Math 341: Probability
Sixteenth Lecture (11/5/09)

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Summary for the Day

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- Generating Functions:
  - Review Definitions.
  - Properties.
  - Example (Poisson Sums).
- Complex Analysis:
  - Warnings, good/bad examples.
  - Complex functions and differentiability.
  - Definitions.
  - Accumulation point theorem.

Generating Functions

Definitions

Generating Function

Given a sequence $\{a_n\}_{n=0}^{\infty}$, we define its generating function by

$$G_a(s) = \sum_{n=0}^{\infty} a_n s^n$$

for all s where the sum converges.

Useful choice is $a_n = \mathrm{Prob}(X = n)$ if $X \in \{0, 1, 2, \dots\}$.

Leads to $G_X(s) = \mathbb{E}[s^X]$.

Results

Uniqueness Theorem

Let $\{a_m\}_{m=0}^{\infty}$ and $\{b_m\}_{m=0}^{\infty}$ be two sequences of numbers with generating functions $G_a(s)$ and $G_b(s)$ which converge for $|s| < r$. Then the two sequences are equal (i.e., $a_i = b_i$ for all i) if and only if $G_a(s) = G_b(s)$ for all $|s| < r$. We may recover the sequence from the generating function by differentiating: $a_m = \frac{1}{m!}\frac{d^m G_a(s)}{ds^m}$.

Other results:

- $\mathbb{E}[X] = G_X'(1)$.
- $\mathrm{Var}(X) = G_X''(1) + G_X'(1) - G_X'(1)^2$.

Equivalent formulations: t imaginary $\Longrightarrow$ use complex analysis

Probability Generating Function

X r.v., probability generating function is $G_X(s) = \mathbb{E}[s^X]$.

Moment Generating Function

X r.v., moment generating function is $M_X(t) = \mathbb{E}[e^{tX}]$.

Key results:

- $M_X(t) = G_X(e^t)$.
- X, Y independent: $G_{X+Y}(s) = G_X(s)G_Y(s)$ and $M_{X+Y}(t) = M_X(t)M_Y(t)$.

Theorem: Let X be a random variable with moments μ'_k.

1. $$M_X(t) = 1 + \mu'_1 t + \frac{\mu'_2 t^2}{2!} + \frac{\mu'_3 t^3}{3!} + \cdots;$$
   in particular, $\mu'_k = d^k M_X(t)/dt^k \Big|_{t=0}$.
2. α, β constants: $M_{\alpha X+\beta}(t) = e^{\beta t} M_X(\alpha t)$. Also $M_{X+\beta}(t) = e^{\beta t} M_X(t)$, $M_{\alpha X}(t) = M_X(\alpha t)$, $M_{(X+\beta)/\alpha}(t) = e^{\beta t/\alpha} M_X(t/\alpha)$.
3. X_i's indep. r.v., MGF $M_{X_i}(t)$ converge for $|t| < r$ then $M_{X_1+\cdots+X_N}(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_N}(t)$; if i.i.d.r.v. equals $M_X(t)^N$.

Poisson Example

- X_i Poiss(λ_i):

$$f(n) \ = \ \text{Prob}(X_i = n) \ = \ \frac{\lambda_i^n e^{-\lambda_i}}{n!}$$

for $n \geq 0$, and 0 otherwise.

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- Does this imply sum of Poissons is Poisson?Yes because...

Uniqueness Theorem

Uniqueness of moment generating functions for discrete random variables

X and Y discrete random variables on $\{0, 1, 2, \dots\}$) with MGFs $M_X(t)$ and $M_Y(t)$ converging for $|t| < \delta$. Then X and Y have the same distribution iff $M_X(t) = M_Y(t)$ for $|t| < \delta$.

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- Proof: follows from uniqueness of generating functions as $M_X(t) = G_X(e^t)$.
- Proof: If only take on finite values, consider

$$p_1 x_1^k + \cdots + p_m x_m^k \;=\; q_1 y_1^k + \cdots + q_n y_n^k.$$

Warning!

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Exist distinct probability distributions with same moments. Standard example: for $x \geq 0$,

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{2\pi x^2}}\, e^{-(\log^2 x)/2} \\ f_2(x) &= f_1(x)\left[1 + \sin(2\pi \log x)\right]. \end{aligned}$$

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\end{aligned}
$$

Plan of attack:

- Explore what goes wrong with the functions.
- Discuss properties needed to prevent this. Soln involves complex analysis, tells us when a MGF uniquely determines a distribution.

Complex Analysis

Warning from Real Analysis

Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using the definition of the derivative and L'Hopital's rule, we can show that f is infinitely differentiable, and all of its derivatives at the origin vanish.

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Ridiculous! Taylor series (which converges for all x!) only agrees with the function when $x = 0$.

A Taylor series does not uniquely determine a function!
Both $\sin x$ and $\sin x + g(x)$ have the same Taylor series about $x = 0$.

