^{c-(b-a)+(b-a)} \\ &\geq a^{a+(b-a)} b^b c^{c-(b-a)} \\ &= a^b b^b c^{(c-b)+b-(b-a)} \\ &\geq a^b b^{b+(c-b)} c^{b-(b-a)} = a^b b^c c^a. \end{aligned}$$

Note that all the exponents are positive, and the inequalities are true as we replace larger numbers in the product with smaller ones.

For another good inequality to know, see Jensen's inequality:

[http://www.artofproblemsolving.com/Wiki/index.php/Jensen's\\_Inequality](http://www.artofproblemsolving.com/Wiki/index.php/Jensen's_Inequality)

**Homework #7: Due March 17, 2017: #1, #2, #3, #4 (counts as four problems): Show that any decomposition of  $N$  as a sum of Fibonacci numbers cannot have fewer summands than the Zeckendorf decomposition. Is there a monovariant that can help?**

**ALSO: Read: From the textbook (Famous Puzzles): Find a problem or section you enjoy, read about it, write me a short note by Wednesday on it and let me know if you are willing to present.**

**ALSO: For Monday, think about which is larger:  $e^\pi$  or  $\pi^e$ . You are NOT allowed to use a computer to calculate anything; try to prove elementarily which wins.**

## 6. HW #7: DUE MARCH 17, 2017

**Homework #7: Due March 17, 2017: #1, #2, #3, #4 (counts as four problems): Show that any decomposition of  $N$  as a sum of Fibonacci numbers cannot have fewer summands than the Zeckendorf decomposition. Is there a monovariant that can help?**

**Solution:** The solution below is a note I wrote to myself in summer 2014 after talking to one of my SMALL students as we drove back from a talk I gave at Hampshire College. I find it useful to jot things down and email them so I have them on file; pieces of paper are easily lost! (If you write it up and don't want to TeX it up, just scan the papers and email yourself the .pdf.) I then discussed this further with some colleagues a year later at West Point, and we submitted a short note on it a few weeks ago. This past summer I gave the problem to some of my then current SMALL students and they ran with it, greatly generalizing the result (see <https://arxiv.org/pdf/1608.08764.pdf>).

Given a decomposition of  $m$  into a sum of Fibonacci numbers, consider the sum of indices of terms in the decomposition (start  $F_1 = 1, F_2 = 2$ ). If you ever have two adjacent summands you do not increase the index sum by combining. If you have  $F_1$  twice use  $F_2$ . If you have  $F_2$  twice use  $F_1$  and  $F_3$ . In general, if you have  $F_k$  twice use  $2F_k = F_{k-2} + F_{k-1} + F_k = F_{k-2} + F_{k+1}$ , which has decreased the index sum for  $k \geq 3$  and you now have a larger Fibonacci summand. You can only do this a bounded number of times or you'll end up with Fibonacci number larger than the largest Fibonacci number less than  $m$ , so when you terminate you cannot have any repeats or adjacencies, and thus must be a legal Zeckendorf decomposition!

**No written HW over spring break.**

## 7. HW #8: DUE APRIL 7, 2017

**Homework #8: #1:** There are four people who want to cross a bridge. They all begin on the same side. You have 17 minutes to get all of them across to the other side. It is night. There is one flashlight. A maximum of two people can cross at one time. Any party who crosses, either one or two people, must have the flashlight with them. The flashlight must be walked back and forth, it cannot be thrown, etc. Each walks at a different speed. A pair must walk together at the rate of the slower. Their times are 1, 2, 5 and 10 minutes to cross. How do you get them across in 17 minutes? **#2: Generalizing Conway's Checker Problem:** Consider the three-dimensional version, where we again have an integer lattice and now have checkers at all points with coordinates  $(x,y,z)$  with  $z$  non-positive. We jump as before, but can now do north/south, east/west, and up/down. Prove there is a positive number  $N$  such that we cannot get a checker to a point  $(x,y,N)$  in finitely many moves. (Note: you can probably see how to generalize to arbitrarily many dimensions, the bound  $N$  will grow the dimensions, but the bound and the truth could be far apart). **#3: 1989A2** Evaluate  $\int_0^a \int_0^b \exp(\max\{b^2x^2, a^2y^2\}) dydx$  where  $a, b$  are positive.

**Solution:** Send 1 and 2 over, send either back (say 1), send 5 and 10 over, send 2 back, send 1 and 2 over. The trick is to realize you need 5 and 10 to go over together, and thus you need someone on the other side so you don't have to send 5 or 10 back with the flashlight.

**Solution:** Let  $\omega < 1$  be our weight to be determined. Given a point at location  $(x, y, z)$  attach a weight of  $\omega^{|x|+|y|+|N-z|}$ . If we sum over all  $x, y \in \mathbb{Z}$  and all non-positive integral  $z$  we get a finite number; by taking  $N$  sufficiently large we can make this number less than 1; note 1 is the weight assigned to the point  $(0, 0, N)$ . By the same logic as before, the weight attached to the board does not increase as we do legal moves, and thus a similar monovariant argument as before shows that there is no way to move to some sufficiently large  $N$ .

