Math 341: Probability
Fifteenth Lecture (11/3/09)

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Summary for the Day

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- Distributions from Normal:
  - Sample mean and variance.
  - Central Limit Theorem and Testing.
- Generating Functions:
  - Definition.
  - Properties.
  - Applications.

Section 4.10
Distributions from the Normal

Standard results and definitions

- $X \sim N(0,1)$ then X^2 is chi-square with 1 degree of freedom.
- Sample mean: $\overline{X} := \frac{1}{N}\sum_{i=1}^{n} X_i$.
- Sample variance: $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

Main theorem

Sums of normal random variables

Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$. Then

- $\overline{X} = N(\mu, \sigma^2/n)$.
- $(n-1)S^2$ is a chi-square with $n-1$ degrees of freedom. (Easier proof with convolutions?)
- $\overline{X}$ and S^2 are independent.
- Central Limit Theorem: $\overline{X} \sim N(\mu, \sigma^2/n)$.

Generating Functions

Definitions

Generating Function

Given a sequence $\{a_n\}_{n=0}^{\infty}$, we define its generating function by

$$G_a(s) = \sum_{n=0}^{\infty} a_n s^n$$

for all s where the sum converges.

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Examples

- $a_n = 1/n!$ or $a_n = 2^n$ or $a_n = n!$.
- $a_n = (1-p)^{n-1}p$.
- Cookie problem, Goldbach, ...

Why generating functions

- Makes algebra easier (example: telescoping sums, diagonalizing matrices).
- Means, variances and moments....

Results

Uniqueness Theorem

Let $\{a_m\}_{m=0}^{\infty}$ and $\{b_m\}_{m=0}^{\infty}$ be two sequences of numbers with generating functions $G_a(s)$ and $G_b(s)$ which converge for $|s| < r$. Then the two sequences are equal (i.e., $a_i = b_i$ for all i) if and only if $G_a(s) = G_b(s)$ for all $|s| < r$. We may recover the sequence from the generating function by differentiating: $a_m = \frac{1}{m!} \frac{d^m G_a(s)}{ds^m}$.

Other results:

- $\mathbb{E}[X] = G_X'(1)$.
- $\mathrm{Var}(X) = G_X''(1) + G_X'(1) - G_X'(1)^2$.

Equivalent formulations: Why do we need both?

Probability Generating Function

X r.v., probability generating function is $G_X(s) = \mathbb{E}[s^X]$.

Moment Generating Function

X r.v., moment generating function is $M_X(t) = \mathbb{E}[e^{tX}]$.

Equivalent formulations: t imaginary $\Longrightarrow$ use complex analysis

Probability Generating Function

X r.v., probability generating function is $G_X(s) = \mathbb{E}[s^X]$.

Moment Generating Function

X r.v., moment generating function is $M_X(t) = \mathbb{E}[e^{tX}]$.

Key results:

- $M_X(t) = G_X(e^t)$.
- X, Y independent: $G_{X+Y}(s) = G_X(s)G_Y(s)$ and $M_{X+Y}(t) = M_X(t)M_Y(t)$.

Theorem: Let X be a random variable with moments μ'_k.

1. $$M_X(t) \;=\; 1 + \mu'_1 t + \frac{\mu'_2 t^2}{2!} + \frac{\mu'_3 t^3}{3!} + \cdots;$$
   in particular, $\mu'_k = d^k M_X(t)/dt^k \Big|_{t=0}$.
2. α, β constants: $M_{\alpha X+\beta}(t) \;=\; e^{\beta t} M_X(\alpha t)$. Also $M_{X+\beta}(t) = e^{\beta t} M_X(t)$, $M_{\alpha X}(t) = M_X(\alpha t)$, $M_{(X+\beta)/\alpha}(t) = e^{\beta t/\alpha} M_X(t/\alpha)$.
3. X_i's indep. r.v., MGF $M_{X_i}(t)$ converge for $|t| < r$ then $M_{X_1+\cdots+X_N}(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_N}(t)$; if i.i.d.r.v. equals $M_X(t)^N$.