Math 341: Probability
Thirteenth Lecture (10/27/09)

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Summary for the Day

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- Change of variable formulas:
  - Review of Jacobians.
  - Joint density of functions of random variables.
- Sums of random variables:
  - Convolution.
  - Properties of convolution.
  - Poisson example.
- Distributions from Normal:
  - Sample mean and variance.
  - Central Limit Theorem and Testing.
  - Pepys' Problem.

Section 4.7
Functions of Random Variables

One-dimension

Change of variable formula

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Proof:
$\mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \leq g^{-1}(y)) = F_X(g^{-1}(y)) = F_X(h(y))$.

$f_Y(y) = F'_X(h(y))h'(y) = f_X(h(y))h'(y)$.

As $g(h(y)) = y$, $g'(h(y))h'(y) = 1$ or $h'(y) = 1/g'(h(y))$.

Review of Jacobian

Definition of the Jacobian

Given variables (x_1, x_2) that are transformed to (y_1, y_2) by

$$T(x_1, x_2) = (y_1(x_1, x_2), y_2(x_1, x_2))$$

and inverse mapping

$$T^{-1}(y_1, y_2) = (x_1(y_1, y_2), x_2(y_1, y_2)).$$

The Jacobian is defined by

$$J(y_1, y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_2} \end{vmatrix}.$$

Review of Jacobian

- Note $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- Use: $dx_1 dx_2 \to |J| dy_1 dy_2$ (tells us how the volume element is transformed).

Example of Jacobian

Polar Coordinates

$$x_1(r,\theta) = r\cos\theta, \quad x_2(r,\theta) = r\sin\theta.$$

Calculating gives

$$J = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

Thus $dx_1 dx_2 \to r dr d\theta$.

Change of Variable Theorem

Theorem

f_{X_1,X_2} joint density of X_1 and X_2, $(Y_1, Y_2) = T(X_1, X_2)$ with Jacobian J. For points in the range of T,

$$f_{Y_1,Y_2}(y_1, y_2) \;=\; f_{X_1,X_2}\left(x_1(y_1,y_2), x_2(y_1,y_2)\right)\left|J(y_1,y_2)\right|.$$

Example: X_1, X_2 independent $\mathrm{Exponential}(\lambda)$. Find the joint density of $Y_1 = X_1 + X_2$, $Y_2 = X_1/X_2$. Answer is

$$f_{Y_1,Y_2}(y_1,y_2) \;=\; \lambda^2 y_1 e^{-\lambda y_1} \cdot \frac{1}{(1+y_2)^2}.$$

If instead had $Y_1 = X_1 + X_2$ and $Y_3 = X_1 - X_2$, would find

$$f_{Y_1,Y_3}(y_1,y_3) \;=\; \frac{\lambda^2}{2} e^{-\lambda y_1} \;\text{ for }\; |y_3| \le y.$$

Strange Example

Let X_1, X_2 be independent $\text{Exponential}(\lambda)$. Compute the conditional density of $X_1 + X_2$ given $X_1 = X_2$.

One solution is to use Y_1, Y_2 from above; another is to use Y_1, Y_3.

Note $\{X_1 = X_2\}$ is a null event, these two describe it differently.

Sections 3.8 and 4.8
Sums of Random Variables

Example

X_1, X_2 independent $\mathrm{Uniform}(0, 1)$. What is $X_1 + X_2$?

- Build intuition: extreme examples.
- Consider discrete analogue: die.

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- Build intuition: extreme examples.
- Consider discrete analogue: die.
- Answer: triangle from 0 to 2 with maximum at 1.

Convolution

Definition

$$(f * g)(x) := \int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

Interpretation: X and Y with densities f and g then density of $X + Y$ is $f * g$.

Revisit sum of uniforms.

Properties of the convolution

Lemma

- $f * g = g * f$.
- $(\widehat{f * g})(x) = \widehat{f}(x) \cdot \widehat{g}(x)$, where

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}$$

  is the Fourier transform.
- $f * \delta = f$ where δ is the Dirac delta functional.
- $f * (g * h) = (f * g) * h$.

