② Fourier transforms of the densities for the uniform distribution on [0,1] and uniform distribution on [-1/2, 1/2]

$$\hat{f}(y) = \int_0^1 f_X(x)\, e^{-2\pi i x y}\, dx$$

$$= \int_0^1 (1)\, e^{-2\pi i x y}\, dx$$

$$= \int_0^1 [\cos(-2\pi xy) + i \sin(-2\pi xy)]\, dx$$

$$= \frac{-1}{2\pi y} \left(-\sin(-2\pi xy) + i\cos(-2\pi xy)\right)\Big|_0^1$$

$$= \frac{-1}{2\pi y}\left(-\sin(-2\pi y) + i\cos(-2\pi y) - i\right) \quad \#$$

<u>November 6, 2009</u>

Prob Generating Function : $G_X(s) = E[s^X]$.

Moment Generating fn : $M_X(t) = E(e^{tX})$.

$t=0$; $M_X(0) = E[e^{0X}] = E[1] = 1$

$X \sim Exp(\lambda)$

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\, dx \quad \Rightarrow \quad f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_0^{\infty} e^{tx} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \int_0^{\infty} \lambda\, e^{-\lambda(x - \frac{t}{\lambda}x)}\, dx$$

$$= \int_0^{\infty} \lambda \cdot e^{-\lambda(1-\frac{t}{\lambda})x}\, dx$$

$1 - \frac{t}{\lambda} > 0$

$t < \lambda$

$$= \frac{1}{(1-\frac{t}{\lambda})} \int_0^{\infty} \lambda(1-\frac{t}{\lambda})\, e^{-\lambda(1-\frac{t}{\lambda})x} = (1-\frac{t}{\lambda})^{-1} = \frac{1}{1-\frac{t}{\lambda}}$$

Margin (cut off, from facing page): $\Rightarrow w = \frac{t}{c}$; dw; ⇒ just a constant!; square ... *; ~~distribution~~

Date : ○○○ Proof $M_{X+Y} = M_X(t) \cdot M_Y(t)$

$$M_{X+Y} = E\left[e^{t(X+Y)}\right]$$
$$= E\left[e^{tX} \cdot e^{tY}\right] \Rightarrow X, Y \text{ independent}, \ e^{tX} \text{ and } e^{tY} \text{ are independent}, s$$
$$= E[e^{tX}] \cdot E[e^{tY}]$$
$$= M_X \cdot M_Y \quad \#$$

Convolutions

- $(a_2x^2 + a_1x + a_0)(b_1x + b_0) = a_2b_1x^3 + (a_2b_0 + a_1b_1)x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$
- $(a_2x^2 + a_1x + a_0)(b_3x^3 + b_2x^2 + b_1x + b_0) = a_2b_3x^5 + (a_2b_2 + a_1b_3)x^3 + (a_2b_1 + a_1b_2 + a_0b_3)x^3 + \dots$

$$\Downarrow$$

$$\therefore \left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum C_k x^k$$

$$\sum_{l=0}^{k} a_{k-l}\, b_l = C_k$$

$$G_X(s) = \sum \text{Prob}(X = n) \cdot s^n$$
$$G_Y(s) = \sum \text{Prob}(Y = n) \cdot s^n$$
$$G_X(s) \cdot G_Y(s) = \sum_k \left[\sum_l \text{Prob}(X = k-l) \cdot \text{Prob}(Y = l)\right] s^k$$
$$= G_{X+Y}(s) \quad \#$$

Aside $A(x) = f(x) \cdot g(x)$

Proof $A'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

① $A'(x) = \lim_{x \to 0} \dfrac{A(x+h) - A(x)}{h}$

② substitute for $A(x)$ and $A(x+h)$

⇒ Then, add and subtract zero.

③ algebra

Proof that $M_{\alpha X + \beta}(t) = e^{\beta t} \cdot M_X(\alpha t)$

step ① Definition: Recall that $M_X(t) = E(e^{tX})$

step ② $M_{\alpha X + \beta}(t) = E\left[e^{t(\alpha X + \beta)}\right]$, $Y = \alpha X + \beta$
$$= E\left[e^{t\alpha \cdot X} \cdot e^{\beta t}\right]$$
$$= e^{\beta t}\, E\left[e^{t\alpha \cdot X}\right]$$
$$= e^{\beta t} \cdot M_X(\alpha t) \quad \#$$

Ex Poisson Example

$X_i \sim \text{Poisson}(\lambda)$:

$$f(n) = \text{Prob}(X_i = n) = \frac{\lambda_i^{\,n} \cdot e^{-\lambda_i}}{n!} \quad \text{for } n \geq 0 \text{ and } 0 \text{ otherwise}$$

$$M_X(t) = \sum_{n=0}^{\infty} e^{tn} \cdot f_n = e^{\lambda(e^t - 1)}$$

Proof Moment Generating Function for Poisson (λ)

$$M_X(t) = \sum_{n=0}^{\infty} e^{tn} \cdot \frac{\lambda^n \cdot e^{-\lambda}}{n!}$$

Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$M_X(t) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{e^{tn} \cdot \lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(e^t \cdot \lambda)^n}{n!}$$

$$= e^{-\lambda} \cdot e^x \quad \text{when } x = \lambda e^t$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)} \quad \#$$

$$M_X(t) = e^{\lambda(e^t - 1)}$$

$$M_{X_1}(t) \cdot M_{X_2}(t) = e^{\lambda_1(e^t - 1)} \cdot e^{\lambda_2(e^t - 1)}$$

$$= e^{(\lambda_1 + \lambda_2)(e^t - 1)}$$

For discrete random variable in $\{0, 1, 2, \ldots\}$ in Posson, the moment generating function is unique!

Take $\lambda = \lambda_1 + \lambda_2$

Conclude sum of two independent Poissons is a Poisson with parameter $\lambda = \lambda_1 + \lambda_2$

Date : ○○○

Dream Theorem: A prob distribution is uniquely determined by its moments

↳ **NOT TRUE!**

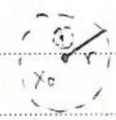
* There exists prob. distributions with same moments.

