$Var(X - X) \neq Var(X) + Var(X)$

$Prob(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$

$X = X_1 + \ldots + X_n$ each X_i is $Bern(p)$

Say X_i's are iidrv (independently identically distributed random variables)

$E[X] = E[X_1] + \ldots + E[X_n]$

$E[X_i] = 1 \cdot p + 0(1-p) = p$

$E[X] = \underbrace{p + \ldots + p}_{np} = np$

$X = X_1 + \ldots + X_n$

all of X_i are independent : $Var(X) = \sum_{i=1}^{n} Var(X_i)$

$= \sum_{i=n}^{n} E[(X_i - p)^2]$

$= \sum_{i=1}^{n} \left[(1-p)^2 p + (0-p)^2 (1-p) \right]$

$= p(1-p) \sum_{i=1}^{n} [p + 1 - p]$

$= p(1-p)\, n$

$\therefore Var(X_i) = p(1-p)$

$Bin(n,p)$
$\mu = np$
$\sigma^2 = np(1-p)$
$\sigma = \sqrt{n} \cdot \sqrt{p(1-p)}$

$\Rightarrow$ in this problem $n = 4{,}000{,}000$

$p = 1/2$ (expect 2,000,000 H)

fluctuations $\approx 2000 \cdot \frac{1}{2} = 1{,}000$

Date :

Chebyshev Theorem

X random variable with finite mean μ and finite variance σ^2.

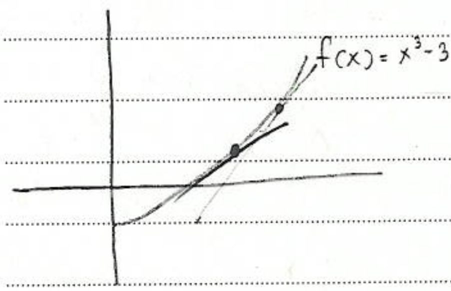
Then, $\text{Prob}(|X - \mu| \geq k\sigma^2)$ is at most $\frac{1}{k^2}$ (k is any non-negative number)

length (m) m^2

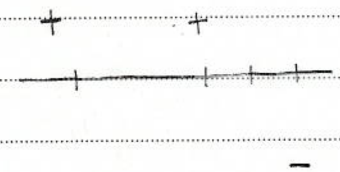
so this should be $\text{prob}(|X - \mu| \geq k\sigma)$ to have the same unit.

Aside

Newton's Method => better technique



Divide and Conquer



errors : $\frac{1}{2^{10}} \approx \frac{1}{10^3}$

Proof :

$$\mathbb{P}(|X - \mu| > k\sigma) = \int_{|X-\mu| > k\sigma} f(x)\,dx$$

$$\leq \int_{|X-\mu| > k\sigma} \left(\frac{X - \mu}{k\sigma}\right)^2 f(x)\,dx$$

$$\leq \frac{1}{k^2\sigma^2} \underbrace{\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx}_{\sigma^2} = \frac{1}{k^2} \quad \underline{\underline{\text{Ans}}}$$

Fermat Primes

$$F_n = 2^{2^n} + 1$$

F_n is prime iff $n \in \{0, 1, 2, 3, 4\}$

Prime Number Theorem (PNT)

$$\pi(x) = \#\{\text{primes } p \le x\}$$

As $x \to \infty$, $\pi(x) \approx \dfrac{x}{\log x}$

% prime at most $x = \dfrac{\pi(x)}{x} \approx \dfrac{1}{\log x} \to 0$

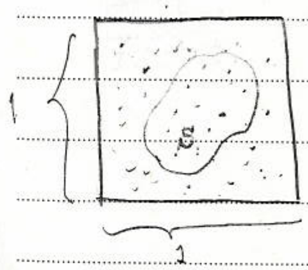
% perfect square at most $x \approx \dfrac{\sqrt{x}}{x} = \dfrac{1}{\sqrt{x}}$

Let $X = \sum_{n=0}^{\infty} X_n$

$$E[X] = \sum_{n=0}^{\infty} \frac{1}{\log F_n} = \sum_{n=0}^{\infty} \frac{1}{\log(2^{2^n}+1)}$$

$$< \sum_{n=0}^{\infty} \frac{1}{2^n \cdot \log 2}$$

Monte-Carlo Integration



$$\frac{\#\text{hits}}{\#\text{tosses}} \xrightarrow[\#\text{tosses} \to \infty]{} \text{Area}$$

$A(S) = p$, n tosses

$X_1, \ldots, X_n$ iid rv

$$X_i = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases}$$

$E(\sum X_i) = np$

St Dev$(\sum X_i) = \sqrt{n} \cdot \sqrt{p(1-p)}$

Date : ○○○

Chebyshev : Probability being more than $n^{1/4}$ standard deviations away

$$\left(\frac{1}{n^{3/4}}\right)^n = \frac{1}{\sqrt{n}}$$

$$\frac{np + \text{Error at most } n^{1/4}}{n} = p + \text{Error size } \frac{1}{n^{1/4}}$$

October 20, 2009

Norminalization Constant

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-\mu)^2/2\sigma^2} dx = 1$$ => use polar coordinate to prove this

Change of vars :

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \cdot \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-y^2/2} dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)} dx dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$

