Math 341: Probability
Fourteenth Lecture (10/29/09)

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Summary for the Day

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- General advice from the Millers:
- Sums of random variables:
    - Convolution.
    - Properties of convolution.
    - Poisson example.
- Distributions from Normal:
    - Sample mean and variance.
    - Central Limit Theorem and Testing.

General Advice

General advice from Jeff Miller

Three tips for college

- Drink less than those that are flunking out. (You don't have to be faster than the bear....)

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Three tips for college

- Drink less than those that are flunking out. (You don't have to be faster than the bear....)
- Learn to manage your time, because no one else wants to. (Critical life lesson.)
- Don't be afraid to ask for help, office hours is the most under utilized resource. (In industry you'll beg for mentoring and won't get it, yet many don't take advantage of it when it's free and plentiful in college.)

Additional advice

My two cents.

- Get to know at least one professor well a semester.
- Think about the facts you've learned and will forget, and the techniques you will constantly reuse.
- Always know your audience and have something to say to anyone.
- Anticipate questions, but don't be afraid to ask for time to think.

Sections 3.8 and 4.8
Sums of Random Variables

Example

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- Build intuition: extreme examples.
- Consider discrete analogue: die.
- Answer: triangle from 0 to 2 with maximum at 1.

Convolution

Definition

$$(f * g)(x) := \int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

Interpretation: X and Y independent random variables with densities f and g then density of $X + Y$ is $f * g$.

Revisit sum of uniforms.

Properties of the convolution

Lemma

- $f * g = g * f$.
- $(\widehat{f * g})(x) = \widehat{f}(x) \cdot \widehat{g}(x)$, where

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

  is the Fourier transform.
- $f * \delta = f$ where δ is the Dirac delta functional.
- $f * (g * h) = (f * g) * h$.

Example

X_1, X_2 Poisson(λ_1) and Poisson(λ_2), then $X_1 + X_2$ is Poisson$(\lambda_1 + \lambda_2)$

Proof: Evaluate convolution, using binomial theorem.

Section 4.10
Distributions from the Normal

Standard results and definitions

- $X \sim N(0,1)$ then X^2 is chi-square with 1 degree of freedom.
- Sample mean: $\overline{X} := \frac{1}{N}\sum_{i=1}^{n} X_i$.
- Sample variance: $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

Main theorem

Sums of normal random variables

Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$. Then

- $\overline{X} \;=\; N(\mu, \sigma^2/n)$.
- $(n-1)S^2$ is a chi-square with $n-1$ degrees of freedom. (Easier proof with convolutions?)
- $\overline{X}$ and S^2 are independent.
- Central Limit Theorem: $\overline{X} \sim N(\mu, \sigma^2/n)$.