Chapter 1: Events and their Probabilities

Section 1.1: Introduction

Model world, axiomatize to theory

Section 1.2: Events as Sets

Definitions:

- Sample Space (Ω): all possible outcomes
  - ↳ Example: toss coin three: {HHH, ..., TTT}
    - ↳ haven't assigned probabilities to events
  - ↳ Toss coin till head: {H, TH, TTH, ...} infinite
- Events: subsets of sample space
  - ↳ {HHH, HHT, HTH, THH} in first (at least 2 heads)
- Complement: $A^c = \Omega - A$
- Certain/impossible event: Ω The certain event, $\emptyset$ The impossible event

Will not assign prob to all events: which ones?

Key, desired properties: Assign probabilities to collection subsets $\mathcal{F}$ st

↳ $A, B \in \mathcal{F} \Rightarrow A \cup B$ and $A \cap B \in \mathcal{F}$

↳ $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

↳ $\emptyset \in \mathcal{F}$ (which implies $\Omega \in \mathcal{F}$)

If satisfy these properties called a field; called a σ-field

if ↳ $\emptyset \in \mathcal{F}$

↳ $A_i \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ (infinite union, countable)

↳ $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

Note: will see later 2^{Ω} (power sets) is not a σ-field

Section 1.2: HW

Page 4

#2) $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

$A \Delta B = \{x : x \in A \text{ and } x \notin B\} \cup \{x : x \in B \text{ and } x \notin A\}$

$= (A \setminus B) \cup (B \setminus A)$ (The symmetric difference)

#5) Does $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?

↳ Standard proof: show LHS $\subset$ RHS and RHS $\subset$ LHS

$\Longrightarrow$ (Showing LHS $\subset$ RHS)

Let $x \in A \cup (B \cap C)$

Then $x \in A$ ~~and~~ OR $x \in B \cap C$

↳ Thus $x \in A$ ~~and~~ OR ($x \in B$ and $x \in C$ (or both))

Case 1: $x \in A$ implies $x \in A \cup B$ and $A \cup C$

so $x \in (A \cup B) \cap (A \cup C)$

Case 2: $x \notin A$ Thus $x \in B$ and $x \in C$

So $x \in A \cup B$ and $x \in A \cup C$

so $x \in (A \cup B) \cap (A \cup C)$

$\Longleftarrow$ Showing (RHS $\subset$ LHS)

Let $x \in (A \cup B) \cap (A \cup C)$

So $x \in A \cup B$ and $x \in A \cup C$

Case 1: If $x \in A$ then $x \in A \cup (B \cap C)$ and done

If $x \notin A$ then $x \in B$ and $x \in C$ so $x \in B \cap C$

and thus $x \in A \cup (B \cap C)$ ∎

HW: DO: #2, #3, #5d

Optional: #1, #4, #5bc

Section 1.3: Probability

Run experiment, in N trials observe event A total of $N(A)$ times

↳ As $N \to \infty$, $N(A)/N$ should approx $\text{Prob}(A) = \mathbb{P}(A)$

↳ basis Monte Carlo integration

↳ how most integration done (finance)

↳ A, B disjoint: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$, $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$

Probability should be <u>finitely additive</u>: $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$

- Call $\mathbb{P}$ <u>probability measure</u> on $(\Omega, \mathcal{F})$ if $\mathbb{P}: \mathcal{F} \to [0,1]$

Satisfies: (1) $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$

(2) $\{A_i\}$ pairwise disjoint ($i \neq j \Rightarrow A_i \cap A_j = \emptyset$)

$$\text{Then } \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Call triple $(\Omega, \mathcal{F}, \mathbb{P})$ a <u>probability space</u>.

<u>Example: Coin Toss</u>

Get Heads prob p, tails prob $1-p$, toss once.

$\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$, $\Omega = \{H, T\}$

with $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(H) = p$, $\mathbb{P}(T) = 1-p$, $\mathbb{P}(\Omega) = 1$

SECTION 1.3: PROBABILITY (CONT)

LEMMA: Probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

(1) Law of total prob: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

(2) $B \supseteq A$ Then $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) \geq \mathbb{P}(A)$

(3) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

(4) $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k)$

$- \cdots + (-1)^{n+1} \mathbb{P}(A_1 \cap \cdots \cap A_n)$ Inclusion Exclusion Principle

Ask class which are to prove from (1), (2), (3)

Give proof of 4

LEMMA: $A_1 \subseteq A_2 \subseteq \cdots$ and $B_1 \supseteq B_2 \supseteq \cdots$ Then

- If $A = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i$ Then $\mathbb{P}(A) = \lim_{i \to \infty} \mathbb{P}(A_i)$
- If $B = \bigcap_{i=1}^{\infty} B_i = \lim_{i \to \infty}$

Proof: $A = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \cdots$

<u>Notation</u>: Event A is <u>null</u> if $\mathbb{P}(A) = 0$, and A occurs almost surely if $\mathbb{P}(A) = 1$.