Ex $f_1(x) = \dfrac{1}{\sqrt{2\pi x^2}} \cdot e^{-(\log^2 x)/2}$

$$f_2(x) = f_1(x)\left[1 + \sin(2\pi \log x)\right]$$

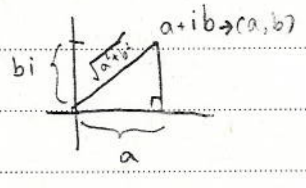
Complex Analysis

Open set, Closed set

$B(x_0, r) = \{x \text{ such that } |x - x_0| < r\}$ $\Rightarrow$ 

open subset

$|a+ib| = \sqrt{a^2+b^2}$



Nov. 10, 2009 : Limit Point

Ex $1, \frac{1}{2}, 2, \frac{1}{4}, \ldots$

$z_1 = 1,\ z_2 = \frac{1}{2},\ z_3 = 2 \ \ldots\ z_{nk}$

Ex $h(x) = x^3 \sin(1/x)$

⇓

$\dfrac{1}{x_n} = 2\pi n \Rightarrow x_n = \dfrac{1}{2\pi n} \Rightarrow$ to make $\sin(\ ) = 0$

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{(z+h)^3 \cdot \sin\left(\frac{1}{z+h}\right) - z^3 \sin\frac{1}{z}}{h}$$

When $z=0$; $\displaystyle\lim_{h\to 0} \frac{h^3 \sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h^2 \sin\frac{1}{h}$ ← *

Recall: $e^{i\theta} = \cos\theta + i\sin\theta$ or $\sin\theta = (e^{i\theta} - e^{-i\theta})/2$

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ plug in z

$= \lim\limits_{h\to 0} \dfrac{h^2 \cdot e^{i/h} - e^{-i/h}}{2i}$

$z = a+ib$

$\bar{z} = a-ib$

If $h_2 = 0$; $= \lim\limits_{h_1\to 0} \dfrac{h^2 \cdot e^{i/h_1} - e^{-i/h_1}}{2i} = 0$

$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2+b^2}$

$e^{ih} \cdot e^{-ih} = e^0 = 1$

If $h_1 = 0$; $= \lim\limits_{h_2\to 0} \dfrac{(ih_2)^2 (e^{i/h_2} - e^{-i/h_2})}{2i} = \infty$

Check $\lim\limits_{h\to 0} h^2 \cdot e^{1/h}$ $x = \frac{1}{h}$

$\lim\limits_{x\to\infty} \frac{1}{x^2} \cdot e^x$ L'HOSPITAL

$= \lim\limits_{x\to\infty} \dfrac{e^x}{2x}$ L'HOSPITAL AGAIN

$= \lim\limits_{x\to\infty} \dfrac{e^x}{2} = \infty$ $\Rightarrow$ not complex differentiable

Laplace and Fourier Transform

General framework: Given $K(t,s)$ consider

$$g(s) = \int_{-\infty}^{\infty} f(t) \cdot K(t,s)\, dt$$

$M_X(t) = E[e^{tX}] \Rightarrow$ Moment Generating Function

$$= \int_{-\infty}^{\infty} f_X(t) \cdot e^{tx}\, dt$$

Laplace transform: $f(t) = \begin{cases} 0 & \text{otherwise} \\ e^{-t} & t \geq 0 \end{cases}$

$$(\mathcal{L}f)(s) = \int_0^{\infty} e^{-t} e^{-st}\, dt$$

If $s = -2$, $\int_0^{\infty} e^{t}\, dt = +\infty$ $\Rightarrow$ Laplace transform does not have to exist for all value of x

Date :

* BUT The Fourier transform always exists for all value of x.

Schwartz Space $\Rightarrow \sup_{x \in \mathbb{R}} \left| (1+x^2)^m \frac{d^n f}{dx^n} \right| < \infty$

Ex $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Ex $f(x) = e^{-x}$; $x \geq 0$

$g(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ $\Rightarrow$ fails for $m > 0$, $(n=0)$

MGF = moment generating function

Ex $x_1, x_2, \dots$ $p_1, p_2, \dots$

$n=1$, mean $\mu = p_1 x_1$

↳ take on one point only

$n=2$, mean $\mu = p_1 x_1 + p_2 x_2 = \frac{1}{2}x_1 + \frac{1}{2}x_2$

2^{nd} moment $\mu_2' = p_1 x_1^2 + p_2 x_2^2 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

$n=3$, mean $\mu = \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3$

2^{nd} moment $\mu_2' = \frac{1}{3}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{3}x_3^2$

3^{rd} moment $\mu_3' = \frac{1}{3}x_1^3 + \frac{1}{3}x_2^3 + \frac{1}{3}x_3^3$

$\Bigg\} \; f(x) = \begin{cases} \frac{1}{3} & x \in (x_1, x_2, x_3) \\ 0 & \end{cases}$

$\Downarrow$

density function can be found through knowing the moment

November 13, 2009 : Accumulation Points and Moments

Cauchy : $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{e^{tx}}{1+x^2} \, dx$

$\hookrightarrow$ exists only for $t = 0$

Exponential : $m_X(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} \cdot e^{-\lambda x} \, dx$; if $t < \lambda$, exists

Laplace Distribution : $f_X(x) = e^{-|x|/2}$

$$M_X(t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{tx} \cdot e^{-|x|} \, dx$$

$t < 1$ $\qquad$ $t > -1$

$$= \frac{1}{2}\int_0^{\infty} e^{tx} \cdot e^{-x} \, dx + \frac{1}{2}\int_{-\infty}^{0} e^{tx} \cdot e^{x} \, dx \quad ; \text{ so } |t| < 1$$

Cauchy

$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ $\Rightarrow$ slow decay (to infinity)

$f(e^t) = \frac{1}{\pi} \cdot \frac{1}{1+e^{2t}}$ $\Rightarrow$ fast decay (to infinity)

Accumulation Points and Moments

Proof $h(x) = f(x) - g(x)$

By assumption, h is bounded.

$$A(z) = \int_0^{\infty} x^z \cdot h(x) \, dx$$

$$|A(z)| \leqslant \int_0^{\infty} x^{\text{Re}(z)} |h(x)| \, dx$$

real number

$$\leqslant \int_0^{\infty} x^{\text{Re}(z)} f(x) \, dx + \int_0^{\infty} x^{\text{Re}(z)} \cdot g(x) \, dx$$

Aside: $x^{r} = e^{r \log x}$

$x^{a+ib} = x^a \cdot x^{ib} = x^a \cdot e^{ib \log x}$

$|x^{a+ib}| = x^a$

$$x^{\text{Re}(z)} \leqslant x^{\lfloor \text{Re}(z) \rfloor} + x^{\lceil \text{Re}(z) \rceil}$$

floor $\qquad$ ceiling

x_1, x_2, x_3)

be found

moment

Date : ○○○

$A(n) = 0$; n pos int

$A(r_n) = 0$; r_n sequence

$$A'(z) = \int_0^\infty \left(\frac{d}{dz} x^z\right) h'(x)\,dx \qquad \Rightarrow x^z = e^{z\ln x} = \sum_{k=0}^{\infty} \frac{(z\ln x)^k}{k!}$$

$$\frac{d}{dz} x^z = (\ln x) e^z = (\ln x) \cdot x^z$$

$$A'(z) = \int_0^\infty (\ln x) \cdot x^z h(x)\,dx$$

$\rightarrow$ x big, $\ln x \le x^2$ at ∞

x small, $x^z \ln x$ is ok as $x \rightarrow 0$

Complex Analysis $\Rightarrow A(z) \equiv 0 \quad Re(z) > 0$

$A(c + 2\pi i y) \qquad h(t)dt = h(e^t) e^t dt$

$$A(z) = \int_{-\infty}^{\infty} e^{tz} h(t)\,dt$$

$$0 = A(c + 2\pi i y) = \int_{-\infty}^{\infty} e^{-2\pi i y t} \left[e^{cty} h(t)\right] dt$$