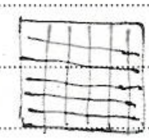
$dx dy = r dr d\theta$

$$I^2 = \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2/2} r dr d\theta$$

$$= \int_{r=0}^{\infty} e^{-r^2/2} \cdot r dr$$

$$= e^{-r^2/2}\Big]_{\infty}^{0} = 1$$

Note $\iint_A f(x) dx dy$



$I^2 = 1$

$\epsilon = \frac{1}{e}, \frac{1}{e^2}, \ldots$

away

Gamma Function

$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx \qquad Re(s) > 0 \qquad s = \sigma + it \ (\sigma > 0)$

$$\lim_{\epsilon \to 0} \int_\epsilon^1 x^r dx = \begin{cases} \lim_{\epsilon \to 0} \left(\frac{x^{r+1}}{r+1} \right) \Big|_\epsilon^1 & \text{if } r \neq -1 \Rightarrow \frac{1}{r+1}\left(1 - \lim_{\epsilon \to 0} \epsilon^{r+1}\right) \\ \lim_{\epsilon \to 0} \ln x \Big|_\epsilon^1 & \text{if } r = -1 \Rightarrow -\lim_{\epsilon \to 0} \ln \epsilon \end{cases}$$

Note $x^{s-1} < e^{x/2}$ as x is big, so this function exist.

$\Gamma(s+1) = \int_0^\infty e^{-x} x^s dx \Rightarrow$ By Part $u = x^s$, $du = s x^{s-1}$

$dv = e^{-x} dx$, $v = -e^{-x}$

$$= e^{-x} x^s \Big|_0^\infty + \int_0^\infty s e^{-x} x^{s-1} dx$$

$\Gamma(s+1) = s\Gamma(s)$

$$\left.\begin{aligned} \Gamma(1) &= \int_0^\infty e^{-x} \cdot x^{1-1} dx = 1 = 0! \\ \Gamma(2) &= \Gamma(1+1) = 1 \cdot \Gamma(1) = 1 = 1! \\ \Gamma(3) &= \Gamma(2+1) = 2\Gamma(2) = 2 = 2! \\ \Gamma(4) &= \Gamma(3+1) = 3\Gamma(3) = 6 = 3! \\ \Gamma(5) &= \Gamma(4+1) = 4\Gamma(4) = 24 = 4! \end{aligned}\right\} \Gamma(n+1) = n!$$

$\Gamma(\frac{1}{2}) = -\frac{1}{2}\Gamma(-\frac{1}{2}) = -\frac{1}{2}!$

$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx \Rightarrow$ By Part $x = t^2$, $dx = 2t dt$, $t = \sqrt{x}$

$x^{-\frac{1}{2}} dx = 2 dt$

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$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t^2}\, 2\, dt$$

$$= \underbrace{\left[\int_{-\infty}^{\infty} e^{-t^2/2\left(\frac{1}{\sqrt{2}}\right)^2} dt \cdot \frac{1}{\sqrt{2\pi\left(\frac{1}{2}\right)^2}}\right]}_{=1 \text{ (from nominalization constant)}} \sqrt{2\pi\left(\tfrac{1}{2}\right)^2}$$

$$= \sqrt{\pi}$$

Joint Density and Independence

Prob Mass function => Discrete case : $f(x) = P(X = x)$

Prob Density function => Continuous : $F(x) = \int_{-\infty}^{\infty} f(u) \cdot du$

Main Lemma on Independence

Random variables X and Y are independent iff $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

Case 1 Factors

$$P(A \cap B) = \int_{x=a}^{b} \int_{y=c}^{d} f_{X,Y}(x,y)\, dx\, dy$$

$$= \underbrace{\int_{x=a}^{b} f_X(x)\, dx}\ \underbrace{\int_{y=c}^{d} f_Y(y)\, dy}$$

$$= P(A) \cdot P(B)$$

Case 2 : Doesn't Factors

Choose (x_0, y_0) such that $f_X(x_0, y_0) - f_X(x_0) \cdot f_Y(y_0) > \varepsilon > 0$

(WLOG without lost of generality)

By continuity, there are $\delta_1, \delta_2, \delta_3$ and $\delta = \min(\delta_1, \delta_2, \delta_3)$ if $|(x,y) - (x_0, y_0)| < \delta$. Then, $f_{X,Y}(x,y) - f_X(x) \cdot f_Y(y) > \frac{\varepsilon}{2}$

$$\iint_{I \times J} f_{X,Y}(x,y)\, dx\, dy - \int_I f_X(x)\, dx \int_J f_Y(y)\, dy > \frac{\varepsilon}{2} |I|\, |J|$$

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Contradicts Independence

Ex $\int_0^1\int_0^1 x e^{xy}\,dx\,dy = \int_{x=0}^1 \left[\int_{y=0}^1 e^{xy}\cdot x\,dy\right] dx$

$= \int_{x=0}^1 e^{xy}\Big]_0^1 dx$

$= \int_{x=0}^1 (e^x-1)\,dx \quad = e^x \Big|_0^1 - 1 \quad = e-1-1 \quad = e-2$

Cauchy-Schwartz Inequality

$$\left(\int_{-\infty}^{\infty} |f(x)g(x)|\,dx\right)^2 \leq \left(\int_{-\infty}^{\infty} |f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} |g(x)|^2\,dx\right)$$

If $g = f$, True!

