Math 341: Probability
Twentieth Century Fox Lecture (11/19/09)

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Summary for the Day

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- Central Limit Theorem:
  - ◇ Proof with Fourier analysis.
  - ◇ Discuss rate of convergence.
- Special Topics:
  - ◇ Gambling.
  - ◇ Benford's Law

Central Limit Theorem and Fourier Analysis

Convolutions

Convolution of f and g:

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X_1 and X_2 independent random variables with probability density p.

$$\mathrm{Prob}(X_i \in [x, x+\Delta x]) = \int_{x}^{x+\Delta x} p(t)\mathrm{d}t \approx p(x)\Delta x.$$

$$\mathrm{Prob}(X_1 + X_2) \in [x, x+\Delta x] = \int_{x_1=-\infty}^{\infty} \int_{x_2=x-x_1}^{x+\Delta x - x_1} p(x_1)p(x_2)\mathrm{d}x_2\mathrm{d}x_1.$$

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As $\Delta x \to 0$ we obtain the convolution of p with itself:

$$\mathrm{Prob}(X_1 + X_2 \in [a, b]) = \int_a^b (p * p)(z)\mathrm{d}z.$$

Exercise to show non-negative and integrates to 1.

Statement of Central Limit Theorem

- WLOG p has mean zero, variance one, finite third moment and decays rapidly so all convolution integrals converge: p infinitely differentiable function satisfying

$$\int_{-\infty}^{\infty} x p(x) \mathrm{d}x = 0, \quad \int_{-\infty}^{\infty} x^2 p(x) \mathrm{d}x = 1, \quad \int_{-\infty}^{\infty} |x|^3 p(x) \mathrm{d}x < \infty.$$

- $X_1, X_2, \dots$ are iidrv with density p.
- Define $S_N = \sum_{i=1}^{N} X_i$.
- Standard Gaussian (mean zero, variance one) is $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

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Central Limit Theorem Let X_i, S_N be as above and assume the third moment of each X_i is finite. Then $S_N/\sqrt{N}$ converges in probability to the standard Gaussian:

$$\lim_{N\to\infty} \mathrm{Prob}\left(\frac{S_N}{\sqrt{N}} \;\in\; [a,b]\right) \;=\; \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\mathrm{d}x.$$

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$$\widehat{g}'(y) = \int_{-\infty}^{\infty} -2\pi i x \cdot g(x) e^{-2\pi ixy} \mathrm{d}x;$$

g prob. density, $\widehat{g}'(0) = -2\pi i \mathbb{E}[x]$, $\widehat{g}''(0) = -4\pi^2 \mathbb{E}[x^2]$.

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- We Taylor expand $\widehat{p}$ (need technical conditions on p):

$$\widehat{p}(y) = 1 + \frac{p''(0)}{2} y^2 + \cdots = 1 - 2\pi^2 y^2 + O(y^3).$$

  Near origin, $\widehat{p}$ a concave down parabola.

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- $\text{Prob}\left(\frac{X_1 + \cdots + X_N}{\sqrt{N}} = x\right) \ = \ (\sqrt{N}p * \cdots * \sqrt{N}p)(x\sqrt{N})$.

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- $\text{FT}\left[(\sqrt{N}p * \cdots * \sqrt{N}p)(x\sqrt{N})\right](y) \;=\; \left[\widehat{p}\left(\frac{y}{\sqrt{N}}\right)\right]^N$.

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$$\left[1 - \frac{2\pi^2 y^2}{N} + O\left(\frac{y^3}{N^{3/2}}\right)\right]^N.$$

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- For any **fixed** y,

$$\lim_{N\to\infty}\left[1 - \frac{2\pi^2 y^2}{N} + O\left(\frac{y^3}{N^{3/2}}\right)\right]^N = e^{-2\pi^2 y^2}.$$

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- Fourier transform of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ at y is $e^{-2\pi^2 y^2}$.

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We have shown:

- the Fourier transform of the distribution of S_N converges to $e^{-2\pi^2 y^2}$;
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Therefore the distribution of S_N equalling x converges to $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

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Therefore the distribution of S_N equalling x converges to $\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$.
We need complex analysis to justify this inversion. Must be careful: Consider

$$g(x) \;=\; \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

All the Taylor coefficients about $x = 0$ are zero, but the function is not identically zero in a neighborhood of $x = 0$.