Lecture 24 (12/8/09)

Summary of the day

- **Sabermetrics**
  - Closed form expression
- **Economics**
  - Fractals vs Random Walk
- **Gambling**
  - Blackjack (card counting)

$f(n) = a p^n \quad n = 0, 1, 2, \ldots$

$\sum_{n=0}^{\infty} f(n) = 1 \rightarrow a = \frac{1}{1-p} \Rightarrow f(n) = p^n (1-p)$

Mean $\sum_{n=0}^{\infty} n \cdot p^n (1-p) = (1-p) \sum_{n=0}^{\infty} n p^n$

↳ now $\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$ apply $x \frac{d}{dx}$

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

So Mean is $(1-p) \frac{p}{(1-p)^2} = \frac{p}{1-p}$

$\frac{p}{1-p} = RS$

$p = RS - pRS$

$p = \frac{RS}{1+RS}$

Say $g(m) = q^m (1-q)$ with mean $\frac{q}{1-q}$

$$P(X > Y) = \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} p^n (1-p) \cdot q^m (1-q)$$

$$= (1-p)(1-q) \sum_{n=1}^{\infty} p^n \frac{1-q^n}{1-q}$$

$$= (1-p) \sum_{n=1}^{\infty} p^n (1-q^n)$$ note zero at $n=0$

$$= (1-p) \left[\sum_{n=0}^{\infty} p^n - \sum_{n=0}^{\infty} (pq)^n \right]$$

$$= (1-p) \left[\frac{1}{1-p} - \frac{1}{1-pq} \right]$$

$$= (1-p) \frac{(1-pq) - (1-p)}{(1-p)(1-pq)} = \frac{p - pq}{1-pq} = p(1-q)$$

$$P(X = Y) = \sum_{n=0}^{\infty} (1-p)(1-q)(pq)^n = \frac{(1-p)(1-q)}{1-pq}$$

By symmetry: $P(Y > X) = \frac{q - pq}{1-pq}$

$$P(\text{tie}) = \frac{(1-p)(1-q)}{1-pq} = \frac{1-p-q+pq}{1-pq}$$

$$\frac{RS^\gamma}{RS^\gamma + RA^\gamma} = \frac{RS^\gamma}{RS^\gamma + (RS - (RS - RA))^\gamma}$$