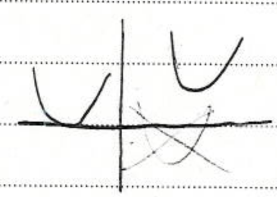
Proof $0 \leq \int_{-\infty}^{\infty} (af(x) + b\,g(x))^2\,dx$

$$0 \leq \int_{-\infty}^{\infty} (a^2 f(x)^2 + 2ab\,f(x)g(x) + b^2 g(x)^2)\,dx$$

$$0 \leq \overbrace{a^2\int_{-\infty}^{\infty} f(x)^2\,dx}^{\gamma} + \overbrace{2ab\int_{-\infty}^{\infty} f(x)g(x)\,dx}^{\beta} + \overbrace{b^2\int_{-\infty}^{\infty} g(x)^2\,dx}^{\alpha}$$

Assume $\int |f|^2, \int |g^2| \neq 0, \infty$ (else trivial)

Take $a=1$, quadratic polynomial in b that's always ≥ 0



ax^2+bx+c

roots $= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$: b^2-4ac = discrimenant.

$$0 \geq \beta^2 - 4\alpha\gamma = 4\left|\int_{-\infty}^{\infty} f(x)g(x)\,dx\right|^2 - 4\left[\int_{-\infty}^{\infty} f(x)^2dx \cdot \int_{-\infty}^{\infty} g(x)^2 dx\right]$$

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$$\int f(x)^2 dx \int g(x)^2 dx \geq \left| \int f(x) g(x) dx \right|^2$$

Review session : H.W.

(3.6) ⑦ $f_{X,Y}(x,y) = \log_{10}\left(1 + \frac{1}{10x+y}\right) = \frac{\ln\left(1+\frac{1}{10x+y}\right)}{\ln 10}$

$f_X(x) = \sum_y f_{X,Y}(x,y)$

$= \sum_{y=0}^{9} \log_{10}\left(\frac{10x+y+1}{10x+y}\right)$ or $= \log_{10}\left(\prod_{y=0}^{9} \frac{10x+y+1}{10x+y}\right)$

$= \sum\left(\log_{10}(10x+y+1) - \log_{10}(10x+y)\right)$

$= \log_{10}(10x+1) - \log_{10}(10x) + \log_{10}(10x+2) - \log_{10}(10x+1) + \log(10x+3) - \log(10x+2)$
$+ \ldots$

$E(x) = \sum x \cdot f_X(x)$

② $P(N_i = n_i \text{ for } 1 \le i \le t) = \frac{n!}{n_1! n_2! \ldots n_t!} \cdot p_1^{n_1} \cdot p_2^{n_2} \ldots p_t^{n_t}$

$f_{X_t}(x) = \binom{n}{x} \cdot p_t^x \cdot \underbrace{(1-p_t)^{n-x}}_{\text{all the earlier one}}$

$$\boxed{P(X_t = x) = \left[\sum_{\substack{n_1, n_2, \ldots, n_{t-1} \\ n_1 + \ldots + n_{t-1} = n-x}} \sum \ldots \sum \frac{n!}{n! \ldots n_{t-1}!} p_1^{n} \ldots p_{t-1}^{n_{t-1}}\right) \frac{1}{x!} p_t^x}$$

$= \frac{n!}{(n-x)!} (1-p_t)^{n-x} \cdot \frac{1}{x!} p_t^x$

$$\left(\underbrace{\sum\sum \cdots \sum \frac{(n-x)!}{n_1! \cdots n_{t-1}!} p_1^{n_1} \cdots p_{t-1}^{n_{t-1}}}_{(p_1 + \cdots p_{t-1})^{n-x} \binom{n}{x} p_t^x} \right) \frac{n(n-1)\cdots(n-(x-1))}{(n-x)!} \frac{(n-x)!}{x!} \cdot p_t^x$$

see before

$$\underbrace{(p_1 + \cdots p_{t-1})}_{1-p_t}$$

(3.5) N coins $\Rightarrow$ expected number of heads $= pN$

X = the number of coins that we tossed and gave head. , N = numbers of coins

Poiss(λ) $\quad P(N=n) = \dfrac{\lambda^n e^{-\lambda}}{n!}$

$X \sim Bin(N, p)$

$$P(X=m) = \sum_{n=m}^{\infty} P(X=m \mid N=n) \cdot P(N=n)$$

have m heads

has to start w/ m b/c. want m heads

$$= \sum_{n=m}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} \cdot \frac{\lambda^n \cdot e^{-\lambda}}{n!}$$

$$= \sum_{n=m}^{\infty} \frac{\cancel{n!}}{(n-m)!\, m!} \cdot p^m (1-p)^{n-m} \cdot \frac{\lambda^n \cdot e^{-\lambda}}{\cancel{n!}}$$

$$= \frac{p^m \cdot e^{-\lambda}}{m!} \sum_{n=n}^{\infty} \frac{1}{(n-m)!} (1-p)^{n-m} \cdot \lambda^{n-m+m}$$

$$= \frac{\lambda^m \cdot p^m \cdot e^{-\lambda}}{m!} \sum_{n=m}^{\infty} \frac{1}{(n-m)!} (1-p)^{n-m} \cdot \lambda^{n-m}$$

Let $k = n-m$

$n = k+m$

$$= \frac{\lambda^m \cdot e^{-\lambda}}{m!} \cdot p^m \sum_{k=0}^{\infty} \frac{(\lambda(1-p))^k}{k!} = e^{\lambda(1-p)}$$

$$= \frac{(\lambda p)^m \cdot e^{-\lambda p}}{m!} \sim \text{Poiss}(\lambda p)$$

$)-\log(10x+2)$

Date : ○○○

October 21, 2009

Cauchy - Schwartz Inequality

$$\left(\int_{-\infty}^{\infty} |f(x)\cdot g(x)|\,dx\right)^2 \leq \left(\int_{-\infty}^{\infty} |f(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} |g(x)|^2 dx\right)$$

$\Downarrow$

$$\left(\int\cdots\int_A |f(\vec{x})\cdot g(\vec{x})|\,d\vec{x}\right)^2 \leq \left(\int\cdots\int_A |f(\vec{x})|^2 d\vec{x}\right)\left(\int\cdots\int_A |g(\vec{x})|^2 d\vec{x}\right)$$

with $\vec{x} = (x_1, \ldots, x_n)$, $d\vec{x} = dx_1 \ldots dx_n$

Application : Covariance $(x,x) = Var(x)$

Covariance X and Y is defined to be

$$CoVar(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

Algebra yields

$$CoVar(X, Y) = E[XY] - E[X]\cdot E[Y]$$

$$Var[X] = E[(X - \mu)^2] = E[X^2] - E[X]^2$$

Note that ① $Cov(X, X) = Var(X)$

② If $Cov(X, Y) = 0$, say uncorrelated.