Therefore, $e^{ct} h(t) = 0$

$h(t) = 0 \Rightarrow$ implies that $h(x) = 0$

END OF THE PROOF

Ex two densities

$$f_1(x) = \frac{1}{\sqrt{2\pi x^2}} \cdot e^{-(\log^2 x)/2}$$

$$f_2(x) = f_1(x)\left[1 + \sin(2\pi \log x)\right]$$

$$** \; e^{-(\log^2 x)/2} = \left(e^{-\log x}\right)^{\frac{\log x}{2}} = \left(\frac{1}{e^{\log x}}\right)^{\frac{\log x}{2}} = \left(\frac{1}{x}\right)^{\frac{\log x}{2}}$$

These two functions $\Rightarrow$ same integral moments e^{tk}

a^{th} moment of $f_2 = e^{a^2/2} + e^{(a-2i\pi)^2/2} (1 - e^{4ia\pi})$

Look at $1 - e^{4\pi i a} = 1 - \cos(4\pi a) + i\sin(4\pi a)$

$a = \frac{m}{2}, m \in \mathbb{Z}$ $\qquad$ $a = \frac{n}{4}, n \in \mathbb{Z}$

* Normalization of r.v.

$$Y = \frac{X - E(X)}{\text{stDev}(X)} = \frac{X - \mu}{\sigma}$$

For standard normal, $E(Y) = 0$ and $\text{st.Dev } Y = 1$

CLM $\quad \bar{X}_n = \frac{x_1 + \dots + x_n}{N}$

$$Z_n = \frac{\bar{x}_n - \mu}{\sigma / \sqrt{N}}$$

Note $\log_b x = \frac{\ln x}{\ln b} = \frac{\log x}{\log b}$

Kurtosis $\Rightarrow$ how much mass you have at ∞

Moment Generating fn of standard normal

$X \sim N(0,1)$

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \qquad \Rightarrow \text{Complete the square}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x^2 - 2tx)}{2}} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2tx + t^2 - t^2)} dx$$

$$M_{\bar{X}}(t) = e^{t/p}\left[\frac{pe^{pt/\sqrt{n(1-p)}}}{1-(1-p)e^{pt/\sqrt{n(1-p)}}}\right]^n$$

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$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} \cdot e^{t^2/2}\,dx$$

$$= e^{t^2/2} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}}\,dx}_{N(t,\sigma)}$$

$$= e^{t^2/2} \quad \#$$

November 16, 2009

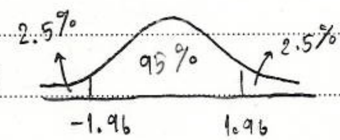
Central Limit Theorem

$X_1, \dots X_n$ be independent, idrv

$$\bar{X}_n = \frac{X_1 + \dots + X_n}{n}$$

$P(|Z| < 1) \approx 68.2\%$

$P(|Z| \leq 1.96) \simeq 95\%$ $\Rightarrow$



MGF of normal distributions

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}, \quad \mu = 0, \sigma = 1$$

$$M_Z(t) = e^{t^2/2}$$

Proof we prove earlier that $M_Z(t) = e^{t^2/2}$

From $M_{aX+b}(t) = e^{bt} \cdot M_X(at)$

S·M·I·L·E

$X \sim N(\mu, \sigma^2)$; $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

$X = \sigma Z + \mu$

$$M_X(t) = M_{\sigma Z + \mu}(t)$$
$$= e^{\mu t} \cdot e^{(\sigma t)^2/2}$$ # end of proof !!

Poisson Example

$\bar{X}_n = \dfrac{X_1 + \dots + X_n}{n}$, $Y = \dfrac{\bar{x} - E(\bar{x})}{StDev(\bar{x})}$ => If $n \to \infty$, Y converges to having the standard normal distribution

Recall Poisson (λ)

density $f(n) = \begin{cases} \dfrac{\lambda^n e^{-\lambda}}{n!} & ; n \in \{0,1,2,\dots\} \\ 0 & ; \text{otherwise} \end{cases}$ => Mean λ, StDev $\sqrt{\lambda}$

General Proof via MGF

X_i's iidrv.

$$Z_n = \frac{\bar{x} - \mu}{\sigma / \sqrt{N}} = \sum_{n=1}^{N} \frac{X_i - \mu}{\sigma \sqrt{N}}$$

M.G.F is :

$$M_{Z_n}(t) = \prod_{n=1}^{N} e^{\frac{-\mu t}{\sigma \sqrt{N}}} \cdot M_X\left(\frac{t}{\sigma \sqrt{N}}\right) = e^{\frac{-\mu t \sqrt{N}}{\sigma}} M_X\left(\frac{t}{\sigma\sqrt{N}}\right)^N$$ #

* $$M_{Z_n}(t) = M_{\sum \frac{x_i - \mu}{\sigma \sqrt{N}}}(t) = \prod_{n=1}^{N} M_{\frac{X_i - \mu}{\sigma\sqrt{N}}}(t)$$
$$= \prod_{n=1}^{N} e^{-\mu t / \sigma\sqrt{N}} M_X\left(\frac{t}{\sigma\sqrt{N}}\right)$$

$$M_X(t) = E\left[e^{tX}\right]$$
$$M_X(0) = E[1] = 1 = \int_{-\infty}^{\infty} e^{0 \cdot x} f(x)\,dx$$

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Recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

extra credit => show that these two are the same (hint: use binomial power)

$$e^x = \lim_{N\to\infty} \left(1 + \frac{x}{N}\right)^N$$

Aside $\lim_{N\to\infty} \left(1 + \frac{x}{N^2}\right)^N = \lim_{N\to\infty} \left(1 + \frac{(x/N)}{N}\right)^N$

$$\leq \lim_{N\to\infty} \left(1 + \frac{x/m}{N}\right)^N \quad \text{fixed } m$$

$$= e^{x/m}$$

Aside $\log(1-u) = -\left(u + \frac{u^2}{2} + \frac{u^3}{3} + \frac{u^4}{4}\right)$

$$\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots$$

$$\lim_{N\to\infty} \left(1 + \frac{x}{N^2}\right)^N$$

$$p_N = \left(1 + \frac{x}{N^2}\right)^N$$

$$\log p_N = N \log\left(1 + \frac{x}{N^2}\right), \quad u = \frac{x}{N^2}$$

$$= N\left(\frac{x}{N^2} - \frac{x^2}{2N^4} + \frac{x^3}{3N^6} - \dots\right)$$

$$= \frac{x}{N} - \frac{x^2}{2N^3} + \frac{x^3}{3N^5} + \dots$$

the same

$p_N \sim e^{x/N}$ N big

$p_N \sim e^{x/N - x^2/2N^3 + \dots}$

$\sim e^{x/N} \cdot e^{-x^2/2N^3}$

Taking logarithms:

$$\log M_{Z_N}(t) = -\frac{\mu t \sqrt{N}}{\sigma} + N \log M_X\left(\frac{t}{\sigma\sqrt{N}}\right)$$