$$= \frac{1}{1 + \left(1 - \frac{RS - RA}{RS}\right)^\gamma}$$

replace RS in denom with league average A

$$= \left[1 + \left(1 - \frac{RS - RA}{A}\right)^\gamma\right]^{-1}$$

$$\cong \left[1 + 1 - \gamma \frac{RS - RA}{A} + H.O.T\right]^{-1}$$

H.O.T. is ~~lower~~ higher order terms

$$= \left[2 - \frac{\gamma}{A}(RS - RA)\right]^{-1}$$

$$= \frac{1}{2}\left(1 - \frac{\gamma}{2A}(RS - RA)\right)^{-1}$$

$$\cong \frac{1}{2}\left(1 + \frac{\gamma}{2A}(RS - RA) + H.O.T.\right)$$

$$= \frac{1}{2} + \frac{\gamma}{4A}(RS - RA)$$

Note $RS = RA \Rightarrow$ prob win is 1/2

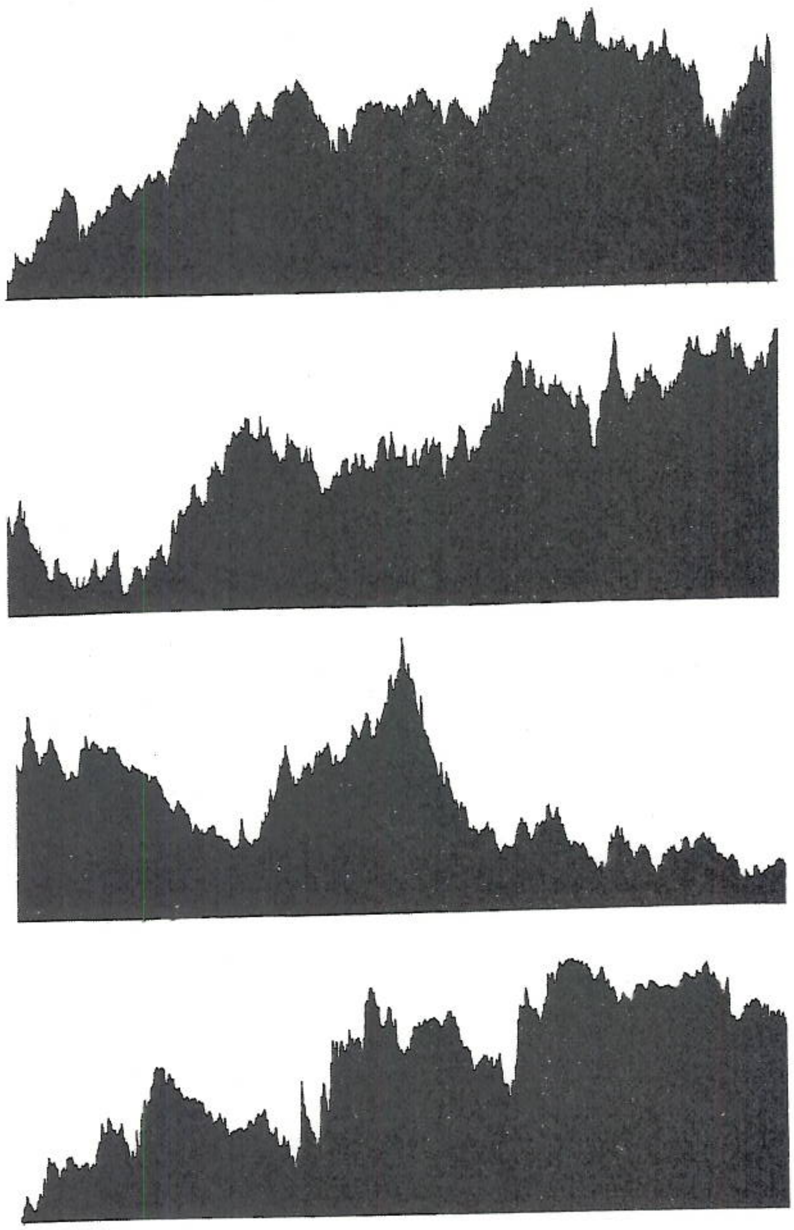
A is average scoring in league, MUCH larger than RS − RA.

So much for conventional market wisdom. As we know now, the International Monetary Fund patched Russia, the Federal Reserve stabilized Wall Street, and the bull market ran another few years. In fact, by the conventional wisdom, August 1998 simply should never have happened; it was, according to the standard models of the financial industry, so improbable a sequence of events as to have been impossible. The standard theories, as taught in business schools around the world, would estimate the odds of that final, August 31, collapse at one in 20 million—an event that, if you traded daily for nearly 100,000 years, you would not expect to see even once. The odds of getting three such declines in the same month were even more minute: about one in 500 billion. Surely, August had been supremely bad luck, a freak accident, an "act of God" no one could have predicted. In the language of statistics, it was an "outlier" far, far, far from the normal expectation of stock trading.

Or was it? The seemingly improbable happens all the time in financial markets. A year earlier, the Dow had fallen 7.7 percent in one day. (Probability: one in 50 billion.) In July 2002, the index recorded three steep falls within seven trading days. (Probability: one in four trillion.) And on October 19, 1987, the worst day of trading in at least a century, the index fell 29.2 percent. The probability of that happening, based on the standard reckoning of financial theorists, was less than one in 10^{50}—odds so small they have no meaning. It is a number outside the scale of nature. You could span the powers of ten from the smallest subatomic particle to the breadth of the measurable universe—and still never meet such a number.

So what's new? Everyone knows: Financial markets are risky. But in the careful study of that concept, risk, lies knowledge of our world and hope of a quantitative control over it.