Example

X_1, X_2 $\text{Poisson}(\lambda_1)$ and $\text{Poisson}(\lambda_2)$, then $X_1 + X_2$ is $\text{Poisson}(\lambda_1 + \lambda_2)$

Proof: Evaluate convolution, using binomial theorem.

Section 4.10
Distributions from the Normal

Standard results and definitions

- $X \sim N(0,1)$ then X^2 is chi-square with 1 degree of freedom.
- Sample mean: $\overline{X} := \frac{1}{N}\sum_{i=1}^{n} X_i$.
- Sample variance: $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

Main theorem

Sums of normal random variables

Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$. Then

- $\overline{X} = N(\mu, \sigma^2/n)$.
- $(n-1)S^2$ is a chi-square with $n-1$ degrees of freedom. (Easier proof with convolutions?)
- $\overline{X}$ and S^2 are independent.
- Central Limit Theorem: $\overline{X} \sim N(\mu, \sigma^2/n)$.

Clicker Questions

Pepys' Problem

Problem Statement

Alice and Bob decide wager on the rolls of die. Alice rolls $6n$ fair die and wins if she gets at least n sixes, while Bob wins if she fails. What n should Alice choose to maximize her chance of winning?

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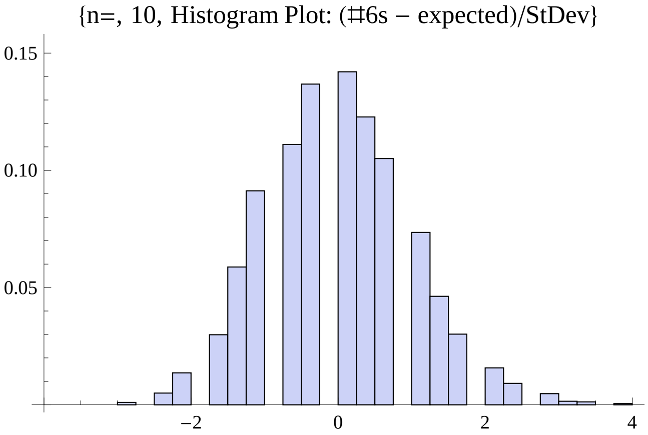
- (a) 1
- (b) 2
- (c) 6
- (d) 10
- (e) 20
- (f) 341
- (g) The larger n is, the greater chance she has of winning.

Pepys' Problem (continued): 1000 simulations, binsize = .25



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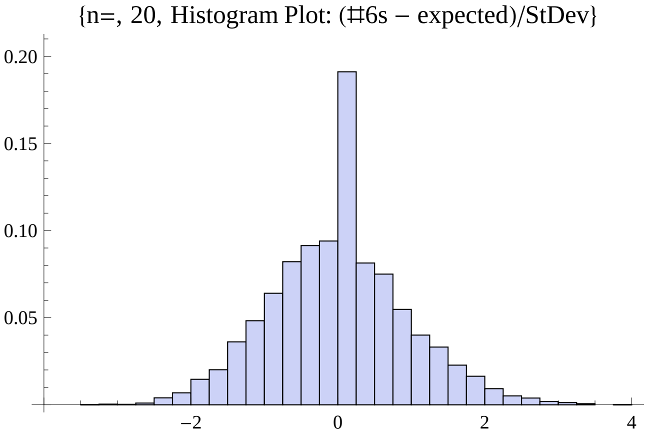

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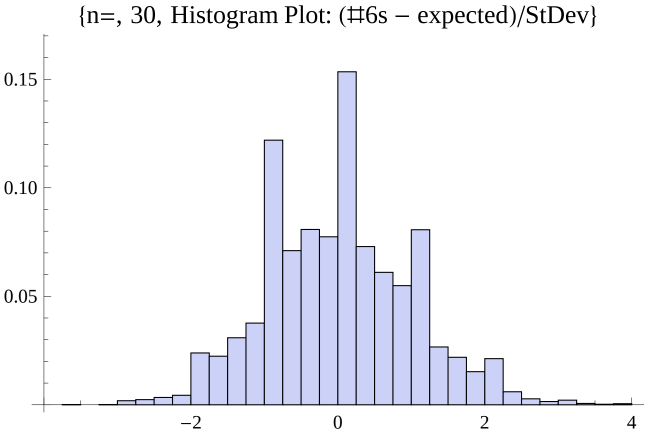