Use Poisson (λ)

$$\log M_{Z_N}(t) = -\sqrt{\lambda} t N + N\lambda\left(e^{\frac{t}{\sqrt{\lambda}\sqrt{N}}} - 1\right)$$, std of poisson $= \sqrt{\lambda}$

$$= -\sqrt{\lambda} t N + N\lambda\left(1 + \frac{t}{\sqrt{\lambda}\sqrt{N}} + \frac{1}{2}\left(\frac{t}{\sqrt{\lambda}\sqrt{N}}\right)^2 + \frac{1}{3!}\left(\frac{t}{\sqrt{\lambda}\sqrt{N}}\right)^3 + \dots - 1\right)$$

$$= \cancel{-t\sqrt{N}\sqrt{\lambda}} + \cancel{t\sqrt{N}\sqrt{\lambda}} + \frac{t^2}{2} + \frac{1}{3!}\frac{t^3}{\sqrt{N\lambda}} + \dots$$

$$= \frac{t^2}{2} + \frac{1}{3!}\frac{t^3}{\sqrt{\lambda N}} + \dots$$

$$M_{Z_N}(t) = \underbrace{e^{\frac{t^2}{2}}}_{e^{t^2/2}} \cdot \underbrace{e^{\frac{1}{3!}\frac{t^3}{\sqrt{\lambda N}}}}_{\Downarrow\, 1}$$

Expansion of MGF

$$M_X(t) = 1 + \mu t + \frac{\mu_2' t^2}{2!} + \dots = 1 + t\left(\mu + \frac{\mu_2' t}{2} + \dots\right)$$

from $\log(1+U) = U - \frac{U^2}{2!} + \frac{U^3}{3!} + \dots$

Combining gives:

$$\log M_X(t) = t\left(\mu + \frac{\mu_2' t}{2} + \dots\right) - \frac{t^2\left(\mu + \frac{\mu_2' t}{2} + \dots\right)^2}{2} + \dots$$

$$= \mu t + \left(\frac{\mu_2' - \mu^2}{2}\right)t^2 + \text{terms in } t^3 \text{ or higher}$$

Date :

$$\log m_X\left(\frac{t}{\sigma\sqrt{N}}\right) = \frac{\mu t}{\sigma\sqrt{N}} + \frac{\sigma^2}{2}\frac{t^2}{\sigma^2 N} + \text{term in } t^3/N^{3/2} \text{ or lower in } N$$

Denote lower order terms by $O(N^{-3/2})$. Collecting gives:

$$\log M_{Z_N}(t) = \frac{-\mu t\sqrt{N}}{\sigma} + N\left(\frac{\mu t}{\sigma\sqrt{N}} + \frac{t^2}{2N} + O(N^{-3/2})\right)$$

$$= -\frac{\mu t\sqrt{N}}{\sigma} + \frac{\mu t\sqrt{N}}{\sigma} + \frac{t^2}{2} + O(N^{-1/2})$$

$$= \frac{t^2}{2} + O(N^{-1/2})$$

$$M_{Z_N}(t) = \underbrace{e^{t^2/2}}_{e^{t^2/2}} \cdot \underbrace{e^{O(N^{-1/2})}}_{1}$$

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\left(f_{S+Norm}(x) - f_{Z_n}(x)\right)e^{tx}\,dx = 0$$

$$\text{Bin}(N,\tfrac{1}{2}) = \text{Bern}(\tfrac{1}{2}) + \dots + \text{Bern}(\tfrac{1}{2})$$

$$f(n) = \underbrace{\binom{2N}{n}}_{\frac{2N!}{n!(2N-n)!}}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{2N-n} = \binom{2N}{n}\frac{1}{2^{2N}}$$

Stirling

November 19, 2009

Last proof of Central Limit Theorem

Convolution of f and g: $h(y) = (f*g)(y)$

$$= \int_{\mathbb{R}} f(x)\cdot g(y-x)\,dx$$

$$= \int_{\mathbb{R}} f(x-y)\cdot g(x)\,dx$$

X_1 and X_2 are independent random variables with probability p.

$$\text{Prob}(x_i \in (x, x+\Delta x)) = \int_x^{x+\Delta x} p(t)\,dt$$

$$\approx p(x)\,dx$$

$$\text{Prob}(x_1 + x_2) \in [x, x+\Delta x] = \int_{x_1=-\infty}^{\infty} \int_{x_2 = x - x_1}^{x+\Delta x - x_1} p(x_1)\,p(x_2)\,dx_2\,dx_1$$

$$p(t) = p(x) + p'(x)\cdot(t-x) + \dots$$

$$\int_x^{x+\Delta x} p(t)\,dt = \int_x^{x+\Delta x} [p(x) + p'(x)\cdot(t-x) + \dots]\,dt$$

$$= p(x)\,\Delta x + p'(x)\cdot \frac{(\Delta x)^2}{2} + \dots$$

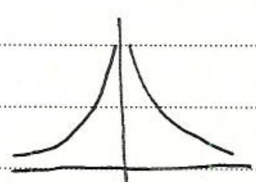
Taylor with Remainder

From $\dfrac{p(x) - p(t)}{x - t} = p'(x)$

$$p(x) = p(t) + p'(x)(x - t)$$

↳ If p' bounded means $|p'(x)| \leq B$

<u>Aside</u> Laplace / Exponential



$\Rightarrow \dfrac{e^{-|x|}}{2}$

$f(x) = e^{-x}$

$f'(x) = -e^{-x}$

$\Downarrow$

$x \to 0,\ f'(x) \to -1$

$f(x) = e^{x}$

$f'(x) = e^{x}$

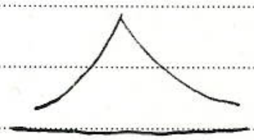
$\Downarrow$

$x \to 0,\ f'(x) \to 1$

NOT DIFFERENTIABLE

Date :

Cauchy $p(x) = \dfrac{1}{\pi(1+x^2)}$ Find mean



$$\int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)}\,dx = \lim_{A\to\infty} \int_{-A}^{A} \frac{x}{\pi(1+x^2)}\,dX = 0$$

but $\displaystyle\lim_{A\to\infty} \int_{A}^{2A} \frac{x}{\pi(1+x^2)}\,dx \neq 0 \Rightarrow \approx \frac{\ln 2}{\pi}$

PROBLEM! It shall be path independent

Proof of CLT.

