MATH 341: PROBABILITY: FALL 2009
QUIZ 3: CHAPTERS 3 AND 4: SECTIONS 3.2, 3.3, 4.2 AND 4.3

STEVEN J. MILLER (SJM1@WILLIAMS.EDU)

1. PROBLEMS

The point of these problems is to help you prepare for class by highlighting ***some*** of the key definitions and results, and giving you a simple problem or two to test your reading. You are on the honor system to make sure you attempt these problems before each class; if you do not or if you do not understand one of the solutions, you must let me know. You are welcome to hand in your solutions (I often find the act of writing things down helps me remember), but you are not required to do so.

Question 1: Define expected value. Assume a standard deck of 52 cards is shuffled so that each card is equally likely to be anywhere. If we are dealt the first seven cards, what is the expected number of kings we are dealt?

Question 2: Does every distribution have a finite expected value? If yes, prove this; if no, give a counter-example.

Question 3: Define the k^{th} moment and the k^{th} centered moment of a distribution. Calculate the variance of X if X is a uniform random variable on $[-1, 1]$.

Question 4: Prove or disprove: Assume X and Y are random variables with finite means and variances. Then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Solutions on the next pages.

Date: October 4, 2009.

Question 1: Define expected value. Assume a standard deck of 52 cards is shuffled so that each card is equally likely to be anywhere. If we are dealt the first seven cards, what is the expected number of kings we are dealt?

Solution 1: If X is discrete then $\mathbb{E}[X] = \sum_x x f_X(x)$ while if X is continuous then $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$, where f_X is the probability density / mass function. To calculate the expected value from the definition, we need to know the probability of having 0, 1, 2, 3 or 4 kings in our hand. The probability of having k kings (with $k \in \{0,1,2,3,4\}$ in 7 cards is $\binom{4}{k}\binom{48}{7-k}/\binom{52}{7}$. Thus the expected number of kings is just

$$\frac{0\binom{4}{0}\binom{48}{7}}{\binom{52}{7}} + \frac{1\binom{4}{1}\binom{48}{6}}{\binom{52}{7}} + \frac{2\binom{4}{2}\binom{48}{5}}{\binom{52}{7}} + \frac{3\binom{4}{3}\binom{48}{4}}{\binom{52}{7}} + \frac{4\binom{4}{4}\binom{48}{3}}{\binom{52}{7}} \ = \ \frac{72037840}{133784560}.$$

Can you find a general formula for the expected number of kings when we draw n cards? What properties should this formula have?

Question 2: Does every distribution have a finite expected value? If yes, prove this; if no, give a counter-example.

Solution 2: No. Consider the Cauchy distribution,

$$f_X(x) \ = \ \frac{1}{\pi}\frac{1}{1+x^2}.$$

The fact that this integrates to 1 is proved in Calculus I or II using the arctan function. While we need to know $\arctan'(x) = 1/(1+x^2)$ to show the integral is 1, it isn't too hard to see the integral converges and thus there is some normalization constant which, when multiplied, gives us an area of 1. Further, the density is clearly non-negative, so this is a density. The reason the expected value does not exist is that, since this is an improper integral (the bounds are $\pm\infty$), we require the integral of the absolute value of the integrand to be finite for the integral to exist. The integral here depends on how we tend to infinity. In particular,

$$\lim_{A\to\infty}\int_{-A}^{A} \frac{xdx}{\pi(1+x^2)} \ \neq \ \lim_{A\to\infty}\int_{-A}^{2A} \frac{xdx}{\pi(1+x^2)};$$

the first integral is just 0, while the second is basically

$$\int_A^{2A} \frac{1}{\pi}\frac{dx}{x} \ = \ \frac{\log x}{\pi}\Big|_A^{2A} \ = \ \frac{\log 2}{\pi}$$

for A large.

If you don't like this example, consider instead the half-Cauchy density

$$f_Y(x) \ = \ \begin{cases} \frac{2}{\pi}\frac{1}{1+x^2} & \text{if } x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that this has an infinite expected value, as

$$\mathbb{E}[Y] \ = \ \int_0^{\infty} \frac{2xdx}{\pi(1+x^2)}.$$

For large x the integrand looks like $1/x$, which means the anti-derivative looks like $\log x$, and thus the integral diverges.

Question 3: Define the k^{th} moment and the k^{th} centered moment of a distribution. Calculate the variance of X if X is a uniform random variable on $[-1, 1]$.
Solution 3: The k^{th} moment is $\mathbb{E}[X^k]$ and the k^{th} centered moment is $\mathbb{E}[(X - \mu)^k]$, where $\mu = \mathbb{E}[X]$. The variance is $\mathbb{E}[(X - \mu)^2]$. Clearly the uniform distribution on $[-1, 1]$ has density function $1/2$ for $-1 \leq x \leq 1$ and 0 otherwise. The mean μ is zero, and hence the variance is just

$$\mathbb{E}[X^2] \ = \ \int_{-1}^{1} x^2 \cdot \frac{1}{2} dx \ = \ \frac{x^3}{6}\Big|_{-1}^{1} \ = \ \frac{1}{3}.$$

Question 4: Prove or disprove: Assume X and Y are random variables with finite means and variances. Then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.
Solution 4: False. Imagine $X = 1$ with probability 1/2 and -1 with probability 1/2, and set $Y = -X$. Then $XY = -1/4$ with probability 1, but $\mathbb{E}[X] = \mathbb{E}[Y] = 0$. Note we are using the definition

$$E[XY] \ := \ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x, y) dx dy.$$