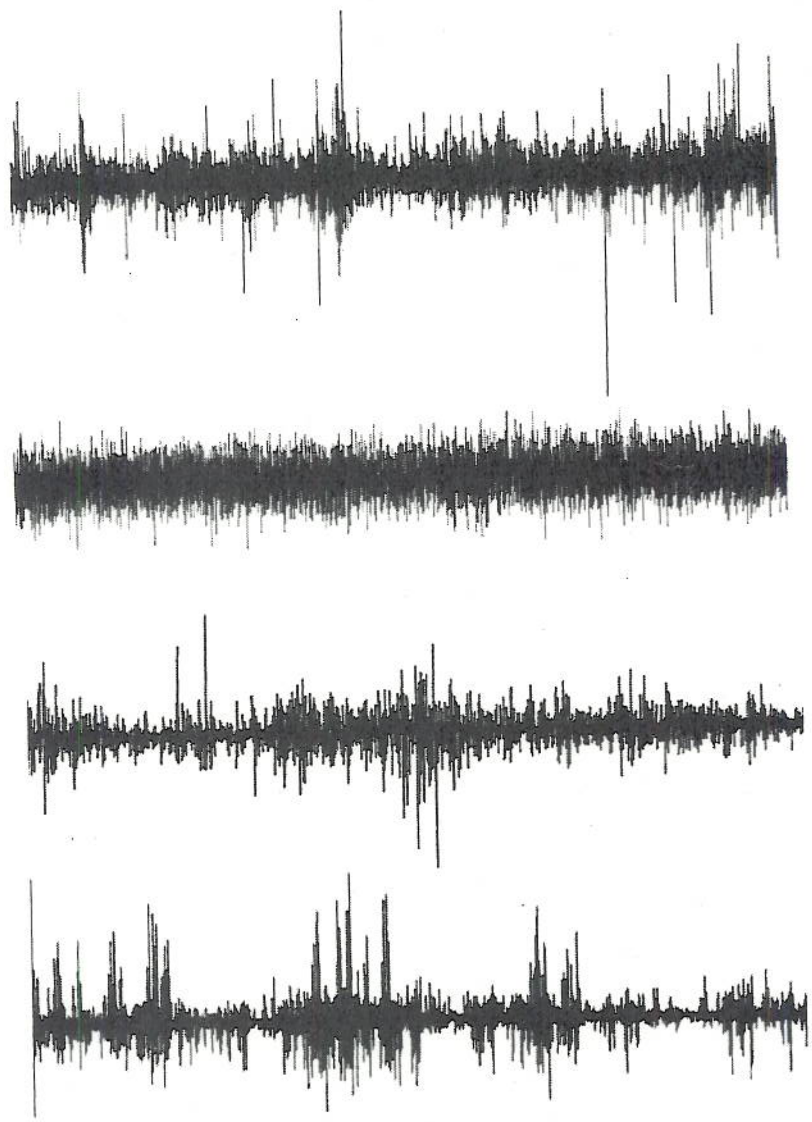
For more than a century, financiers and economists have been striving to analyze risk in capital markets, to explain it, to quantify it, and, ultimately, to profit from it. I believe that most of the theorists have been going down the wrong track. The odds of financial ruin in a free, global-market economy have been grossly underestimated. In



Four charts: Which are real, which are fake?

All fairly similar, many readers will say. Indeed, stripped of legends, axis labels, and other clues to context, most price "fever charts," as they are called in the financial press, look much the same. But pictures can deceive better than words.