HW: Do: #2, #3, #5

Optional: #1, #6

INTRODUCTION TO COMBINATORICS

↳ Not in book, reviewing what need, take 251 more details

Probability of event = # times happen / # options

↳ often hard to count, but techniques elementary

CHOOSINGS

- $n! = n(n-1)\cdots 2\cdot 1$ = # ways arrange n people, order counts
- $_nP_r = \dfrac{n!}{(n-r)!}$ = # ways to choose r from n, order matters

  ↳ Think offices in government

- $_nC_r = \dfrac{n!}{r!(n-r)!}$ = # ways choose r from n, order doesn't matter

  ↳ Think student rep

- $0! = 1$ (one way to do nothing)
- Binomial Thm: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

  ↳ very important in prob/math

  ↳ proofs: induction: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

  Combinatorial: choose k x's and thus $n-k$ y's

  ↳ application: $\frac{d}{dx} x^n = n x^{n-1}$ if n pos integer

  ↳ note this method fails if $n \in \mathbb{Q}$ (can do $f(x) = x^{p/q} \to x^p = f(x)^q$)

  but for x^r use $x^r = \exp(r \log x)$ and chain rule

- Cookie problem: 5 people, 10 cookies: $\binom{10+5-1}{5-1}$ or $\binom{\#C + \#P - 1}{\#P - 1}$

  ↳ do with dividers: $\circ\circ \mid\, \mid \circ\circ\circ \mid \circ\circ \mid \circ\circ\circ$

  $2 \quad 0 \quad 3 \quad 2 \quad 3$

  ↳ Can you compute other stuff?

Introduction to Combinatorics (cont)

Proving Identities

- Way prove is count two ways, do one and get other

$$\sum_{k=0}^{n} \binom{k+r-1}{r-1} = \binom{n+r}{r}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

↳ ask class which to prove / do both if interest

- Advanced: hypergeometric fns (see paper @ Murty)

Two Problems

- lottery: Choose 6 of 50, any order, no repeats: $\binom{50}{6}$

  Choose 6 of 50, any order, repeats allowed: see notes

  ↳ prob winning in first case is $1/\binom{50}{6}$

- matchings
  - $2n$ couples, randomly matched in pairs, what prob matched correctly?
  - n people, ordered $1\,2\,3 \ldots n$, randomly rearrange, prob at least one in right spot?

    ↳ later: prob exactly r in correct order

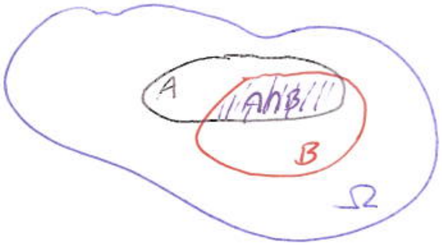
    expected value

HW:

SECTION 1.4: CONDITIONAL PROBABILITY

- INTUITION

↳ What is Prob A happens given B happens? Denote $\mathbb{P}(A|B)$.



As B happens, only care about part inside B

If $N(E)$ how often observe E in N trials:

$$\frac{N(A\cap B)}{N(B)} = \frac{N(A\cap B)/N}{N(B)/N} \longrightarrow \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$$

Defn: Conditional probability: If $\mathbb{P}(B) > 0$ then the cond prob of A occuring given B occurs is $\mathbb{P}(A|B) = \mathbb{P}(A\cap B)/\mathbb{P}(B)$

Ex1: Prob of a 7 given first roll fair die is a 3; prob of an 11?