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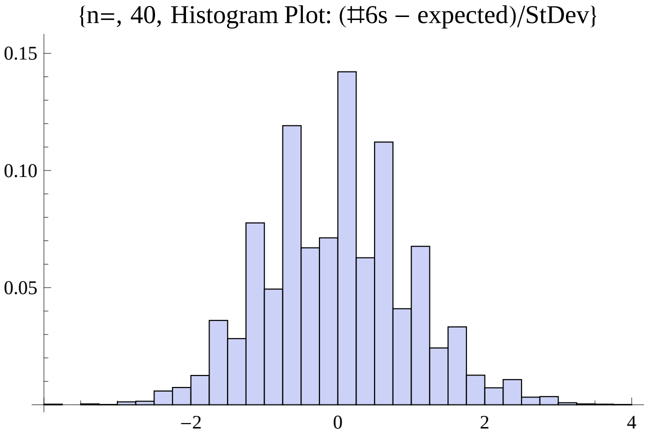

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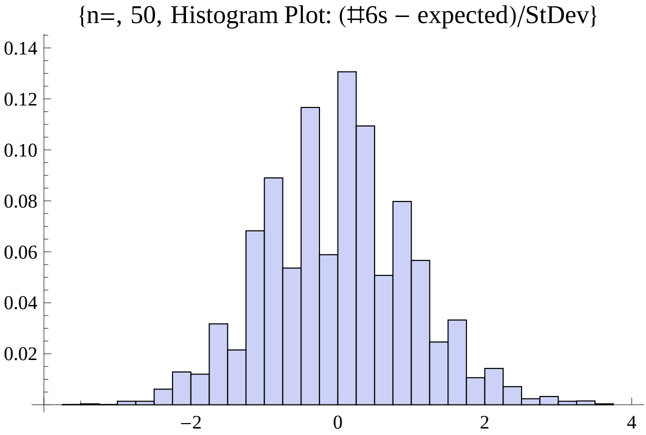

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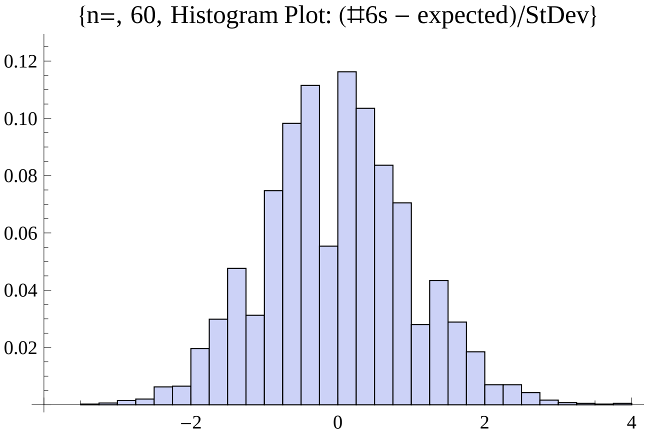

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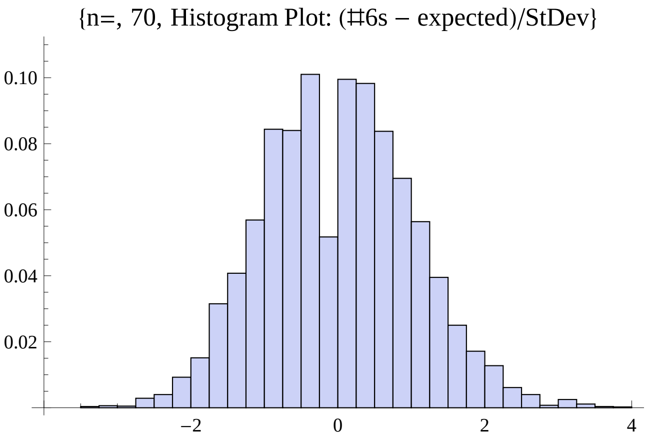

