MATH 341: PROBABILITY: FALL 2009: HOMEWORK #11

STEVEN J. MILLER (SJM1@WILLIAMS.EDU)

Due Thursday November 19 (though you may place in my mailbox up till 10am on Friday 11/20):

DO ANY FIVE OF THE PROBLEMS BELOW. *If you choose to do either problem 6 or 7 (you of course may elect to do both), you must email me your .tex file and .pdf, and note on the homework you submit to the grader which of these problems you elected to do.*

(1) Let $X, X_1, \dots, X_N$ be independent exponential random variables with parameter λ. Find the moment generating function for X_i. Directly using the moment generating function, prove the central limit theorem for $X_1 + \cdots + X_N$ (i.e., mimic what we did for the Poisson).

(2) Let $f(x)$ be a Schwartz function on $(-\infty, \infty)$. In particular, this means that f is a k times continuously differentiable probability density for any positive integer j. In other words, the first k derivatives of f exist and each of these derivatives is continuous. Prove there is some constant C (depending on f, f', ..., $f^{(k)}$ such that as $|y| \to \infty$, $|\widehat{f}(y)| \le C/|y|^k$.

(3) We say f is a continuous probability density supported on $[-B, B]$ if $f(x) = 0$ if $|x| > B$; equivalently if X is a random variable with density f we say X is supported on $[-B, B]$ if f is supported on $[-B, B]$. For example, if $X \sim \text{Unif}(2, 5)$ then X is supported on $[-5, 5]$, while if $X \sim \text{Exp}(1)$ then there is no B such that X is supported on $[-B, B]$.

$\diamond$ Prove or disprove: if f is supported on $[-B, B]$ then the $2k^{\text{th}}$ moment of f is at most B^{2k}.

$\diamond$ Prove or disprove: Let μ'_{2k} denote the $2k^{\text{th}}$ moment of f. Assume that

$$\lim_{k\to\infty} \mu'^{1/2k}_{2k} \ \le\ B.$$

Then f is supported on $[-B, B]$. (In other words, the probability of f taking on a value x with $|x| > B$ is zero.)

$\diamond$ Prove or disprove: Assume μ'_{2k}, the $2k^{\text{th}}$ moment of f, satisfies

$$(2k)!! \ \le\ \mu'_{2k} \ \le\ (2k)!.$$

Then there is some finite B such that f is supported on $[-B, B]$.

Date: November 20, 2009.

(4) Is the following argument correct: Consider

$$\lim_{N\to\infty}\left[\left(1+\frac{x}{N}\right)^{N^2}\cdot\left(1-\frac{x}{N}\right)^{N^2}\right].$$

For large N the first factor looks like e^{xN} since

$$\left(1+\frac{x}{N}\right)^{N^2} \ = \ \left(\left(1+\frac{x}{N}\right)^{N}\right)^{N} \ \longrightarrow \ (e^x)^N \ = \ e^{xN}.$$

Similarly we see that the second factor looks like e^{-xN}, and thus the product tends to 1 as $N\to\infty$. If this argument is wrong, what should the limit be? In other words, find

$$\lim_{N\to\infty}\left[\left(1+\frac{x}{N}\right)^{N^2}\cdot\left(1-\frac{x}{N}\right)^{N^2}\right]$$

if the argument above is incorrect.

(5) Let A be an arithmetic progression of n integers with common difference d; this means there is some n_0 such that

$$A \ = \ \{n_0, n_0+d, n_0+2d, \dots, n_0+nd\}.$$

Prove $|A+A| = |A-A|$, where

$$\begin{aligned} |A+A| \ &= \ \{a_1+a_2 : a_1, a_2 \in A\} \\ |A-A| \ &= \ \{a_1-a_2 : a_1, a_2 \in A\}. \end{aligned}$$

This implies that arithmetic progressions are 'balanced' (their sumset $A+A$ is as large as their difference set $A-A$). *Hint: show without loss of generality that we may take* $n_0=0$ *and* $d=1$ *when we count the number of differences or sums.*

(6) Write up a problem of your choosing and a solution. You must have someone from the class check it. If the problem is unclear or the solution is wrong, unlike previous homework assignments this time you will lose points.

(7) Read a paper involving probability and give a one page summary.