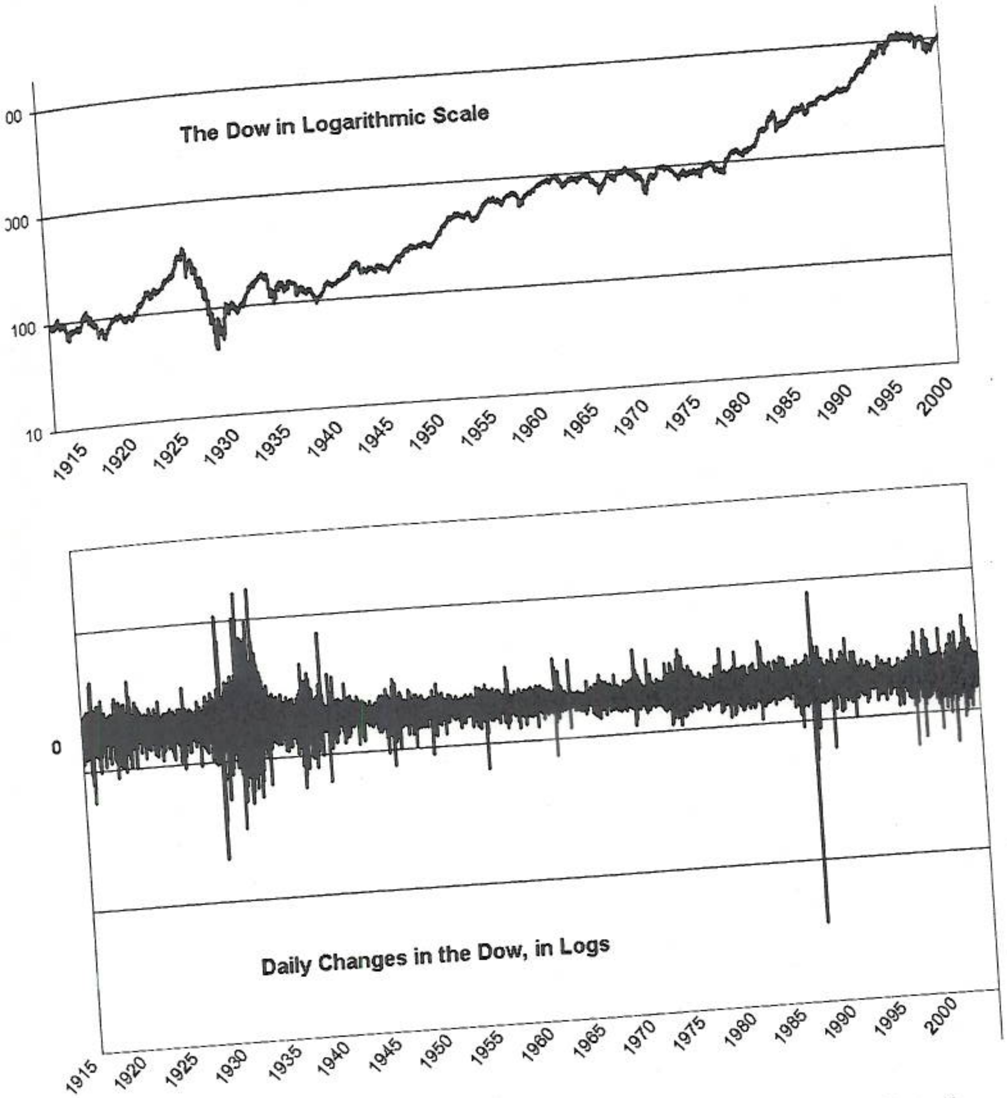
For the truth, look at the next set of charts. These show, rather than the prices themselves, the changes in price from moment to



**The "daily changes" in the four charts.
Again, which are fake?**

ding, later chapters will elaborate and show the model to be extremely parsimonious.

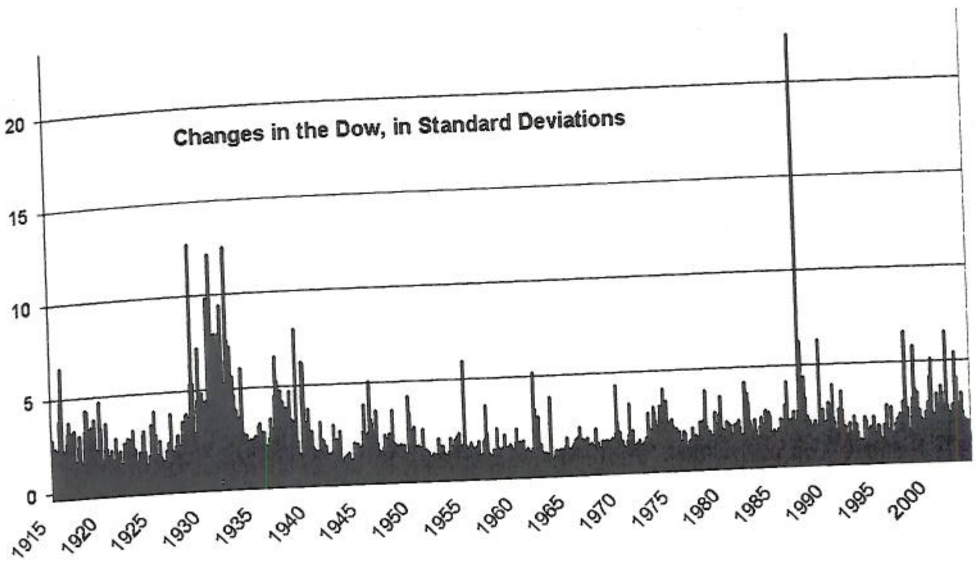
How does it work? It is based on my fractal mathematics, which subsequent chapters will elucidate. It is a model still in development. What I know cannot yet be used to pick stocks, trade derivatives, or value options; time, and further research by others, will determine whether it ever can. But to be able to imitate reality is a