Correlation Coefficient

Define the correlation coefficient by

$$\rho(X,Y) = \frac{CoVar(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}}$$

If $Y = X$; $\rho(X,X) = \dfrac{Cov(X,X)}{\sqrt{Var(X)\cdot Var(X)}} = \dfrac{Var(X)}{Var(X)} = 1$

If $Y = -X$; $\rho(X,Y) = \dfrac{Cov(X,-X)}{\sqrt{Var(X), -Var(X)}} = -1$

$\Bigg\}\ -1 \leq \rho(X,Y) \leq 1$

S·M·I·L·E

Proof that $-1 \le \rho(x,y) = 1$

probability density

Key Input: $f_{X,Y}(x,y) = \sqrt{f_{X,Y}(x,y)} \cdot \sqrt{f_{X,Y}(x,y)}$

Allows us to write $E[(X-\mu_X)(Y-\mu_Y)]$ as

$$\text{CoVariance} = \boxed{\iint \underbrace{(x-\mu_x)(y-\mu_y)}_{A(x,y)} \; \underbrace{f_{X,Y}(x,y)}_{B(x,y)} \, dx\,dy}$$

=> Bad Idea

$$\le \iint |x-\mu_x||y-\mu_y| \cdot f_{X,Y}(x,y)\,dx\,dy$$

$$\le \left(\iint A(x,y)^2\,dx\,dy\right)^{\frac{1}{2}} \left(\iint B(x,y)^2\,dx\,dy\right)^{1/2} \Rightarrow \text{NOT A GOOD IDEA}$$

Aside $f_X(x) = \dfrac{1/2}{\sqrt{x}}$, $0 \le x \le 1$

$$\int_0^1 f_X(x)\,dx = \int_0^1 \frac{1/2}{\sqrt{x}}\,dx = 1$$

$$\int_0^1 f_X(x)^2\,dx = \int_0^1 \frac{1}{4}\cdot\frac{1}{x}\,dx = \frac{1}{4}\ln x\Big|_0^1 = -\infty$$

* although $f_X(x)$ = prob. distribution, $f_X(x)^2$ may not be the prob. distribution.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|^2 \cdot |y|^2\,dx\,dy \quad \times \quad \text{BAD IDEA AGAIN}$$

* Idea $\int x^2\,dx \Rightarrow \text{BAD}$, $\int f(x)^2\,dx \Rightarrow \text{GOOD}$

$$\le \iint |x-\mu_x|\sqrt{f_{X,Y}(x,y)} \cdot |y-\mu_y|\sqrt{f_{X,Y}(x,y)}\,dx\,dy$$

$$\le \left(\iint |x-\mu_x|^2 f_{X,Y}(x,y)\,dx\,dy\right)^{\frac{1}{2}} \left(\iint |y-\mu_y| f_{X,Y}(x,y)\,dx\,dy\right)^{\frac{1}{2}}$$

$\Downarrow$ do y first to get the marginal

$\Downarrow$ do x first to get the marginal

$y) \le 1$

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Aside $\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = f_X(x)$

$$\left(\int_{-\infty}^{\infty} (x-\mu_x)^2 f_X(x)\,dx\right)^{1/2} = \mathrm{Var}(x)^{1/2}$$

$$|\text{CoVariance}| \leq \sqrt{\mathrm{Var}(x)\mathrm{Var}(y)}$$

$$\frac{|\text{coVariance}|}{\sqrt{\mathrm{Var}(x)\mathrm{Var}(y)}} \leq 1$$

<u>Three Hat Problem</u> : Clicker Question

See 2 different => silent

See 2 same color => say opposite

WWW	—	—	—
WWB			+
WBW		+	
BWW	+		
WBB	+		
BWB		+	
BBW			+
BBB	—	—	—

<u>Suggested problem 3.3.8</u> : If roll 1, stop. Can stop at any time. Score is the last roll. Strategy w/ best expected value.

<u>Strategy</u> => See 6, stop!

=> See 1, forced to stop

average expected value = $3\frac{1}{2}$ =

6 ↑ stop
5
4
3
2 ↓ Go.
1

strategy k, stop at k

$E[S_6] = 3.5 \quad \left[\frac{1}{2}6 + \frac{1}{2}1\right]$

Strategy 5 => have 3 number 1,5,6

$E[S_5] = \frac{1}{3}(6) + \frac{1}{3}(5) + \frac{1}{3}(1) = \frac{12}{3}, 4$

$E[S_4] = \frac{1}{4}6 + \frac{1}{4}5 + \frac{1}{4}4 + \frac{1}{4}1 = \frac{16}{4} = 4$

Suppose that you roll, and the payoff is the square of your roll:

$E[S_6^2] = \frac{1}{2}6^2 + \frac{1}{2}1 = \frac{37}{2} = 18.5$

$E[S_5] = \frac{1}{3}6^2 + \frac{1}{3}5^2 + \frac{1}{3}1^2 = \frac{62}{3} \approx 20.67$

$E[S_4] = \frac{1}{4}6^2 + \frac{1}{4}5^2 + \frac{1}{4}4^2 + \frac{1}{4}1^2 = \frac{78}{4} < 20$

October 27, 2009

- Change of variable formulars
- Sums of random variables
- Distribution from Normal

X_i is r.v. with mean μ and var σ^2 (all independent)

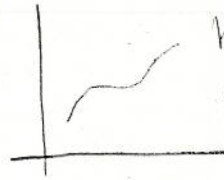
$$\frac{\frac{X_1 + \dots + X_n}{n} - \mu}{\sigma\sqrt{n}} = \frac{Y - E(Y)}{\text{stDev.}(Y)}$$

the

$$\mathrm{Var}\left(\frac{X_1 + \dots + X_n}{n} - \mu\right) = \mathrm{Var}\left(\frac{X_1 + \dots + X_n}{n} - 6\right)$$

$$= \mathrm{Var}\left(\frac{X_1 + \dots + X_n}{n}\right)$$

$$= \mathrm{Var}\left(\frac{X_1}{n}\right) + \dots + \mathrm{Var}\left(\frac{X_n}{n}\right) = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$$



$h(g(x)) = x$

$g(h(y)) = y$

$\Rightarrow$ Ex $g(x) = x^2$ on $(0, \infty)$

$h(y) = \sqrt{y}$

$h(g(x)) = \sqrt{x^2} = x$

Date : ○○○

Change of variable formula

$Y = g(X)$ so $h(Y) = X$

$F_Y(y) = P(Y \le y)$

$= P(g(X) \le y)$ $\quad$ g is a strictly increasing fn.

$= P(X \le g^{-1}(y))$

$= F_X(h(y))$ $\quad$ since $g^{-1}(y) = h(y)$

$f_Y(y) = F_X'(h(y)) \cdot h'(y)$

$= f_X(h(y)) \cdot h'(y)$

$g(h(y)) = y$ take derivative

$g'(h(y)) \cdot h'(y) = 1$

$$h'(y) = \frac{1}{g'(h(y))}$$

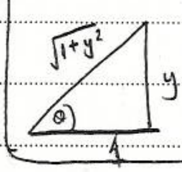
Ex $\arctan y = y$

$$\arctan'(y) = \frac{1}{\tan'(\arctan y)} \Rightarrow$$

$$\boxed{A(x) = \frac{\sin x}{\cos x} \Rightarrow A'(x) = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x}}$$

$$= \cos^2(\arctan y) \Rightarrow$$

$$= \frac{1}{1+y^2}$$

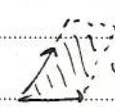


$\cos\theta = \frac{1}{\sqrt{1+y^2}}$

$\cos^2\theta = \frac{1}{1+y^2}$

Definion of the Jacobian

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$



(x_1, x_2) transform to (y_1, y_2) by

$T(x_1, x_2) = (y_1(x_1, x_2), y_2(x_1, x_2))$ and inverse $T^{-1}(y_1, y_2) = (x_1(y_1, y_2), x_2(y_1, y_2))$

$$\text{Jacobian} = J(y_1, y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Ex $x_1(y_1, y_2) = y_1^3 e^{y_2}$

$$\frac{\partial x_1}{\partial y_1} = 3y_1^2 e^{y_2} \quad \Bigg| \quad \frac{\partial x_1}{\partial y_2} = y_1^3 e^{y_2}$$

Ex $x_1 = r\cos\theta \implies r = \sqrt{x_1^2 + x_2^2}$

$x_2 = r\sin\theta \qquad \theta = \arctan(x_2/x_1)$

Say $\left.\begin{array}{l}\theta = [0, 2\pi) \\ r = [0, \infty)\end{array}\right\}$ to eliminate the problem when diff.

** Jacobian : Use $dx_1 \cdot dx_2 = |J|\, dy_1 dy_2$

$J = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r \qquad$ So, $dx_1 dx_2 = r\,dr\,d\theta$

Change of variable Theorem

f_{X_1,X_2} joint density of X_1, X_2 $(Y_1, Y_2) = T(X_1, X_2)$ with Jacobian J. For points in the range of

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1(y_1,y_2), x_2(y_1,y_2))\,|J(y_1,y_2)|$$

$$T(X_1, X_2) = (\overset{Y_1}{X_1 + X_2}, \overset{Y_2}{X_1/X_2})$$

$$T^{-1}(Y_1, Y_2) = (X_1(Y_1,Y_2), X_2(Y_1,Y_2))$$

$Y_1 = X_1 + X_2, \quad Y_2 = X_1/X_2$

$X_1 = X_2 \cdot Y_2$

$\implies Y_1 = X_2 \cdot Y_2 + X_2$

$Y_1 = X_2(Y_2 + 1)$

$$X_2 = \frac{Y_1}{Y_2 + 1}$$

$$X_1 = \frac{Y_1 Y_2}{Y_2 + 1}$$

Then, $J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} = \frac{Y_2}{Y_2+1} & \frac{\partial X_2}{\partial Y_1} = \frac{1}{y_2+1} \\ \cdots & \cdots \end{vmatrix}$

find the density function

$$f_{X_1,X_2}(x_1,x_2) = \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2}$$

$$\text{Ctm.} = \frac{X_1 + \ldots + X_{6n} - n}{\text{std.}(X_1 + \ldots X_n)}$$

Clicker Question Roll 6n die

$$P(\text{Alice loses}) = \sum_{n=0}^{n-1} \binom{6n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{6n-k}$$

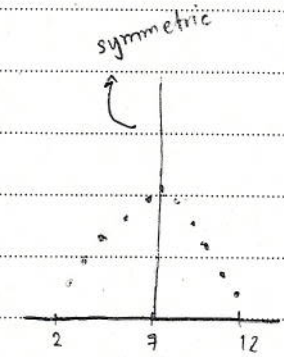
→area

$_2(y_1, y_2))$

October 29, 2009

Section 3.8 and 4.8

X_1, X_2 independent uniform $(0,1)$. What is X_1+X_2?