- The Fourier Transform: $\displaystyle\hat{p}(y) = \int_{-\infty}^{\infty} p(x)\cdot e^{-2\pi i x y}\,dx$

- Derivative of $\hat{g}$ of the Fourier transform of $2\pi i x g(x)$; differentiation (hard) is converted to multiplication (easy).

$$\hat{g}'(y) = \int_{-\infty}^{\infty} 2\pi i x \cdot g(x)\, e^{-2\pi i x y}\,dx$$

g prob. density, $\hat{g}'(0) = 2\pi i\, E(x)$

$\hat{g}''(0) = -4\pi^2 E(x^2)$

Proof : $\displaystyle\hat{g}(y) = \int_{-\infty}^{\infty} g(x)\, e^{-2\pi i x y}\,dx$

$$\hat{g}'(y) = \frac{d}{dy}\int_{-\infty}^{\infty} g(x)\cdot e^{-2\pi i x y}\,dx$$

$$= \int_{-\infty}^{\infty} (-2\pi i x)\, g(x)\, e^{-2\pi i x y}\,dx$$

$\Rightarrow \dfrac{d}{dy} e^{ay} = a e^{ay}$

$\cos(2\pi x y) - i\sin(2\pi x y) = -2\pi x \sin(2\pi x y) - i 2\pi x \cos(2\pi x y)$

$= -2\pi i x\,[\cos(2\pi x y) - i\sin(2\pi x y)] = -2\pi i x e^{-2\pi i x y}$

PROBLEM

It should be path independent

Proof $\hat{g}(y) = \int_{-\infty}^{\infty} g(x) \cdot e^{-2\pi i x y} \, dx$

$$\hat{g}''(y) = \int_{-\infty}^{\infty} (-2\pi i x)^2 \, g(x) \cdot e^{-2\pi i x y} \, dx$$

$$\hat{g}''(0) = -4\pi^2 \int_{-\infty}^{\infty} x^2 g(x) dx = -4\pi^2$$

MGF evaluated at $t/\sigma\sqrt{N}$

↳ $N \cdot \log M_X\left(\frac{t}{\sigma\sqrt{N}}\right)$ ⇒ only look at the first few terms

We taylor expand $\hat{p}$

$$\hat{p}(y) = 1 + \frac{\hat{p}''(0)}{2} y^2 + \ldots = 1 - 2\pi^2 y^2 + O(y^3) \quad , \hat{p}'(0) = 0 , \ \hat{p}''(0) = 4\pi^2$$

Now $P(X_1 + \ldots + X_n \in [a,b]) = \int_a^b (p \cdot \ldots \cdot p)(z) dz$

$X_1 + X_2 \longleftrightarrow p * p$

$(X_1 + X_2) + X_3 \longleftrightarrow (p*p)*p$

- $FT[p * \ldots * p](y) = \hat{p}(y) \ldots \hat{p}(y)$, $FT[f](y)$ = Fourier transform of f evaluated at y

- do not want distribution $X_1 + \ldots + X_n = x$, but rather $S_n = \frac{X_1 + \ldots + X_n}{\sqrt{n}} = x$
- If $B(x) = A(cx)$ for some fixed $c \neq 0$, then $\hat{B}(y) = \frac{1}{c} \hat{A}\left(\frac{y}{c}\right)$

- $\text{Prob}\left(\frac{X_1 + \ldots + X_n}{\sqrt{N}} = x\right) = (\sqrt{N} p * \ldots * \sqrt{N} p)(x\sqrt{N})$

$e^{-2\pi i x y}$

Date : ○○○

$$FT\left[(\sqrt{N}p * \dots * \sqrt{N}p)(x\sqrt{N})\right](y) = \left[\hat{p}(y/\sqrt{N})\right]^N$$

Study $\left(\hat{p}\left(\frac{y}{\sqrt{N}}\right)\right)^N$

$$\hat{p}(t) = 1 - 2\pi^2 t^2 + \mathcal{O}(t^3)$$

$$\hat{p}(y) = \int_{-\infty}^{\infty} p(x) \cdot e^{-2\pi i x y}\, dx$$

Thus study,

$$\left[\hat{p}\left(\frac{y}{\sqrt{N}}\right)\right]^N = \left[1 - \frac{2\pi^2 t^2}{N} + O\left(\frac{y^3}{N^{3/2}}\right)\right]^N$$

<u>Perfect World</u>

$$\lim_{N\to\infty}\left(1 - \frac{2\pi^2 y^2}{N}\right)^N = e^{-2\pi^2 y^2}$$

<u>Take logs</u>

$$N\log\left(1 - \left(\frac{2\pi y^2}{N} + O\left(\frac{y^3}{N^{3/2}}\right)\right)\right) = -N\left(\left(\left(\frac{2\pi^2 y^2}{N} + O\left(\frac{y^3}{N^{3/2}}\right)\right) + \frac{1}{2}\left(\frac{2\pi^2 y^2}{N} + O\left(\frac{y^3}{N^{3/2}}\right)\right)^2 + \dots\right)\right)$$

$$= -2\pi^2 y^2 + O\left(\frac{1}{\sqrt{N}}\right)$$

<u>Shown</u> $\left(\hat{p}\left(\frac{y}{\sqrt{N}}\right)\right)^N \approx \underbrace{e^{-2\pi^2 y^2}}_{e^{-2\pi^2 y^2}} \cdot \underbrace{e^{O(1/\sqrt{N})}}_{1}$

For any fixed y, $\lim_{N\to\infty}\left[1 - \frac{2\pi^2 y^2}{N} + O\left(\frac{y^3}{N^{3/2}}\right)\right]^N = e^{-2\pi y^2}$

- Fourier transform of $e^{-2\pi y^2}$ at x is $\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$\nearrow \cos xy + \boxed{i\sin xy}$ => symmetric and sin is odd.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \underline{e^{-2\pi i x y}}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \underline{\cos(2\pi x y)}\, dx$$

$$\Downarrow$$

$$\sum_{k=0}^{\infty} \frac{(2\pi x y)^{2k}}{(2k)!}$$

$Z \sim$ standard Normal

$$E[e^{tZ}] = e^{t^2/2}$$

$$M_Z(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot e^{tx}\, dx$$

If t is complex, $M_Z(-2\pi i y) = e^{-2\pi^2 y^2}$

<u>Benford's law</u> => ตัวอย่างการคำนวณ tax

$$\text{Prob}(1^{st} \text{ digit is } d) = \log_{10}\left(\frac{d+1}{d}\right)$$

$)\left(\frac{y^3}{N^{3/2}}\right)\Big)^2$

<u>November 24</u>

$$e^{\pi i x} = e^{2\pi i (x \bmod 1)}$$

$$\hookrightarrow = \cos 2\pi x + i \sin 2\pi x$$

<u>Assume</u> $\log_{10} 2 = \frac{p}{q}$ ($\log_{10} 2$ is rational)

$$2 = 10^{p/q}$$

$$2^q = 10^p = 2^p \cdot 5^p$$

$$2^{q-p} = 5^p$$

$q - p = 0$ and $p = 0$

$$q = 0$$

$\underbrace{\hphantom{q - p = 0 \text{ and } p = 0}}$

impossible, so $\log_{10} 2$ is irrational. ($\log_{10} 2 \notin \mathbb{Q}$)