↳ $\Omega = \{1,2,\ldots,6\}^2 = \{(1,1),(1,2),\ldots,(6,6)\}$, $\#\Omega = 6^2 = 36$

$B = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$

~~$A_1 = \{(3,4)\}$ $\quad A_2 = \emptyset$~~

$A_1 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$A_2 = \{(5,6),(6,5)\}$

$A_1\cap B = \{(3,4)\} \quad A_2\cap B = \emptyset$

$\mathbb{P}(A_1|B) = \mathbb{P}(A_1\cap B)/\mathbb{P}(B) = \frac{1/36}{6/36} = 1/6$

$\mathbb{P}(A_2|B) = \mathbb{P}(A_2\cap B)/\mathbb{P}(B) = \frac{0/36}{6/36} = 0$

Ex2: See problem in book about children

↳ two slightly different problems

↳ importance of phrasing

Section 1.4: Continued

Defn: Partition: A family of events $B_1, B_2, \ldots, B_n$ is a partition of Ω if

① $B_i \cap B_j = \emptyset$ if $i \neq j$

② $\bigcup_{i=1}^{n} B_i = \Omega$

$\Rightarrow$ each elementary event ω in a unique B_i

LEMMA: If $0 < \mathbb{P}(B) < 1$, for any event A we have

$$\mathbb{P}(A) = \mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\mathbb{P}(B^c)$$

↳ if $B_1, \ldots, B_n$ is a partition with $\mathbb{P}(B_i) > 0$ then

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i)$$

Things to note

↳ See why need $0 < \mathbb{P}(B) < 1$ so $0 < \mathbb{P}(B^c) < 1$

Why must $\mathbb{P}(B_i) > 0$?

↳ Proof: $A = (A \cap B) \cup (A \cap B^c)$ and clearly disjoint union

Thus $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$

by defn conditional prob, $\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\mathbb{P}(B) \ldots$

↳ really using $\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B) / \mathbb{P}(B)$

↳ must avoid division by zero!

↳ Infinite Case?

↳ Always explore conditions in a theorem!

↳ Only useful if easier to compute $\mathbb{P}(A \mid B_i)$ and $\mathbb{P}(B_i)$ than $\mathbb{P}(A \cap B_i)$!

SECTION 1.4: CONTINUED

Example 14 from Section 1.7: False Positives

Rare disease affects 1 in 10^5 people, test shows positive with prob $99/100$ if ill, and shows positive with prob $1/100$ if healthy. If test positive, what is the probability you are ill?

↳ Sec 1.4, #1) Prove if $\mathbb{P}(A)\mathbb{P}(B) \neq 0$ then $\mathbb{P}(A|B)\,\mathbb{P}(B) = \mathbb{P}(B|A)\,\mathbb{P}(A)$

↳ Proof: as $\mathbb{P}(A)\mathbb{P}(B) \neq 0$, both sides are just $\mathbb{P}(A \cap B)$.

↳ let $A = \{\text{ill}\}$, $B = \{\text{test positive}\}$

$$\mathbb{P}(\text{ill}\,|\,+)\,\mathbb{P}(+) = \mathbb{P}(+\,|\,\text{ill})\,\mathbb{P}(\text{ill}) \Rightarrow \mathbb{P}(\text{ill}\,|\,+) = \frac{\mathbb{P}(+\,|\,\text{ill})\,\mathbb{P}(\text{ill})}{\mathbb{P}(+)}$$

Using partition: $\mathbb{P}(+) = \mathbb{P}(+\,|\,\text{ill})\,\mathbb{P}(\text{ill}) + \mathbb{P}(+\,|\,\text{healthy})\,\mathbb{P}(\text{healthy})$

$$\Rightarrow \mathbb{P}(\text{ill}\,|\,+) = \frac{\mathbb{P}(+\,|\,\text{ill})\,\mathbb{P}(\text{ill})}{\mathbb{P}(+\,|\,\text{ill})\,\mathbb{P}(\text{ill}) + \mathbb{P}(+\,|\,\text{healthy})\,\mathbb{P}(\text{healthy})} = \frac{\frac{99}{100}\cdot 10^{-5}}{\frac{99}{100}\,10^{-5} + \frac{1}{100}(1-10^{-5})} \approx \frac{1}{1011}$$

↳ note very likely healthy! Applications medical testing

↳ with above givens, very easy to do it this way

Challenge question: do we want to improve $99/100$ or $1/100$ more, and why?