Ex $Y = X_1 + X_2$

$f(y) = 0$ if $y \in [0,2]$

$$\text{Prob (2 dies get } k) = \begin{cases} \frac{k-1}{36} & \text{if } k \in \{2,\dots,7\} \\ \frac{12-(k-1)}{36} & \text{if } k \in \{8,\dots,12\} \end{cases}$$

$X_1 + X_2$ as likely as $(1-X_1)+(1-X_2) = 2-(X_1+X_2)$

Convolution

Definition: $(f*g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$

Interpretation: x and y with densities f and g then density of $x+y$ is $f*g$.

$Y = X_1 + X_2$, $X_1 \sim$ Uniform $(0,1)$

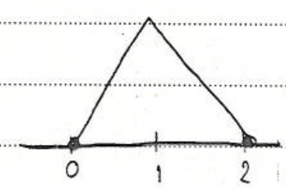
$$(f*g)(y) = \int_{-\infty}^{\infty} f_{X_1}(t) \cdot f_{X_2}(y-t)\, dt$$

$$= \int_0^1 \underbrace{f_{X_1}(t)}_{1} \cdot f_{X_2}(y-t)\, dt \qquad f_{X_1}(u) = \begin{cases} 1 & , 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_0^1 f_{X_2}(y-t)\, dt \quad , 0 \le y \le 2$$

Case 1: $0 \le y \le 1 \Rightarrow \int_0^1 f_{X_2}(y-t)\, dt = \int_0^y f_{X_2}(y-t)\, dt$

$$= \int_0^y 1\, dt = y$$



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$Y = X_1 + X_2$

$\Downarrow$

$$P(Y \le y) = \int_{x_1=-\infty}^{\infty}\int_{x_2=-\infty}^{y-x_1} f_{X_1}(x_1)\cdot f_{X_2}(x_2)\,dx_1 dx_2$$

$$F_X(y) = \int_{x_1=-\infty}^{\infty} f_{X_1}(x_1)\cdot \underbrace{F_{X_2}(y-x_1)}_{F_{X_2}(y-x_1)-F_{X_2}(-\infty)}dx_1$$

$$f_X(y) = \int_{x_1=-\infty}^{\infty} f_{X_1}(x_1)\cdot f_{X_2}(y-x_1)(1)\,dx_1$$

Let $t = x_1$; $f_X(y) = \int_{-\infty}^{\infty} f_{X_1}(t)\cdot f_{X_2}(y-t)\,dt$

Aside: why do we not always be able to exchange two sum?

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} \neq \sum_{n}\sum_{m} a_{mn}$$

$$\left.\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ 0 & 0 & -1 & \vdots \\ 0 & -1 & +1 & \cdot \\ -1 & +1 & 0 & \cdot \\ +1 & 0 & 0 & \cdot \end{matrix}\right\}$$ the sum $0 = 1$. WRONG!!

Properties of Convolution

Proof : $f*g = g*f$

① $(f*g)(x) = \int_{-\infty}^{\infty} f(t)\cdot g(x-t)\,dt$

$\Downarrow$ algebra

$= (g*f)(x)$

density of $x+y$ $=$ density of $y+x$

② $(\widehat{f*g})(x) = \hat{f}(x)\cdot\hat{g}(x)$ where $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\cdot e^{-\pi i x\xi}$ is fourier transform

Definition fourier transform : $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\cdot e^{-2\pi i x\xi}\,dx$

Let $\mathcal{F}(f) = \hat{f}$

$\mathcal{F}(f*g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$

Proof $(\widehat{f*g})(x) = \int_{-\infty}^{\infty} (f*g)(x)\, e^{-2\pi i x \xi}\, dx$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) \cdot g(x-t)\, e^{-2\pi i x \xi}\, dt \right] dx$$

Question Is $\iint f(t)\, g(x-t)\, e^{-2\pi i x \xi}\, dt\, dx$ finite?

Recall : $e^{i\theta} = \cos\theta + i\sin\theta$

$\overline{a+ib} = a - ib$ {conjugate}

$e^{0} = e^{i\theta} \cdot e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$

$= \cos^2\theta + \sin^2\theta = 1$

Define $e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$e^{z} \cdot e^{w} = e^{z+w}$

$$\int_{x=-\infty}^{\infty} \int_{t=-\infty}^{\infty} f(t) \cdot g(x-t)\, dt\, dx = \int_{t=-\infty}^{\infty} f(t) \underbrace{\left[\int_{x=-\infty}^{\infty} g(x-t)\, dx \right]}_{\int_{-\infty}^{\infty} g(u)\, du \,=\, 1} dt$$

$u = x - t$
$du = dx$

$\Rightarrow$ because it's the density function.

$$= \int_{-\infty}^{\infty} f(t)\, dt = 1 \text{ ✓✓ finite}$$

$$= \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} g(x-t)\, e^{-2\pi i x \xi}\, dx \right] dt$$