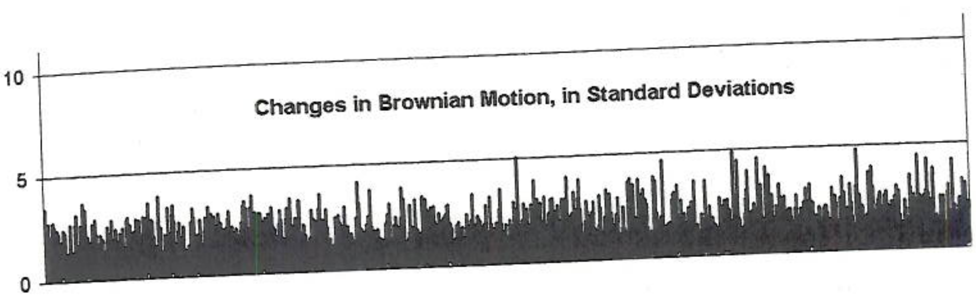


Looking under the varnish. Two charts here: the same daily index values (top chart) and changes (bottom chart) as before—but drawn to a more useful scale, the logarithmic. Logarithms rescale everything, so that a 1 percent change in 1900 will look about the same on our charts as a 1 percent change in 2000. That is just a different way of looking at the data. It makes the charts look the way the market actually felt to someone living through it.

Prominent features: The overall change in the magnitude of the index is no longer overwhelming. The Crash of 1929, the Great Depression, and World War II dominate the picture—just as they dominate our understanding of twentieth-century American economic history. Only the Crash of 1987 rivals those turbulent years. But most price changes merge into a broad strip, which varies in some sort of irregular pattern. The strip alternately narrows and widens, in some apparently haphazard cycle of thin and broad. Also, the spikes seem most likely to cluster together when the strip is wide.

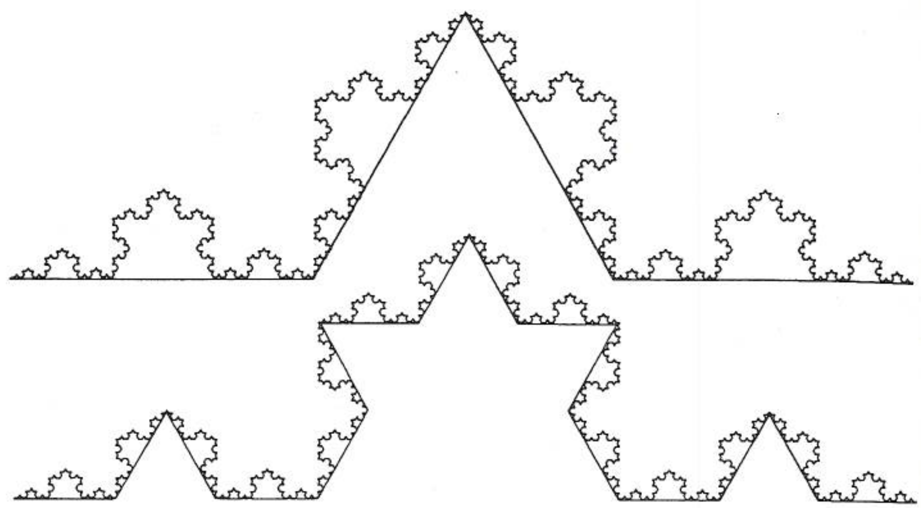
Now we put the Dow to one side, and look at some new data.





Original vs. reproduction—through the analyzer. Here you can see the differences between the Brownian (bottom) and Dow (top) charts more clearly. Instead of using a log scale as before, here we translate each index change into the number of standard deviations it is beyond the average change—in other words, how unusual it is. A very large, rare index movement will have a tall bar on this chart; the common, small changes have short bars.

Prominent features: In the Brownian chart, most changes—in fact, about 68 percent—are small. They are within one standard deviation of the average index change, zero. Mathematicians use the Greek letter sigma, σ, for standard deviation. About 95 percent of the changes are within 2σ, 98 percent within 3σ, and very, very few values are any larger. Next look at the Dow variations. The spikes are huge. Some are 10σ; one, in 1987, is 22σ. The odds of that are something less than one in 10^{50}—so minute that the standard Gaussian tables do not even contemplate it. In other words, virtually impossible. Yet there it is.



Fractal dimension. One of the most significant concepts in fractal geometry is dimension, a numerical measure of the "roughness" of an object. We are familiar with the one dimension of a straight line, or the two dimensions of a plane—but how about a fractional dimension between the two?