**Solution:** Let  $u = bx$ , so  $x : 0 \rightarrow a$  becomes  $u : 0 \rightarrow ab$  and  $dx = du/b$ ; similarly let  $v = ay$  so  $y : 0 \rightarrow b$  becomes  $v : 0 \rightarrow ab$  and  $dy = dv/a$ . Thus our integral becomes  $\int_0^{ab} \int_0^{ab} \exp(\max\{u^2, v^2\}) dvdu/ab$ . The idea is to exploit symmetry. Note that if  $v \leq u$  then the maximum is of course just  $u^2$ ; also we can integrate over the region  $0 \leq v \leq u \leq ab$  and double. Thus our integral equals

$$\frac{2}{ab} \int_{u=0}^{ab} \int_{v=0}^u \exp(u^2) dvdu = \frac{2}{ab} \int_{u=0}^{ab} \exp(u^2) u du = \frac{2}{ab} \frac{1}{2} \exp(u^2) \Big|_{u=0}^{ab} = \frac{\exp(a^2b^2) - 1}{ab}.$$

I tried solving this with Mathematica, typing

```
Integrate[Integrate[Exp[Max[b^2 x^2, a^2 y^2]], {y, 0, b}], {x, 0, a}]
```

(but it took so long I got frustrated). I then typed it again with  $a = 3$  and  $b = 5$ , and it got an answer where it was clear how  $a$  and  $b$  interact. If you do this for several pairs you can sniff out the answer, so even when Mathematica's symbolic integration cannot get the general case, you can find it out....

## 8. HW #9: DUE APRIL 21, 2017

**Homework #9: Due April 21, 2017. #1: Find  $\int_{-\infty}^{\infty} dx/(1+x^4)$ . If you're brave, replace 4 with  $2n$ . #2. Let  $f(z) = A(z)B(z)$  where  $A(z) = a_{-3}/z^3 + a_{-2}z^2 + a_{-1}/z + a_0 + a_1z + \dots$  and  $B(z) = b_0 + b_1z + b_2z^2 + b_3z^3 + b_4z^4 + b_5z^5 + \dots$ . Find the residue of  $f$  at  $z = 0$ . What is the residue of  $C(z) = B(z)^4/(z-3)^2$  at  $z = 2$ ? At  $z = 3$ ? #3 Do not submit, but you are strongly encouraged to write up the Gaussian contour integral.**

**Solution:** The poles are where  $z^4 + 1 = 0$ , or  $z \in \{e^{i\pi/4}, e^{i(\pi/4+\pi/2)}, e^{i(\pi/4+\pi)}, e^{i(\pi/4+3\pi/2)}\}$ . Of these, only the first two are in the upper half plane, so if we do the standard contour we've been doing the answer will be  $2\pi i$  times the sum of the residues at these points. If we call these roots  $r_1, r_2, r_3, r_4$  then we have

$$\frac{1}{z^4 + 1} = \frac{1}{z - r_1} \frac{1}{z - r_2} \frac{1}{z - r_3} \frac{1}{z - r_4},$$

and the residue at  $r_i$  is just

$$\frac{1}{r_1 - r_2} \frac{1}{r_1 - r_3} \frac{1}{r_1 - r_4} = -\frac{1}{4}e^{i\pi/4},$$

and similarly for the other. Thus the answer is

$$2\pi i \left( \frac{1}{r_1 - r_2} \frac{1}{r_1 - r_3} \frac{1}{r_1 - r_4} + \frac{1}{r_2 - r_1} \frac{1}{r_2 - r_3} \frac{1}{r_2 - r_4} \right),$$

which after some algebra simplification is  $\pi/\sqrt{2}$ .

Mathematica code:

```
r[n_] := Exp[I (Pi/4 + n Pi / 2)]
2 Pi I Simplify[1/((r[0] - r[1]) (r[0] - r[2]) (r[0] - r[3]))
+ 1/((r[1] - r[0]) (r[1] - r[2]) (r[1] - r[3]))]
```

**Solution:** The only terms that matter are when the power of the  $z$  term in  $A$  plus the power in the  $z$  term in  $B$  is -1. Thus the residue is  $a_{-3}b_2 + a_{-2}b_1 + a_{-1}b_0$ . As  $C(z)$  is differentiable at  $z = 2$  the residue is zero. It has a pole of order 2 at  $z = 3$ , so we need to find the coefficient of the  $z - 3$  term in the series expansion of  $B(z)^4$  at  $z = 3$ . Let  $H(z) = B(z)^4$ . Then  $H(z) = H(3) + H'(3)(z-3) + H''(3)(z-3)^2 + \dots$ . Thus we need  $H'(3) = 4B(3)^3B'(3)$ . As we have an infinite sum for  $B$  we cannot do better than this.