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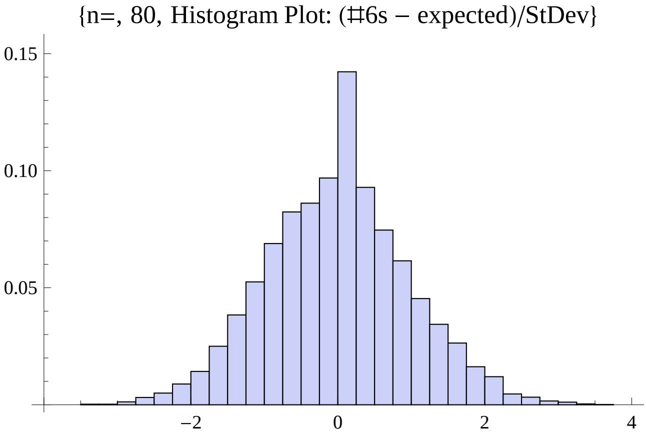

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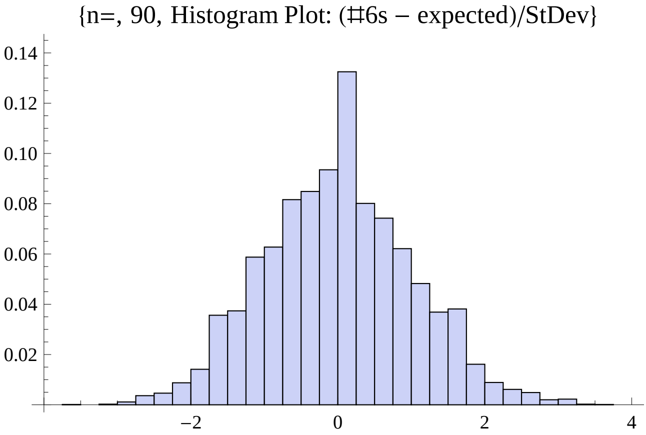

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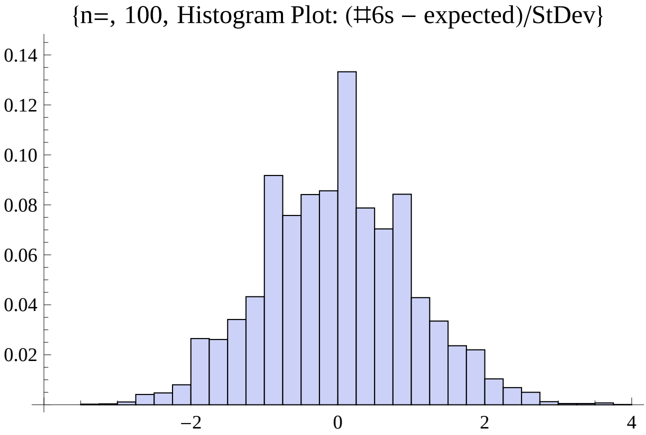

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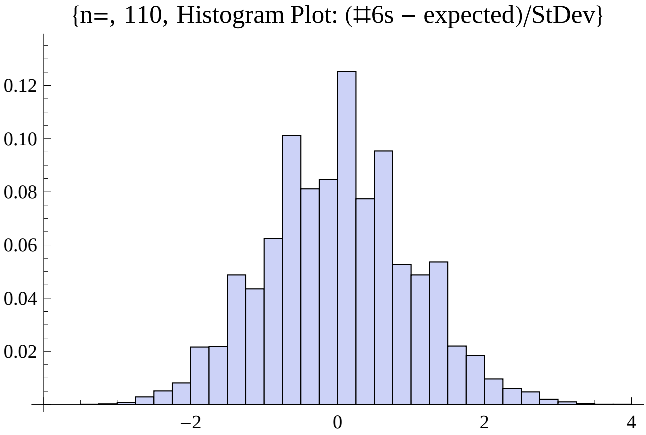