<u>ASIDE</u>

$f(x) = x^{p/q}$

$g(x) = f(x)^q = x^p$

$\Downarrow$

คิด derivative ได้

Date :

Ex 2^n is Benford base as $\log_{10} 2 \notin \mathbb{Q}$

$X_n = 2^n$

$Y_n = \log_{10} X_n = \underbrace{n(\log_{10} 2)}_{\notin \mathbb{Q}}$

Ex Fibonacci numbers are Benford base 10

$a_{n+1} = a_n + a_{n-1}$

Guess $a_n = n^r$ => solve

Roots $r = \dfrac{(1 \pm \sqrt{5})}{2}$

General solution : $a_n = c_1 r_1^n + c_2 r_2^n$

Binet : $a_n = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^n$

Ex $X_n = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n$

$Y_n = \log \dfrac{1}{\sqrt{5}} + n\underbrace{\left(\log_{10} \dfrac{1+\sqrt{5}}{2}\right)}_{\notin \mathbb{Q}}$ => doesn't work

f_1 f_2 => density

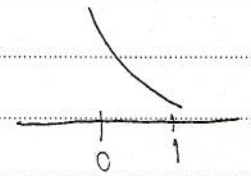
X_1, X_2 mod 1 (live on $[0,1]$)

$\dfrac{3}{4} + \dfrac{1}{2} = \dfrac{1}{4}$

$\dfrac{3}{5} + \dfrac{3}{5} = \dfrac{1}{5}$

$(f_1 * f_2)(x) = \displaystyle\int f_1(t) f_2(x-t)\,dt$ => Assume that X_2 is uniform

$= \displaystyle\int f_1(t)\,dt = 1$



$\frac{3}{4} + (x - \frac{3}{4})$

$\frac{3}{4} + \left(y - \frac{3}{4}\right)$

shift data $\Rightarrow$ doesn't affect the distribution <u>but</u> it may affect the first digit.

<u>Ex</u> $a_{n+1} = 2a_n - a_{n-1}$

If, $a_0 = a_1 = 1 = a_2 = a_3 = a_4$ $\Rightarrow$ doesn't work

If $a_0 = 0$, $a_1 = 1$ $\Rightarrow$ $a_3 = 3$ ... $a_{2009} = 2009$

If <u>Guess</u> $a_n = r^n$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

<u>Central Limit Theorem</u>

↳ sum of random variable converges to standard normal when given random variable:

$$Z = \frac{X - E(X)}{\text{stDev.}(X)}$$

$Y_1, \ldots, Y_M$ live on $[0,1]$

$Y = Y_1 + \ldots + Y_m \mod 1$

<u>Ex</u> $Y_i = \log_{10} X_i$

$$P(Y_i \le y) = P(\log_{10} X_i \le y)$$

$$= P(X_i \le 10^y)$$

$$F_{Y_i}(y_i) = F_{X_i}(10^y)$$

$$f_{Y_i}(y_i) = f_{X_i}(10^y) \cdot 10^y \log_e 10$$

Date :

Let $X_1, \dots, X_n$ iidrv nice and greater than or equal to zero

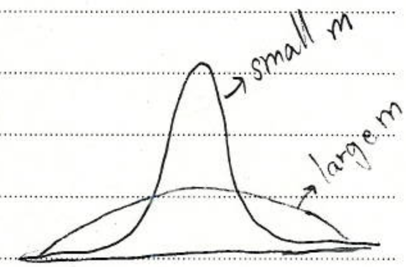
$Y_1 + Y_2 + \dots + Y_m \Longrightarrow$ should converge to normal by CLT

↳ $Y_i = \log X_i$ "nice" iidrv

↳ mean μ

variance σ^2

Therefore, this should converge to $N(m\mu, m\sigma^2)$



December 1, 2009

Poisson Summation Formula $\Rightarrow$ see the slide

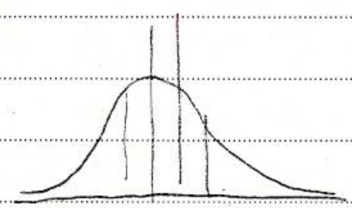
f nice :

Model CLM

$Y_1, \dots, Y_n$ iidrv mod 1

$Y = Y_1 + \dots + Y_n$ mod 1

$X = X_1 \dots X_n$



$\log_{10} X = \log_{10} X_1 + \log_{10} X_n$

$\Downarrow \qquad \Downarrow$

$Y_1 \qquad Y_n$

Lemma : $\boxed{\frac{2}{\sqrt{2\pi\sigma^2}} \int_{\sigma^{1+\delta}}^{\infty} e^{-x^2/2\sigma^2}\, dx \ll e^{-\sigma^{2\delta}/2}}$

Say $f(x) \ll g(x)$ or $f(x) = O(g(|x|))$

if $\exists x_0, C$ st $\forall x \geq x_0$,

$|f(x)| \leq C g(x)$

Ex $2009x \ll 12x\log x$

$\Leftarrow$ probability of being away from the mean

$P(|\bar{X} - \mu| \geq (1+\delta)\sigma)$

$P(|\bar{X} - \mu| \geq \sigma^{\delta} \cdot \sigma)$

by Chevbyshev $\leq \frac{1}{\sigma^{2\delta}}$

we will show $\leq \frac{1}{e^{\sigma^{2\delta}/2}}$

Find $\int_{\sigma^{1+\delta}}^{\infty} e^{-x^2/2\sigma^2} dx \Rightarrow u = \frac{x}{\sigma}, \; du = \frac{dx}{\sigma}$

$= \sigma \int_{\sigma^{\delta}}^{\infty} e^{-u^2/2} du$ Let $w = u - \sigma^{\delta}$, $dw = du$

$= \sigma \int_0^{\infty} e^{-(w+\sigma^{\delta})^2/2} dw$

$= \sigma \int_0^{\infty} e^{-w^2/2} \cdot e^{-w\sigma^{\delta}} \cdot \underbrace{e^{-\sigma^{2\delta}/2}}_{\Downarrow \text{ pull out}} dw$

Including Factor:

$e^{-\sigma^{2\delta}/2} \cdot \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-w^2/2} \cdot e^{-w\sigma^{\delta}} dw$

* Note should think about applying Cauchy-Schwartz whenever you have two fns *

Lemma: As $N \to \infty$, $P_N(x) = \frac{e^{-\frac{\pi x^2}{N}}}{\sqrt{N}}$ becomes equidistributed modulo 1