Complex Analysis Definitions

Complex variable, complex function

Any complex number z can be written as $z = x + iy$, with x and y real. A complex function is a map f from $\mathbb{C}$ to $\mathbb{C}$; in other words $f(z) \in \mathbb{C}$. Frequently one writes $x = \mathfrak{Re}(z)$, $y = \mathfrak{Im}(z)$, and $f(z) = u(x, y) + iv(x, y)$ with u and v functions from $\mathbb{R}^2$ to $\mathbb{R}$.

Complex Analysis Definitions

Differentiable

f is differentiable at z_0 means

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists, h tends to zero along *any* path in the complex plane. Write $f'(z_0)$ if exists. If f is differentiable, then f satisfies the Cauchy-Riemann equations:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

(one direction easy: send $h \to 0$ along the paths $\widetilde{h}$ and $i\widetilde{h}$, with $\widetilde{h} \in \mathbb{R}$).

Complex Analysis Definitions

Many of the theorems below deal with open sets. We briefly review their definition and give some examples.

Open set, closed set

A subset U of $\mathbb{C}$ is an open set if for any $z_0 \in U$ there is a δ such that whenever $|z - z_0| < \delta$ then $z \in U$ (note δ is allowed to depend on z_0). A set C is closed if its complement, $\mathbb{C} \setminus C$, is open.

Examples of Open Sets, Closed Sets

Open sets:

1. $U_1 = \{z : |z| < r\}$ for any $r > 0$. This is usually called the ball of radius r centered at the origin.
2. $U_2 = \{z : \Re(z) > 0\}$.

Closed sets:

1. $C_1 = \{z : |z| \leq r\}$.
2. $C_2 = \{z : \Re(z) \geq 0\}$.

For a set that is neither open nor closed, consider $S = U_1 \cup C_2$.

Holomorphic = Analytic

Holomorphic, analytic

Let U be an open subset of $\mathbb{C}$, and let f be a complex function.

- We say f is holomorphic on U if f is differentiable at every point $z \in U$.
- We say f is analytic on U if f has a series expansion that converges and agrees with f on U. This means that for any $z_0 \in U$, for z close to z_0 we can choose a_n's such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Holomorphic $=$ Analytic

Holomorphic equals Analytic

Let f be a complex function and U an open set. Then f is holomorphic on U if and only if f is analytic on U, and the series expansion for f is its Taylor series.

- If f is differentiable once, it is infinitely differentiable and f agrees with its Taylor series expansion!
- Very different than what happens in the case of functions of a real variable.

Limit points

Limit or accumulation point

We say z is a limit (or an accumulation) point of a sequence $\{z_n\}_{n=0}^{\infty}$ if there exists a subsequence $\{z_{n_k}\}_{k=0}^{\infty}$ converging to z.

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- If $z_n = 1/n$, then 0 is a limit point.
- If $z_n = \cos(\pi n)$ then there are two limit points, namely 1 and -1. (If $z_n = \cos(n)$ then *every* point in $[-1, 1]$ is a limit point of the sequence, though this is harder to show.)

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- If $z_n = (1 + (-1)^n)^n + 1/n$, then 0 is a limit point. We can see this by taking the subsequence $\{z_1, z_3, z_5, z_7, \dots\}$; note the subsequence $\{z_0, z_2, z_4, \dots\}$ diverges to infinity.
- Let z_n denote the number of distinct prime factors of n. Then every positive integer is a limit point!

Limit points

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We say z is a limit (or an accumulation) point of a sequence $\{z_n\}_{n=0}^{\infty}$ if there exists a subsequence $\{z_{n_k}\}_{k=0}^{\infty}$ converging to z.

- If $z_n = n^2$ then there are no limit points, as $\lim_{n\to\infty} z_n = \infty$.
- z_0 any odd, positive integer, set

$$z_{n+1} = \begin{cases} 3z_n + 1 & \text{if } z_n \text{ is odd} \\ z_n/2 & \text{if } z_n \text{ is even.} \end{cases}$$

Conjectured that 1 is always a limit point.

Accumulation points and functions

Theorem

Let f be an analytic function on an open set U, with infinitely many zeros $z_1, z_2, z_3, \ldots$. If $\lim_{n\to\infty} z_n \in U$, then f is identically zero on U. In other words, if a function is zero along a sequence in U whose accumulation point is also in U, then that function is identically zero in U.

Accumulation points and functions

Consider $h(x) = x^3 \sin(1/x)$:

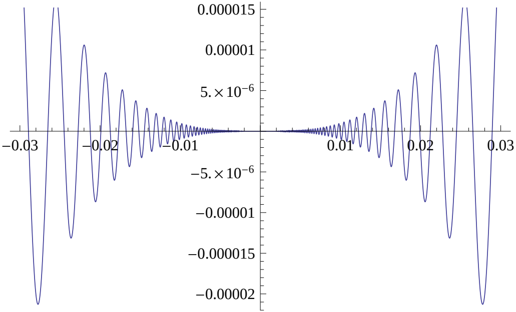

Figure: Plot of $x^3 \sin(1/x)$.

Show $x^3 \sin(1/x)$ is *not* complex differentiable. It will help if you recall $e^{i\theta} = \cos\theta + i \sin\theta$, or $\sin\theta = (e^{i\theta} - e^{-i\theta})/2$.