$\Rightarrow$ ok to switch the order of integral b/c we now know that the integral = finite

$x - t + t$

$e^{-2\pi i ((x-t)+t)\xi} = e^{-2\pi i (x-t)\xi} \cdot e^{-2\pi i t \xi}$

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$$\widehat{(f*g)}(x) = \int_{t=-\infty}^{\infty} f(t)\cdot e^{-2\pi i t\xi}\left[\int_{x-\infty}^{\infty} g(x-t)\, e^{-2\pi i (x-t)\xi}\, dx\right] dt$$

Let $u = x-t \Rightarrow \int_{u=-\infty}^{\infty} g(u)\, e^{-2\pi i u\xi}\, du = \hat{g}(\xi)$

$$= \hat{g}(\xi)\int_{t=-\infty}^{\infty} f(t)\cdot e^{-2\pi i t\xi}\, dt$$

$$= \hat{f}(\xi)\cdot\hat{g}(\xi) \quad \# \quad \underline{\text{TRUE}}$$

③ $f*\delta = f$ where δ is the Diract delta functional

④ $f*(g*h) = (f*g)*h$

<u>Proof of Associaty</u> 2 ways to proof

① Tedious Algebra

② $f*(g*h) = (f*g)*h$

$X + (Y + Z) \sim (X + Y) + Z$

Set : prob densities S

- * Operation : Convolution *

$* : S \to S$

(1) closed

(2) associaty

(3) identity

Challenge inverse

<u>Ex</u> $\widehat{(f*g*h)}(\xi) = \hat{f}(\xi)\cdot\hat{g}(\xi)\cdot\hat{h}(\xi)$

<u>Proof</u> : $\widehat{((f*g)*h)}(\xi) = \widehat{(f*g)}(\xi)\,\hat{h}(\xi)$

$= \hat{f}(\xi)\cdot\hat{g}(\xi)\cdot\hat{h}(\xi)$

Say $f=g=h$, $\widehat{(f*f\ldots*f)}(\xi) = \hat{f}(\xi)^n$

$(f * \dots * f)(x) = \mathcal{F}^{-1}(\mathcal{F}(f)^n)$

<u>Nov 3, 2009</u>

<u>Random Walks</u>

p_n - Prob win when start at n

$p_0 = 0$

$p_N = 1$

Right => prob. = p
Left => prob. = $1-p$

$$p_n = p \cdot p_{n+1} + (1-p)\, p_{n-1}$$

$$p \cdot p_{n+1} - p_n + (1-p)\, p_{n-1} = 0$$

$$p_{n+1} - \frac{1}{p} p_n + \frac{1-p}{p} p_{n-1} = 0$$

<u>By Divine Inspiration</u> Guess $p_n = r^n$

$$r^{n+1} - \frac{1}{p} r^n + \frac{1-p}{p} \cdot r^{n-1} = 0$$

$$r^{n-1} \underbrace{\left(r^2 - \frac{1}{p} r + \frac{1-p}{p} \right)}_{\text{polynomial}} = 0$$

General solution : $p_n = c_1 r_1^n + c_2 r_2^n$ when $r_1 \neq r_2$

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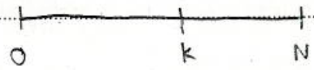

T_n = expected # tosses till hit boundary starting at n.

$$T_n = pT_{n+1} + (1-p)T_{n-1} + 1$$

$$T_{n+1} - \frac{1}{p}T_n + \frac{1-p}{p}T_{n-1} + \frac{1}{p} = 0$$

<u>Guess</u> $T_n = r^n$

$$r^{n+1} - \frac{1}{p}r^n + \frac{1-p}{p}r^{n-1} + \frac{1}{p} = 0$$

<u>Guess</u> $T_n = r^n + a$

$$0 = r^{n+1} - \frac{1}{p}r^n + \frac{1-p}{p}r^{n-1} + a - \frac{1}{p}a + \left(\frac{1-p}{p}\right)a + \frac{1}{p}$$

$$0 = r^{n-1}\left(r^2 - \frac{1}{p}r + \left(\frac{1-p}{p}\right)\right) + \left[a\left(\frac{p-1+1-p}{p}\right) + \frac{1}{p}\right]$$

DOESN'T WORK

<u>Guess</u> $T_n = r^n + na$

$$0 = r^{n-1}\left(r^2 - \frac{1}{p}r + \frac{1-p}{p}\right) + \left((n+1)a - \frac{1}{p}na + \left(\frac{1-p}{p}\right)(n-1)a + \frac{1}{p}\right)$$

$$na\left(\frac{p-1+1-p}{p}\right) + a - \frac{1-p}{p}a + \frac{1}{p} = 0$$

$$0 + a\left(\frac{2p-1}{p}\right) + \frac{1}{p} = 0$$

$$a = \frac{-1}{2p-1} = \frac{1}{1-2p}$$

$\chi^2(n) = C_n \cdot x^{\frac{n}{2}-1} \cdot e^{-x/2}$

Section 4.10

- $\overline{X} \sim N(\underset{\text{mean}=0}{0}, \underset{\text{variance}=1}{1})$ then X^2 is chi-square with 1 degree of freedom

- Sample mean : $\overline{X} = \frac{1}{N} \sum_{i=1}^{n} X_i$

- Sample Variance : $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$

Let $X_1, \ldots, X_n$ be i.i.d $N(\mu, \sigma^2)$. Then

- $\overline{X} = N(\mu, \sigma^2/n)$
- $(n-1)S^2$ is a chi-square with $n-1$ degrees of freedom
- $\overline{X}$ and S^2 are independent
- Central limit theorem : $\overline{X} \sim N(\mu, \sigma^2/n)$