$e^{\frac{-2\pi x^2}{2N}} \cdot \frac{1}{\sqrt{N}} = e^{\frac{-x^2}{2(N/2\pi)}} \cdot \frac{1}{\sqrt{2\pi \cdot \frac{N}{2\pi}}} = N\left(0, \frac{N}{2\pi}\right)$

Study $x \bmod 1 \in [a,b] \subset [0,1]$

$\sum_{n=-\infty}^{\infty} \text{Prob}(x \in [a+n, b+n]) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{N}} \int_{a+n}^{b+n} e^{-\frac{\pi x^2}{N}} dx$

Let $x = x + n$

Date :

$$= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{N}} \int_a^b e^{-\pi(x+n)^2/N} \, dx$$

interchange order of operation

$$= \frac{1}{\sqrt{N}} \int_a^b \left[\sum_{n=-\infty}^{\infty} e^{-\pi(x+n)^2/N} \right] dx$$

Taylor expand $e^{-\pi(x+n)^2/N}$. Then, take the derivative

Apply Poisson Summation

$$\frac{1}{\sqrt{N}} \sum_n e^{-\pi n^2/N} = \sum_n e^{-\pi^2 n N}$$

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 \cdot N} = 1 + \sum_{n \neq 0} e^{-\pi n^2 \cdot N}$$

$$= 1 + 2 \sum_{n=1}^{\infty} \left(e^{-\pi N n}\right)^n$$

$$\left| \sum_{n=-\infty}^{\infty} e^{-\pi n^2 N} - 1 \right| \leq 2 \sum_{n=1}^{\infty} \left(e^{-\pi N}\right)^n$$

$$= \frac{2e^{-\pi N}}{1 - e^{-\pi N}}$$

$$\frac{1}{\sqrt{N}} \int_a^b \left[\sum_{n=-\infty}^{\infty} e^{-\pi(x+n)^2/N} \right] dx \approx \int_a^b 1 \, dx + \text{small}$$

$$= b - a$$

More Sums than Differences : Introduction

Clicker question

What should $\lim_{N \to \infty} \frac{\#MSTD(N)}{2^N}$

New Model: Binomial w/ parameter $p(N)$

- $\frac{1}{N} = o(p(N))$ and $p(N) = o(1)$;

$\Downarrow$

this function is decaying faster

Final => 24 hrs take-home exam

review session : 4pm

December 3, 2009

Range when $p(N) = N^{-\delta}$ $\delta \in (\frac{1}{2}, 1)$

$Bin(N, p^{\gamma}(N))$ $\{1, \ldots N\}$

As $N \to \infty$, $p(N) \to 0$.

$E[|A|] = N \cdot N^{-\delta} = N^{1-\delta}$

Indicator (Bin) random variable. $X_i = \begin{cases} 1 & \text{prob. } p(N) = N^{-\delta} \\ 0 & \text{prob. } 1 - P(N) = 1 - N^{-\delta} \end{cases}$

$X = X_1 + \ldots + X_n$

$E[X_i] = 1 \cdot N^{-\delta} + 0(1 - N^{-\delta})$

$= N^{-\delta}$

$E[X] = \sum_{i=1}^{N} E[X_i] = N \cdot N^{-\delta} \Rightarrow E(X) = N^{1-\delta}$

$Var(X) = \sum_{i=1}^{N} Var(X_i)$

$Var(X_i) = (1 - N^{-\delta})^2 N^{-\delta} + (0 - N^{-\delta})^2 (1 - N^{-\delta})$ From definition $Var(X_i) = E[(X_i - E(X_i))^2]$

$= N^{-\delta}(1 - N^{-\delta})$

Date : ○○○

$\therefore E(Z) = N^{1-\delta}$

$Var(Z) = N^{1-\delta}(1-N^{1-\delta})$

If the number N is large, $E(Z) \approx Var(Z)$

$StDev(Z) = N^{\frac{1-\delta}{2}}(1-N^{-\delta})^{\frac{1}{2}}$

Say $E[Z] = 10^{100}$

$StDev[Z] = 10^{50}$

Lemma : with high probability, we have

(1) $Prob(\frac{1}{2}N^{-\delta} \leq x \leq \frac{3}{2}N^{1-\delta}) = 1 - \text{small}$

(2) $\theta = \{(m,n) \in A : m < n\}$ Then,

$\frac{1}{2} \cdot \frac{N^{1-\delta}}{2} \cdot \frac{N^{1-\delta}-1}{2} \leq \#\theta \leq \frac{1}{2} \cdot \frac{3N^{1-\delta}}{2} \cdot \frac{3N^{1-\delta}-1}{2}$ with Prob $= 1$-small

Prove Lemma 1

Use Cherbyshev

$1 - P(|Z - N^{1-\delta}| > \underbrace{\tfrac{1}{2}N^{\frac{1-\delta}{2}}}_{k} \underbrace{N^{\frac{1-\delta}{2}}}_{\sigma}) = 1 - \frac{4}{N^2}$

N is large, so drop -1

$\frac{1}{2} \cdot \frac{N^{1-\delta}}{2} \cdot \frac{N^{1-\delta}}{2} \leq \#\theta \leq \frac{1}{2} \frac{3N^{1-\delta}}{2} \cdot \frac{3N^{1-\delta}}{2}$

$\therefore \#\theta \sim N^{2-2\delta}$

$D = \{(m,m) \in A\}$

$\#D \leq \frac{3}{2}N^{1-\delta}$ with prob at least $1 - \frac{4}{N^{1-\delta}}$

$*$ $\theta \sim N^{2-2\delta}$, $\theta = \{(m,n) \in A \cdot m < n\}$

(m,n) $m<n$, difference $n-m$

(m',n') $m'<n'$, difference $n'-m'$

w log, assume $m \leq m'$

$$\underset{\text{"Bad"}}{Y_{m,n,m',n'}} \begin{cases} 1 & \text{if } n'-m' = n-m \\ 0 & \text{otherwise} \end{cases}$$

$$Y = \sum_{\substack{m,n,m',n' \\ n \neq m'}} Y_{m,n,m',n'} + \sum_{m,m',n} Y_{m,m',m'',n'}$$

$$E[Y] \leq N^3 \cdot N^{-4\delta} + N^2 \cdot N^{-3\delta} \leq N^{3-4\delta}$$

$\# \theta \sim N^{2-2\delta}$

$\# \text{Bad} \sim N^{3-4\delta}$

We need $2-2\delta > 3-4\delta$

$2\delta > 1$

$$\boxed{\delta > \frac{1}{2}}$$

Aside $Y_{m,n,m',n'} = 1$ $Y_{m',n',m'',n''} = 1$ $\}$ no independence. ?

so, $Y_{m,n,m'',n''}$

Proof of Claim : we cannot use CLT as $Y_{m,n,m',n'}$ are not independent

Use $Var(U+V) \leq 2Var(U) + 2Var(V)$

Note

$$* \quad Var(U+V) = Var(U) + Var(V) + \underbrace{2CoVar(U,V)}_{\substack{\text{Cauchy-Schwartz} \\ \leq \sqrt{Var(U)\cdot Var(V)}}}$$