HW: do: #2, #4

suggested: #5, #6

SECTION ~~1.6~~ 1.5: INDEPENDENCE

Two events indep if knowledge of one happening gives no info on other happening

Defn: A and B *independent* if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

More generally, family $\{A_i : i \in I\}$ indep if

$$\mathbb{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathbb{P}(A_i) \quad \forall \text{ finite subsets } J \text{ of } I$$

↳ Very important that looking at all finite subsets, not just pairwise

↳ Ex: $\Omega = \{1, \dots, 6\}^2$ (fair die)

A = first die is a 1 $\quad \mathbb{P}(A) = 1/6$

B = second die is a 2 $\quad \mathbb{P}(B) = 1/6$

C = sum rolls is a 7 $\quad \mathbb{P}(C) = 1/6$

note $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \dfrac{1/36}{6/36} = \dfrac{1}{6} = \dfrac{\mathbb{P}(A)\,\mathbb{P}(B)}{\mathbb{P}(B)}$

similarly $\mathbb{P}(A|C) = \mathbb{P}(B|A) = \mathbb{P}(B|C) = 1/6 = \mathbb{P}(C|A) = \mathbb{P}(C|B)$

But $\mathbb{P}(A \cap B \cap C) = 0 \neq \left(\frac{1}{6}\right)^3 = \mathbb{P}(A)\mathbb{P}(B)\,\mathbb{P}(C)$

↳ tossing a fair coin twice / spinning wheel on Price Is Right / Roulette

↳ not necc independent!

Defn: Events A, B are conditionally indep given C (with $\mathbb{P}(C) > 0$) if

$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A|C)\,\mathbb{P}(B|C)$$

HW: do : #1, #2, #4 (is it true if p isn't prime?), #8 (used error correcting codes)

suggested: #3, #5, #6

NOT DOING SECTION 1.6

SECTION 1.7: WORKED EXAMPLES

(#4) Symmetric random walk (Gambler's Ruin)

Car costs \$N, starts with \$K ($0<k<N$). Toss fair coin: heads win \$1, tail loses \$1. Play till hits \$0 or \$N. What is prob lose?

Soln: let P_k = prob bankrupted given start with \$k

↳ Clearly $P_0 = 1$, $P_1 = 0$, N even then $P_{N/2} = 1/2$

Conditional prob: A event bankrupted

B event first toss heads

↳ $\mathbb{P}_K$ means prob when start with \$K

$$\underbrace{\mathbb{P}_K(A)}_{P_K} = \underbrace{\mathbb{P}_K(A \mid B)}_{P_{K+1}} \underbrace{\mathbb{P}(B)}_{\frac{1}{2}} + \underbrace{\mathbb{P}_K(A \mid B^c)}_{P_{K-1}} \underbrace{\mathbb{P}_K(B^c)}_{\frac{1}{2}}$$

$$P_K = P_{K+1} \cdot \tfrac{1}{2} + P_{K-1} \cdot \tfrac{1}{2}$$

Get difference eq $P_{K+1} = 2P_K - P_{K-1}$

with initial conds $P_0 = 1$, $P_1 = 0$

↳ Book gives one method to solve, here is a more general one

$P_{k+1} - 2P_k + P_{k-1} = 0$: Guess $P_k = r^k$

↳ Get char polynomial: $r^{k+1} - 2r^k + r^{k-1} = 0$

$$r^{k-1}(r^2 - 2r + 1) = 0$$

$$r = 0 \text{ or } r = 1, 1$$

↳ soln is $r = 1^k$

↳ PROBLEM: Repeated root: have P_k determined once know any two values, general soln should have two free parameters — missing a soln!

Section 1.7: Worked Examples: Cont

(#4) Symmetric Random Walk (Cont)

In DiffEqs learn to guess $P_k = k r^k$ if one repeated root

Now get $(k+1) r^{k+1} - 2k r^k + (k-1) r^{k-1} = 0$

↳ as $r=1 \Rightarrow k+1 - 2k + k-1$ should equal 0, and does

Thus solns to $P_{k+1} - 2P_k + P_{k-1} = 0$ are $P_k = 1, k$

↳ LINEAR difference eq: linear combination of solns is a soln

↳ using a little linear algebra here

↳ Thus for any c_1, c_2 find $P_k = c_1 \cdot 1 + c_2 \cdot k$ is a soln

Choose c_1, c_2 to match boundary conds:

↳ $c_1 \cdot 1 + c_2 \cdot 0 = 1 = P_0$

$c_1 \cdot 1 + c_2 \cdot N = 0 = P_N$

↳ solve with lin alg or substitution

find $c_1 = 1$, $c_2 = -1/N$

↳ General soln: $P_k = 1 - \frac{k}{N}$

HW: do: #1, #3, #4

Suggested: #2, #5

Problems From Section 1.8: Problems

do:	#2	#4	#6	#12	#24	#28	#39
can do after:	§1.3	§1.2	§1.4	§1.2	§1.5	§1.5	§1.5

Note: for #28, must at most 10% be colored, or could more be colored. In 4-dimensions?

suggested: 1, 3, 7, 11, 13, did 14 in class, 16, 18, 20, 23, 29, 31, 32, 35, and are 37 and 38 consistent?

