MATH 341: PROBABILITY: FALL 2009
QUIZ 2: CHAPTER 1: SECTIONS 1.7; CHAPTER 2: SECTIONS 2.1, 2.3

STEVEN J. MILLER (SJM1@WILLIAMS.EDU)

1. Problems

The point of these problems is to help you prepare for class by highlighting ***some*** of the key definitions and results, and giving you a simple problem or two to test your reading. You are on the honor system to make sure you attempt these problems before each class; if you do not or if you do not understand one of the solutions, you must let me know. You are welcome to hand in your solutions (I often find the act of writing things down helps me remember), but you are not required to do so.

Question 1: Define a random variable, and give an example.

Question 2: Define the distribution function of a random variable, and give an example.

Question 3: Define the symmetric random walk (with a fair coin). Solve the problem when $k = N/2$.

Question 4: State what it means for a random variable to be discrete and to be continuous. Give an example.

Solutions on the next pages.

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Question 1: Define a random variable, and give an example.
Solution 1: Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A random variable is a function X from the sample space Ω to the real numbers with the property that $\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$ for each x. For example, let Ω be the space of all tosses of a fair coin five times, and let $X(\omega)$ denote the number of heads in ω. In this case $\mathcal{F} = 2^\Omega$, the set of all subsets of Ω. As there are $2^5 = 32$ elements, there are 2^{32} or about 4,000,000,000 elements in $\mathcal{F}$. Each element of $\mathcal{F}$ is a subset of Ω, and each subset of Ω is an element of $\mathcal{F}$. If we write $F = \{\omega_1, \dots, \omega_k\}$ for an element of $\mathcal{F}$, then $\mathbb{P}(F) = \sum_{i=1}^k \mathbb{P}(\omega_i)$. A straightforward computation (which you do not want to do!) shows that X has the desired property; this is clear as all subsets of Ω are in $\mathcal{F}$! In our problem, imagine $x = 1$. Then

$$\begin{aligned} & \{\omega \in \Omega : X(\omega) \leq 1\} \\ = \ & \{TTTTT, TTTTH, TTTHT, TTHTT, THTTT, HTTTT\}. \end{aligned}$$

If instead we took $x = 4$, then the set would be all outcomes except $HHHHH$.

Question 2: Define the distribution function of a random variable, and give an example.
Solution 2: The distribution function of a random variable $X : \Omega \to \mathbb{R}$ is the function $F : \mathbb{R} \to [0, 1]$ given by $F(x) = \mathbb{P}(X \leq x)$. In other words, it is the probability of observing a value of X of at most x. For our example, consider the previous problem concerning five tosses of a fair coin. We have $F(0) = 1/32$, $F(1) = 6/32$, $F(2) = 16/32$, $F(3) = 26/32$, $F(4) = 31/32$ and $F(5) = 32/32$. Our function is supposed to be defined for all real x, so what we really have is the following: $F(x) = 0$ if $x < 0$, $F(x) = 1/32$ if $0 \leq x < 1$, $F(x) = 6/32$ if $1 \leq x < 2$, and so on.

Question 3: Define the symmetric random walk (with a fair coin). Solve the problem when $k = N/2$.
Solution 3: Also known as the Gambler's ruin, a symmetric random is when we start at \$$k$, and keep flipping a fair coin. Each time we get a heads we increment by \$1, while for each tail we lose \$1. The goal is to reach N before 0. The solution in general involves difference equations; however, if N is even and $k = N/2$, then the probability we reach N before reaching 0 is just 1/2 by symmetry. To prove this 'rigorously', note that any two sequences of heads and tails are equally likely. Given a sequence that reaches N first, if we interchange all heads for tails and tails for heads, we obtain a new sequence that reaches 0 first. Thus we can match the outcomes that give N pairwise with outcomes that give 0.

Question 4: State what it means for a random variable to be discrete and to be continuous. Give an example.

Solution 4: A random variable X is discrete if it takes values in a countable subset $\{x_1, x_2, \dots\}$ of $\mathbb{R}$. It has probability mass function $f : \mathbb{R} \to [0,1]$ given by $f(x) = \mathbb{P}(X = x)$. For example, consider tossing a fair coin until the first head is obtained. Then $\Omega = \{H, TH, TTH, \dots\}$. Let X be the number of tosses needed to obtain the first head. Then X is discrete, taking on the values $\{1, 2, 3, \dots\}$, with the probability X equals n just $1/2^n$.

A random variable X is continuous if its distribution function can be written as $F(x) = \int_{-\infty}^{x} f(u)du$ for some integrable function f (which is called the probability density function of X). For example, let $\Omega = [0,1]$ and let $\mathcal{F}$ be the σ-field generated by the open intervals. (This is the standard σ-field.) Let $X(\omega)$ equal ω^2. If we let Y be uniformly distributed on $[0,1]$, then we see $\mathbb{P}(X \leq x)$ is the same as $\mathbb{P}(Y \leq \sqrt{x})$, which is just $\sqrt{x}$. We are therefore looking for f so that $\sqrt{x} = \int_0^x f(u)du$ for $0 \leq x \leq 1$. Differentiating both sides gives $\frac{1}{2}x^{-1/2} = f(x)$ (note the integral is $\mathfrak{F}(x) - \mathfrak{F}(0)$ with $\mathfrak{F}$ any anti-derivative of f; differentiating yields the claim as $\mathfrak{F}' = f$). We see that for our random variable X, we may take $f(u) = 1/2\sqrt{u}$ for $0 < u \leq 1$ and 0 otherwise.