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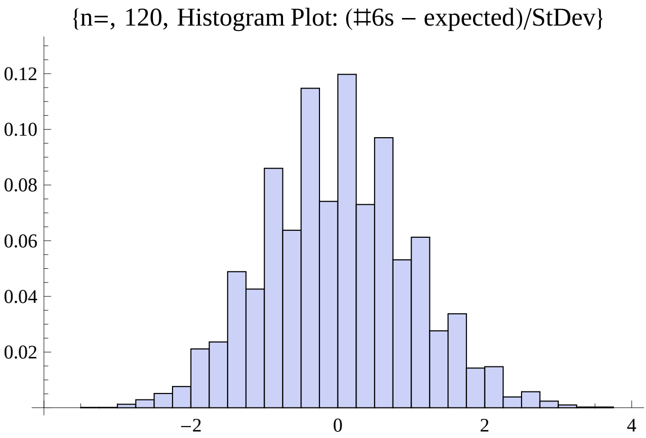

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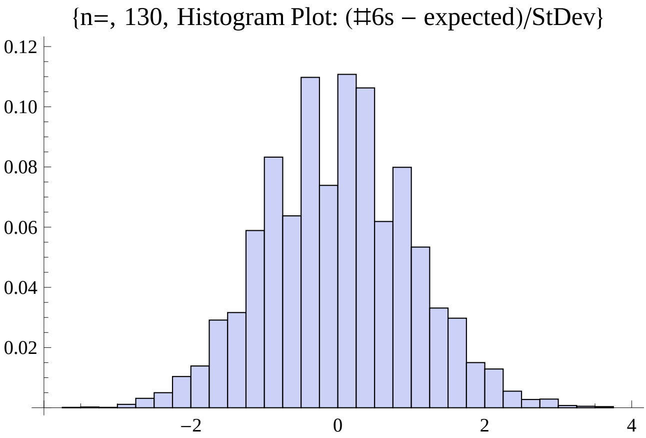

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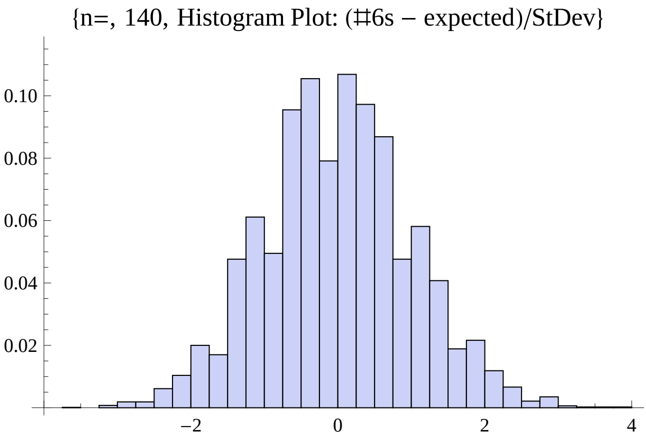

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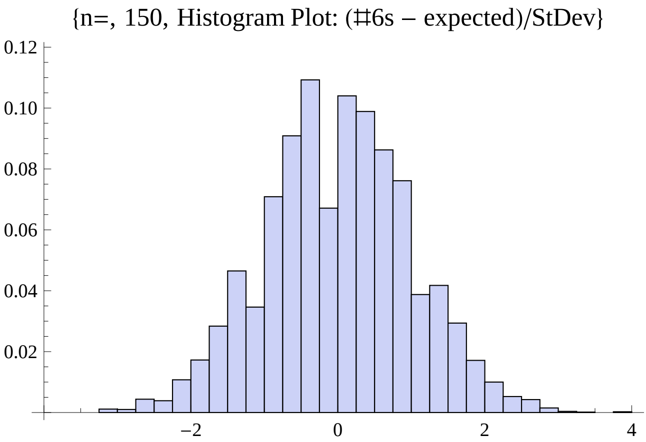

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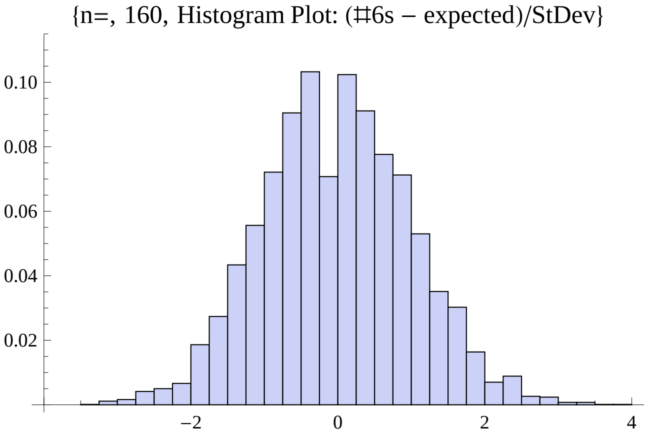