Look at the Koch curve above, and try to measure its length. Start with a ruler one-third the object's breadth. That is the triangular line fitting inside the curve, in the top panel. As you can see, it fits four times. Then shrink the ruler by a third, as in the bottom diagram. Because it can now fit into more crannies of the curve, it measures more distance—in fact, four-thirds as much. Continue the process, shrinking the ruler and measuring. At each stage the length measured is multiplied by the same ratio: 4 to 3. The fractal dimension is defined as the ratio of the logarithm of 4 to the logarithm of 3. A pocket calculator converts that: 1.2618. . . . This makes intuitive sense. The curve is crinkly, so it fills more space than would a one-dimensional straight line; yet it does not completely fill the two-dimensional plane.

Random fractal curves. So far, all the fractals in this gallery have been regular and, once you knew the rule, the constructions were exactly repeatable and the results, predictable. But such constructions are nothing but appetizers. I like to call them cartoons. Adding an element of chance complicates the game, and starts to produce structures that look more like sports of Nature than of man.

The top diagram is the Koch curve again, with luck added. It starts with the same initiator and generator as shown earlier. But whereas the prototypical Koch curve plugs the ever-shrinking generators in exactly the same way at each step, here we toss a coin at each step to decide whether to place the "tent" right side up, or upside down. The result is more irregular and flows more naturally. In fact, it starts to look a bit like a coastline. The bottom diagram, using a more complicated fractal process driven by a computer, starts to look startlingly real—as if traced from a shipping chart.

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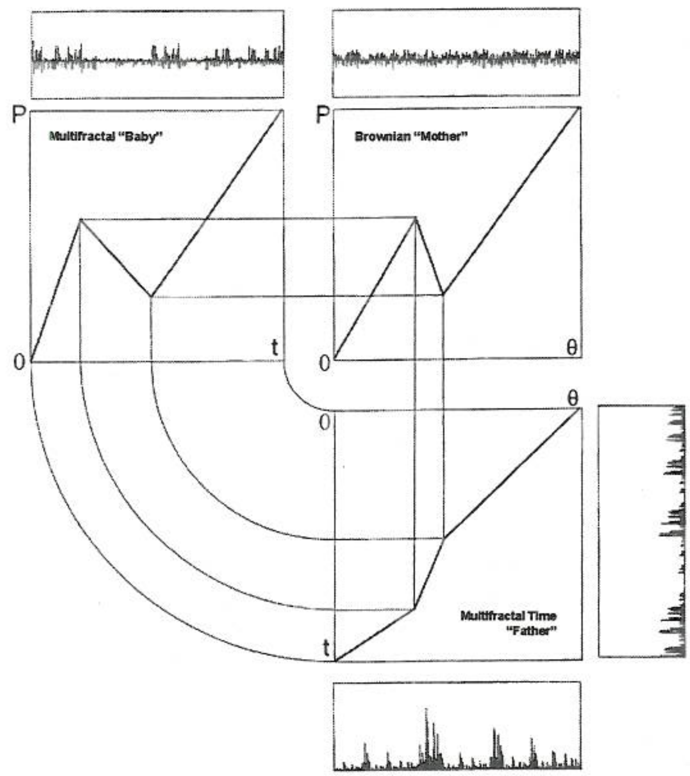

The Baby Theorem. This diagram shows how two generators can pass on traits to a third. The mother generator at top right is a Brownian motion, in conventional clock time—as apparent from the chart of its increments shown above the generator. The father, at bottom right, transforms clock time into a new time-scale, called trading time. By adopting the father's trading time, the mother creates a multifractal baby (top left). Baby's increments, shown above its generator, would pass the "find the fakes" test with flying colors: It is, to all appearances, a genuine price chart. Meanwhile, the uneven, slow-and-fast nature of trading time is shown in the two time-increment charts to the father's bottom and right. And as in the previous, two-page illustration, the horizontal displacement of the generators' break points is the critical step in this particular fractal process. Broadening the gap between the mother's break points yields the baby's generator. I called it the Baby Theorem at first because its mathematical proof was easy, even if its consequences are far-reaching…a common occurrence in science.

Merged together, the baby takes the father's trading-time and converts it into a price by the rules the mother provides. Last step: Use the new, baby generator to make a full fractal price chart that is a variant of one of the panels in the "Panorama of financial multifractal." And there you are: a realistic financial chart, made by stretch-