Section 1.7: Worked Examples

Bonus: More Difference Eqs

(1) Fibonacci: $a_{n+1} = a_n + a_{n-1}$ $a_0 = 0$ $a_1 = 1$

↳ Guess $a_n = r^n$ find characteristic polynomial is

$r^{n+1} - r^n - r^{n-1} = 0$ or $r^{n-1}(r^2 - r - 1) = 0$

so $r = \frac{1 \pm \sqrt{5}}{2}$. Thus general sol'n is

$$a_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

↳ To find c_1 and c_2:

$n=0$: $0 = c_1 \cdot 1 + c_2 \cdot 1$

$n=1$: $1 = c_1 \cdot \frac{1+\sqrt{5}}{2} + c_2 \cdot \frac{1-\sqrt{5}}{2}$

↳ first eq gives $c_2 = -c_1$

second becomes $1 = c_1 \cdot \sqrt{5} \Rightarrow c_1 = \frac{1}{\sqrt{5}}$, $c_2 = -\frac{1}{\sqrt{5}}$

Sol'n: $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

↳ Called Binet's Formula

↳ amazing: a_n always an integer; this doesn't look integral: have divisions and $\sqrt{5}$ all over the place!

Section 1.8: Worked Examples

Bonus: More Difference Eqs

(2) Double Plus One: Great Way to Lose at Roulette!

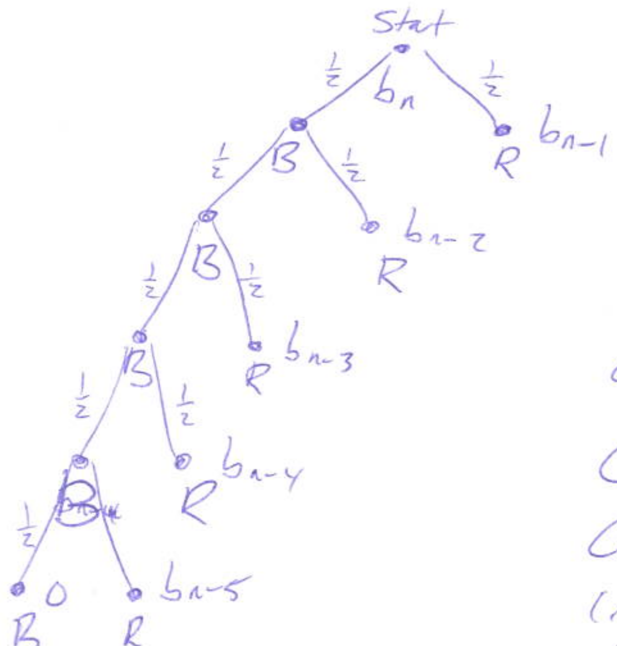
"Friendlier" Roulette: no green, wheel comes up red with prob 1/2 and win, comes up black with prob 1/2 and lose.

Strategy: Bet \$1; win happy else down \$1 so bet \$2
win up \$1 else down \$3 so bet \$4
win up \$1 else down \$7 so bet \$8....
Eventually should be up \$1: what's wrong?

Soln: House limits: say can only double 4 times without exceeding maximum bet. What is prob lose big if play 100 spins?

Let a_k = prob have 5 consec blacks in k tosses
b_k = " don't have " " " " " " $= 1 - a_k$

Technically easier to compute b_k and get a_k from the Law of Total Probability.



Recurrence relation:

$$b_n = \frac{1}{2} b_{n-1} + \frac{1}{2^2} b_{n-2} + \frac{1}{2^3} b_{n-3} + \frac{1}{2^4} b_{n-4} + \frac{1}{2^5} b_{n-5}$$

or $32 b_n - 16 b_{n-1} - \cdots - b_{n-5} = 0$

Guess $b_n = r^n$, find 5 distinct roots
General soln $b_n = c_1 r_1^n + \cdots + c_5 r_5^n$
(more linear algebra), soln controlled by largest root in absolute value, $r_1 =$
Initial conds: $b_0 = b_1 = b_2 = b_3 = b_4 = 0$
Get $b_{100} \approx$