Aside f $F \Rightarrow$ antiderivative of f

Generating Function

$$G_a(s) = \sum_{n=0}^{\infty} a_n s^n \quad \text{for all } s \text{ where the sum converges}$$

$$G(s) = \sum_{n=0}^{\infty} p^n s^n = \frac{1}{1-ps}, \quad |ps| < 1$$
$$|s| < \frac{1}{p}$$

Geometric R.V.

$$\text{Prob}(\underline{X} = n) = (1-p)^n \cdot p$$
$$a_n = (1-p)^{n-1} \cdot p$$
$$G_a(s) = \sum_{n=1}^{\infty} (1-p)^{n-1} \cdot p \cdot s^n$$
$$= \frac{p}{1-p} \sum_{n=1}^{\infty} \left((1-p)s\right)^n = \frac{p}{1-p}\left[\frac{1}{1-(1-p)s} - \underset{\text{when } n=0}{1}\right]$$

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$a_n = \frac{1}{n!}$ Then, $G_a(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot s^n = e^s$

$a_n = n!$ $G_a(s) = \sum_{n=0}^{\infty} n! \cdot s^n$

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Uniqueness Theorem (U.T)

$\{a_m\}_{m=0}^{\infty}$ and $\{b_m\}_{m=0}^{\infty}$: two sequences with generating fn $G_a(s)$ and $G_b(s)$ which converge for $|s| < r$. two sequences are equal iff $G_a(s) = G_b(s)$ for all $|s| < r$. We may recover the sequence from the G. function by differentiating

$$a_m = \frac{1}{m!} \frac{d^m G_a(s)}{ds^m}$$

Application

$a_m = P(X = m)$, $G_a(s)$ exists for $|s| < 1$

Proof of U.T.

$G_a(s) = \sum_{n=0}^{\infty} a_n s^n = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots$

$G_a(0) = a_0$

$G_a'(0) = a_1$

$G_a''(s) = 2 \cdot 1 a_2 + 3 \cdot 2 a_3 s + 4 \cdot 3 \cdot a_4 s^2 + \ldots$

$G_a''(0) = 2! a_2$

$G_a'''(0) = 3! a_3$

If $a_m = b_m$, Clear $G_a(s) = G_b(s)$

If $G_a(s) = G_b(s)$, then $G_a^{(m)}(s) = G_b^{(m)}(s)$

$m! a_m = m! b_m$ //

$G_X(s) = E(s^X)$

Proof of $E[X] = G_X'(1)$

$a_m = P(X = m)$

$G_X(s) = \sum_{m=0}^{\infty} P(X = m) \cdot s^m$ by definition

$G_X'(s) = \sum_{m=0}^{\infty} P(X = m) \cdot m \cdot s^{m-1}$ => take $s = 1$

$G_X'(1) = \sum_{m=0}^{\infty} P(X = m) \cdot m = E(X)$ ✓✓✓

$E[s^X] = \sum_{m=0}^{\infty} s^m \cdot \text{Prob}(X = m)$

$= G_X(s)$ ✓✓✓

$\therefore$ Prob Generating Function is $G_X(s) = E[s^X]$

Moment Generating Function : $M_X(t) = E(e^{tX})$

Theorem X = random variable with moments μ_k'

$M_X t = 1 + \mu_1' t + \frac{\mu_2' t^2}{2!} + \frac{\mu_3' t^3}{3!} + \ldots$

) Gb(s)
or all

Date : ○○○ Review Session

③ $\chi^2(d) : C_d\, x^{d/2 - 1} e^{-x/2}$ $\Rightarrow$ Induction

$Y = X_1^2 + \dots + X_n^2 = \chi^2(n)$

$U = Y + X_i^2 = Y + \chi^2(1)$

$$f_U(u) = \left(f_Y * f_{\chi^2(1)}\right)(u) = \int_{-\infty}^{\infty} f_Y(t) \cdot f_{\chi^2(1)}(u-t)$$

$$= \int_0^U \left(C_n t^{n/2-1} \cdot e^{-t/2}\right)\left(C_1 (u-t)^{-1/2} \cdot e^{-(U-t)/2}\right) dt$$

integral ตัวแปรเป็น (t)

$$= C_1 C_n e^{-U/2} \int_0^U t^{n/2-1} \cdot (u-t)^{-1/2}\, dt \quad \Rightarrow t = Uw \quad \Rightarrow w = \frac{t}{U}$$

$$= C_1 C_2 e^{-U/2} \int_0^1 (Uw)^{n/2-1} (U - Uw)^{-1/2}\, U\, dw$$

$$= e^{-U/2} C_1 C_2 \int_0^1 U^{n/2 - \cancel{1}} \cdot U^{-1/2}\, w^{n/2-1} (1-w)^{-1/2}\, \cancel{U}\, dw$$

$$= e^{-U/2} C_1 C_2\, U^{\left(\frac{n+1}{2}-1\right)} \underbrace{\int_0^1 w^{n/2-1} (1-w)^{-1/2}\, dw}_{\text{no } U \text{ dependent}}$$

$\Rightarrow$ just a constant!

$$= C e^{-U/2} \cdot U^{\left(\frac{n+1}{2}-1\right)} = \chi^2(n+1)$$

<u>Base Case</u> $X_1^2 + X_2^2$

Let $Y = X^2$

$$P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$

*standard normal square is the chi-square *

$\hookrightarrow$ plug in the chi-square distribution will get χ^2