Math 341: Probability
Ninth Lecture (10/8/09)

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Sections 3.3 & 4.3: Expectation

Definition

Moments

Let X be a random variable. We define

- k^{th} moment: $m_k := \mathbb{E}[X^k]$ (if converges absolutely).

Assume X has a finite mean, which we denote by μ (so $\mu = \mathbb{E}[X]$). We define

- k^{th} centered moment: $\sigma_k := \mathbb{E}[(X - \mu)^k]$ (if converges absolutely).

- Be alert: Some books write μ'_k for m_k and μ_k for σ_k.
- Call σ_2 the variance, write it as σ^2 or $\mathrm{Var}(X)$.
- Note $\mathrm{Var}(X) = \mathbb{E}[(X - \mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

Key Results

- Linearity: $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$.
- Independence: X, Y independent then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. If RHS holds say uncorrelated.
- Variance: $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$ if uncorrelated. In general:

$$\begin{aligned} \mathrm{CoVar}(X, Y) &= \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] \\ \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{1 \le i < j \le n} \mathrm{CoVar}(X_i, X_j). \end{aligned}$$

Clicker Questions

Prime divisors

Number of prime divisors

Let N be a large number. If we choose an integer of size approximately N, on average about how many distinct prime factors do we expect N to have (as $N \to \infty$)?
It might be useful to recall the Prime Number Theorem: The number of primes at most x is about $x / \log x$.

- (a) At most 10.
- (b) Around $\log \log \log N$.
- (c) Around $\log \log N$.
- (d) Around $\log N$.
- (e) Around $\log N \log \log N$.
- (f) Around $(\log N)^2$.
- (g) This is an open question.

Fermat Primes

Fermat Primes

If $F_n = 2^{2^n} + 1$ is prime, we say F_n is a Fermat prime. About how many Fermat primes are there less than x as $x \to \infty$?

- (a) 5
- (b) 10
- (c) Between 11 and 20.
- (d) Between 21 and 100.
- (e) $\log \log \log x$.
- (f) $\log \log x$.
- (g) $\log x$.
- (h) More than $\log x$.
- (i) This is an open problem.

$3x+1$ Problem

$3x+1$: Iterating to the fixed point

Define the $3x+1$ map by $a_{n+1} = \frac{3a_n+1}{2^k}$ where $2^k || 3a_n+1$. Choose a large integer N and randomly choose a starting seed a_0 around N. About how many iterations are needed until we reach 1 (equivalently, about how large is the smallest n such that $a_n = 1$)?

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$3x+1$: Iterating to the fixed point

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There is a constant C so that the answer is about

- (a) At most 10.
- (b) Around $C \log\log\log N$.
- (c) Around $C \log\log N$.
- (d) Around $C \log N$.
- (e) Around $C \log N \log\log N$.
- (f) Around $C(\log N)^2$.
- (g) This is an open question.