Date : ○○○

Write $\sum Y_{m,n,m',n'} = \sum U_{m,n,m',n'} + \sum V_{m,n,m',n'}$

Geometric Model : Baseball Game

Ex

$P(X = m) = (1-p)\, p^m \qquad m \in \{0,1,2,\dots\}$

$P(Y = m) = (1-q)\, q^n \qquad n \in \{0,1,2,\dots\}$

flip $p \to 1-p$

start counting at 0.

$$\text{Prob (Tie)} = \sum_{m=0}^{\infty} P(X = Y = m)$$

$$= \sum_{m=0}^{\infty} (1-p)\, p^m (1-q) q^m$$

$$= (1-p)(1-q) \sum_{m=0}^{\infty} (pq)^m$$

$$= \frac{(1-p)(1-q)}{1-pq}$$

X beats Y

$$P(X > Y) = \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} P(X = m)\, P(Y = n)$$

$$= \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} (1-p)\, p^m (1-q) q^n$$

$$= (1-p)(1-q) \left[\sum_{m=1}^{\infty} p^m \sum_{n=0}^{m-1} q^n \right]$$

$$= (1-p)\cancel{(1-q)} \sum_{m=1}^{\infty} p^m \cdot \frac{1-q^m}{\cancel{(1-q)}}$$

$$= (1-p) \sum_{m=0}^{\infty} \left[p^m - (pq)^m \right]$$

at zero, the sum is zero. So, we can just write the same thing with the sum starting from 1.

$$= (1-p)\left[\frac{1}{1-p} - \frac{1}{1-pq}\right]$$

$$= (1-p)\left[\frac{(1-pq)-(1-p)}{(1-p)(1-pq)}\right]$$

$$= \frac{(1-p)(p-pq)}{(1-p)(1-pq)}$$

$$= \frac{p(1-q)}{1-pq}$$

$$\therefore P(Y > X) = \frac{q(1-p)}{1-pq}$$

$$E(X) = \frac{p}{1-p} = RS$$

$$E(Y) = \frac{q}{1-q} = RA$$

$$p = \frac{RS}{1+RS}$$

December 8, 2009 Continue from previous class

$$X: RS = \frac{p}{1-p} \Rightarrow p = \frac{RS}{1+RS}$$

$$Y: RA = \frac{q}{1-q} \Rightarrow q = \frac{RA}{1+RA}$$

Aside TIE : $\frac{(1-p)(1-q)}{1-pq}$

$$P(X > Y) = \frac{p(1-q)}{1-pq}$$

$$P(Y > X) = \frac{q(1-p)}{1-pq}$$

e same

Date :

$$P(X \text{ wins}) = \frac{P(X>Y)}{P(X>Y)+P(Y>X)}$$

$$= \frac{\frac{p(1-q)}{1-pq}}{\frac{p(1-q)}{1-pq} + \frac{q(1-p)}{1-pq}}$$

$$= \frac{p(1-q)}{p(1-q)+q(1-p)}$$ multiply by $\frac{\frac{1}{(p-1)(q-1)}}{\frac{1}{(p-1)(q-1)}}$, we will get:

$$= \frac{\frac{p}{1-p}}{\frac{p}{1-p}+\frac{q}{1-q}}$$

$$= \frac{RS}{RS+RA}$$

Econ Application

Random walk : Efficient Market Hypothesis

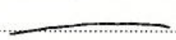
Power Laws

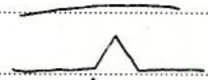
$\text{Prob}(X = x) \sim \frac{C_\alpha}{x^\alpha}$ as x grows, $\alpha > 0$

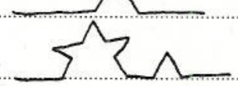
Cauchy Distribution : $\frac{1}{\pi} \frac{1}{1+x^2}$

Benford's law $\sim \frac{1}{x}$

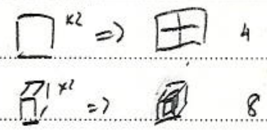
Cotton Crisis

0 : 

1 : 

2 :  => finite area

get:



If double dimension, it expands to become 2^d

Ex 4 copies $\rightarrow$ $4 = 3^d$

$$d = \log_3 4$$

December 10, 2009

Birthday Problem => How many ppl share the B.D.?

- Law of total probability

$$P(\text{none in } n \text{ share}) = \left(1 - \frac{0}{365}\right)\left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\cdots\left(1 - \frac{n-1}{365}\right)$$

$$= \prod_{k=0}^{n-1}\left(1 - \frac{k}{365}\right) = \frac{1}{2}$$

Summify: $\displaystyle\sum_{k=0}^{n-1} \log\left(1 - \frac{k}{D}\right) = -\log 2$

Aside Taylor expansion: $\log(1-u) = -\left(u + \frac{u^2}{2} + \frac{u^3}{3} + \dots\right)$

$$= -\sum_{k=0}^{n-1} \frac{k}{D} \approx -\log 2$$

$$\therefore \sum_{k=0}^{n-1} k \approx D\log 2$$

Aside $\displaystyle\sum_{k=0}^{N} k = \frac{N(N+1)}{2} \xrightarrow{\text{approximate}} \int_0^N x\,dx = \frac{x^2}{2}\Big|_0^N = \frac{N^2}{2}$

$$\sum_{k=0}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

Date : ○○○

<u>Aside</u> $\log(n!) = \log 1 + \log 2 + \dots + \log n$

$$\approx \int_1^n \log t \, dt$$

$$= t\log t - t \Big|_1^n$$

$$\log n! \approx n\log n - n$$

$$n! \approx n^n e^{-n}$$

From $\sum_{k=0}^{n-1} k \approx D\log 2 \approx \frac{(n-\frac{1}{2})^2}{2}$

$$n \approx \frac{1}{2} + \sqrt{2D\log 2}$$

<u>Ex</u> $\binom{n+k-1}{n}(1-p)^n p^k$

<u>Cookie Problem</u>

$$OO\,|\,|\,O\,O\,|\,O\,O\,|\,O\,O\,O\,O$$ $\Rightarrow$ partition

$2 \quad 0 \quad 2 \quad 2 \quad 4$

<u>Ex</u> $X_n = \begin{cases} 1 & \frac{1}{\log n} \\ 0 & 1-\frac{1}{\log n} \end{cases}$

$E\left[\sum_{n=2}^{x-2} X_n \cdot X_{n+2}\right] \sim \frac{x}{\log^2 x}$ $\Rightarrow$ There are some problems with this model

$$E\left[\sum_{n=2}^{x-4} X_n X_{n+2} X_{n+4}\right] = \frac{cx}{\log^3 x}$$

<u>standardization</u> $\Rightarrow$ $\frac{\text{Random variables - means}}{\text{STD.}}$ to compare the same scale.