MATH 341: PROBABILITY: FALL 2009
QUIZ 1: CHAPTER 1: SECTIONS 1.1, 1.2, 1.3

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1. Problems

The point of these problems is to help you prepare for class by highlighting ***some*** of the key definitions and results, and giving you a simple problem or two to test your reading. You are on the honor system to make sure you attempt these problems before each class; if you do not or if you do not understand one of the solutions, you must let me know. You are welcome to hand in your solutions (I often find the act of writing things down helps me remember), but you are not required to do so.

Question 1: Define sample space, field and σ-field.

Question 2: State what it means to be finitely additive and what it means for a triple $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space.

Question 3: Give a formula for the conditional probability of A given B. If we consider the probability space of all rolls of three fair die (each die rolled takes on a value from $\{1, \dots, 6\}$ with probability 1/6), what is the probability that the sum of all the die is 4? What is the probability that the sum of all the die is 4 given that the sum is even?

Question 4: State the law of total probability. Consider again the die problem from the previous exercise. Compute the probability that we the sum of the three fair die is at least 5.

Solutions on the next pages.

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Question 1: Define sample space, field and σ-field.
Solution 1: A sample space Ω is the set of all possible outcomes. A field is a collection of subsets $\mathcal{F}$ of Ω such that if $A, B \in \mathcal{F}$ then so are $A \cup B$, $A \cap B$, A^c (this is the complement of A, frequently denoted $\Omega - A$ or $\Omega \setminus A$) and φ (the empty set; note $\varphi \in \mathcal{F}$ implies that $\Omega \in \mathcal{F}$ as $\Omega = \varphi^c$). To be a σ-field, we need the following additional property: if $A_i \in \mathcal{F}$ for each i then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$. In other words, infinite unions of elements of $\mathcal{F}$ are in $\mathcal{F}$. Note that in a field we are only assured of having finite unions. *As a nice extra credit exercise, find a sample space with a field $\mathcal{F}$ that is not a σ-field.*

Question 2: State what it means to be finitely additive and what it means for a triple $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space.
Solution 2: A probability $\mathbb{P}$ is finitely additive if whenever $A_i \in \mathcal{F}$ we have $\mathbb{P}\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$. A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space if Ω is a set with σ-field $\mathcal{F}$ and a probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$ that satisfies

- $\mathbb{P}(\varphi) = 0$;
- if $\{A_i\}_{i=1}^{\infty}$ is a pairwise disjoing collection of elements of $\mathcal{F}$ (this means $A_i \cap A_j$ is empty if $i \neq j$) then $\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

Question 3: Give a formula for the conditional probability of A given B. If we consider the probability space of all rolls of three fair die (each die rolled takes on a value from $\{1, \dots, 6\}$ with probability 1/6), what is the probability that the sum of all the die is 4? What is the probability that the sum of all the die is 4 given that the sum is even?
Solution 3: If $\mathbb{P}(B) > 0$ then $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$, where $\mathbb{P}(A|B)$ is the conditional probability of A given B (or the probability of A given B). There are $6^3 = 216$ elements of Ω, ranging from $\{1, 1, 1\}$ to $\{6, 6, 6\}$. If the sum is a four, then it must be one of $\{1, 1, 2\}$, $\{1, 2, 1\}$ or $\{2, 1, 1\}$. Thus there are three triples out of 216 that work, for a probability of $3/216$ or $1/72$. If now we assume that the sum is even, then the number of triples that work is still three, but now there are $6^3/2$ or 108 triples (the reason is that half of these have an even sum; the simplest way to see this is to note that half the values of the last roll make the sum even and half make the sum odd). Thus the probability is $3/108$ or $1/36$. To see this using our formula, let A be the event that the sum is 4 and B be the event that the sum is even. Then $A \cap B$ is the event that the sum is 4 *and* even; as the only way to have a sum of 4 is to have the sum even, note $A = A \cap B$, and thus $\mathbb{P}(A \cap B) = 3/216$. Further, $\mathbb{P}(B) = 1/2$ (as half the rolls have an even sum). Thus

$$\mathbb{P}(A|B) \;=\; \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \;=\; \frac{3/216}{1/2} \;=\; \frac{1}{36}.$$

Question 4: State the law of total probability. Consider again the die problem from the previous exercise. Compute the probability that we the sum of the three fair die is at least 5.

Solution 4: The law of total probability states that $\mathbb{P}(A^c = 1 - \mathbb{P}(A)$. Let A be the event that the sum is at most 4. This means our triple must be one of $\{1,1,1\}$, $\{1,1,2\}$, $\{1,2,1\}$ or $\{2,1,1\}$. Thus $\mathbb{P}(A) = 4/216 = 1/54$, and so $\mathbb{P}(A^c) = 1 - \frac{1}{54} = \frac{53}{54}$. Note how much simpler it is to use the law of total probability than to do the following brute force approach: let A_n denote the event that the sum of the three rolls is n. Then the desired probability is

$$\sum_{n=5}^{18} \mathbb{P}(A_n).$$

For example, what is $\mathbb{P}(A_{11})$? Is there an easy way to compute this? In many problems we frequently only care about the probabilities of 'tail' events. For example, imagine that this model represents someone's blood sugar; as long as the blood sugar isn't too low (i.e., at least 5 units), the person will be healthy.