MATH 341: PROBABILITY: FALL 2009: HOMEWORK #10

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Due Thursday November 19 (though you may place in my mailbox up till 10am on Friday 11/20):

(1) Let $X_1, \dots, X_N$ be iidrv with mean 0 and variance 1, and let $\overline{X} = (X_1 + \cdots + X_N)/N$. For any fixed $\epsilon > 0$, prove that as $N \to \infty$ we have
$$\lim_{N\to\infty} \mathrm{Prob}(|\overline{X} - 0| > \epsilon) \ = \ 0.$$
Hint: try using Chebyshev's theorem.

(2) Let
$$f(x) \ = \ \begin{cases} \frac{1}{\sqrt{2\pi x^2}}\ e^{-(\log x)^2/2} & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$
(a) Prove f is continuous.
(b) Let m be any fixed positive integer. Prove that $\lim_{x\to\infty} x^m f(x) \ = \ 0$.

(3) Assume you have a table of probabilities of the standard normal random variable X; in other words, you can easily look up probabilities of the following form: $\Phi(x) = \mathrm{Prob}(X \le a)$. (The cumulative distribution function of the standard normal is used so often it gets a symbol reserved for it, namely Φ.)
(a) Show $\mathrm{Prob}(X \le 0) = 1/2$.
(b) Let $Y \sim N(\mu, \sigma)$ be a normal random variable with mean μ and variance σ^2. Express $\mathrm{Prob}(Y \le a)$ in terms of Φ, μ, σ and of course a.

(4) **DO EXACTLY ONE OF THE FOLLOWING:**

(a) Find any math research paper or expository paper which uses probability and write an at most one page summary (preferably in TeX). As you continue in your careers, you are going to need to read technical papers and summarize them to your superiors / colleagues / clients; this is thus potentially a very useful exercise. Make sure you describe clearly what the point of the paper is, what techniques are used to study the problem, what applications there are (if any). Below is a sample review from MathSciNet; if you would like to see more, you can go to their homepage or ask me and I'll pass along many of the ones I've written. I've chosen this one as it's related to a paper on randomly shuffling cards (this paper is linked in the additional comments from October 27): Bayer, Dave and Diaconis, Persi, *Trailing the dovetail shuffle to its lair*, Ann. Appl. Probab. **2** (1992), no. 2, 294–313.

Rarely does a new mathematical result make both the New York Times and the front page

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Date: November 17, 2009.

of my local paper, and even more rarely is your reviewer asked to speak on commercial radio about a result, but such activity was caused by the preprint of this paper. In layman's terms, it says you should shuffle a deck of cards seven times before playing. More technically, the usual way people shuffle is called a riffle shuffle, and a natural mathematical model of a random shuffle is to assume all possible riffle shuffles are equally likely. With this model one can ask how close is k shuffles of an n-card deck to the uniform distribution on all $n!$ permutations, where 'close' is measured by variation distance. It was previously known that, as $n \to \infty$, one needs $k(n) \sim 32 \log_2 n$ shuffles to get close to uniform. This paper gives an elegant and careful treatment based on an explicit formula for the exact distance $d(k, n)$ to uniformity. To quote the abstract: 'Key ingredients are the analysis of a card trick and the determination of the idempotents of a natural commutative subalgebra in the symmetric group algebra.' – Reviewed by David J. Aldous

(b) The following three problems: Problem #10 in Section 3.11 **and** Problem #17 in Section 3.11 **and** Problem #1 in Section 1.8.

(5) **Extra Credit:** Prove which of the following from lecture converges slowest to the standard normal: uniform, Laplace or Millered Cauchy.

(6) **Extra Credit:** Define

$$f_k(x) \ = \ \frac{C_k(a)}{1 + (ax)^{2k}}$$

where $C_k(a)$ is chosen so that the above is a probability density.
(a) Find a and $C_3(a)$ so that the density above has variance 1.
(b) More generally, for any integer $k \geq 3$ find a and $C_k(a)$ so that the density above has variance 1.