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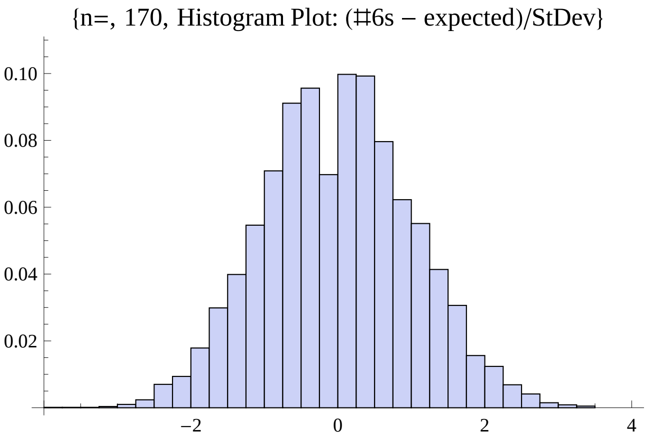

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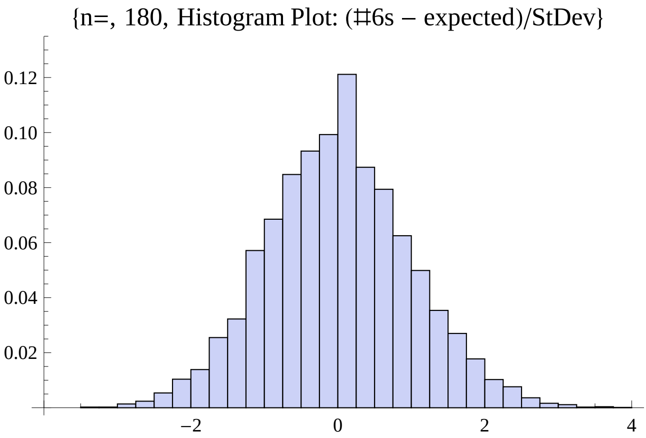

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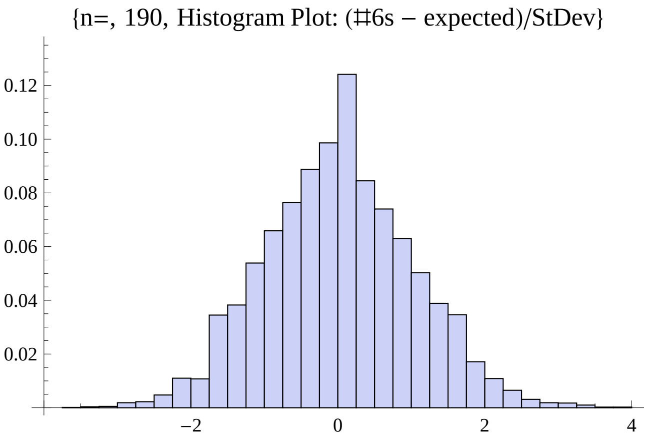

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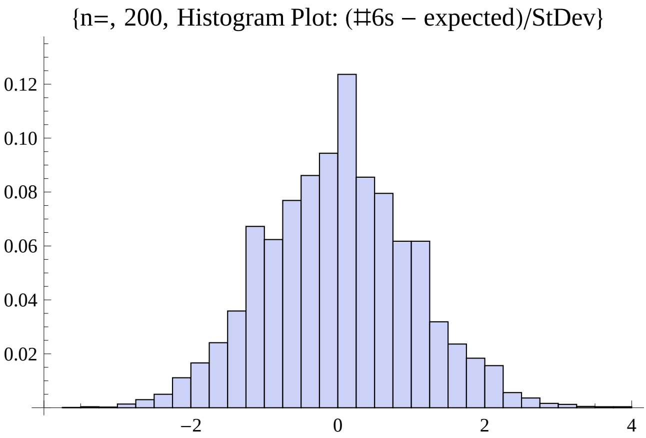

Pepys’ Problem (continued): probability versus n