A world record at an Atlantic City casino and the distribution of the length of the crapshooter's hand

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It was widely reported in the media that, on 23 May 2009, at the Borgata Hotel Casino & Spa in Atlantic City, Patricia DeMauro[1], playing craps for only the second time, rolled the dice for four hours and 18 minutes, finally sevening out at the 154th roll. Initial estimates of the probability of this event ranged from one chance in 3.5 billion [3] to one chance in 1.56 trillion [6]. Consensus was reached within days: one chance in 5.6 billion [1, 5].

According to various sources, this established a new world record, previously held by Stanley Fujitake (118 rolls, May 1989, Las Vegas) and more recently by a gentleman known only as The Captain (148 rolls, July 2005, Atlantic City) [4], though the latter event is not as well documented and was unknown to Borgata officials. Presumably, such events have also occurred in situations where no precise count of the number of rolls was kept.

Background

Craps is played by rolling a pair of dice repeatedly. For most bets, only the sum of the numbers appearing on the two dice matters, and this sum has distribution

$$\pi_j := \frac{6 - |j-7|}{36}, \qquad j = 2, 3, \ldots, 12. \tag{1}$$

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[1]Spelled Demauro in some accounts.

The basic bet at craps is the *pass-line bet*, which is defined as follows. The first roll is the *come-out roll.* If 7 or 11 appears (a *natural*), the bettor wins. If 2, 3, or 12 appears (a *craps number*), the bettor loses. If a number belonging to

$$\mathscr{P} := \{4, 5, 6, 8, 9, 10\}$$

appears, that number becomes the *point.* The dice continue to be rolled until the point is repeated (or *made*), in which case the bettor wins, or 7 appears, in which case the bettor loses. The latter event is called a *seven out.* A win pays even money. The first roll following a decision is a new come-out roll, beginning the process again.

A shooter is permitted to roll the dice until he or she sevens out. The sequence of rolls by the shooter is called the *shooter's hand.* The *length* of the shooter's hand (i.e., the number of rolls) is a random variable we will denote by L. Our concern here is with

$$t(n) := \mathrm{P}(L \geq n), \qquad n \geq 1, \tag{2}$$

the tail of the distribution of L. For example, $t(154) \approx 0.178882 \times 10^{-9}$ is the probability of achieving a hand at least as long as that of Ms. DeMauro; to state it in the way preferred by the media, this amounts to one chance in 5.59 billion, approximately. The 1 in 3.5 billion figure came from a simulation that was not extensive enough. The 1 in 1.56 trillion figure came from $(5/6)^{154}$, which is the right answer to the wrong question.

Two methods

We know of two methods for evaluating the tail probabilities (2). The first is by recursion. As pointed out in [2], $t(1) = t(2) = 1$ and

$$t(n) = \Big(1 - \sum_{j\in\mathscr{P}} \pi_j\Big) t(n-1) + \sum_{j\in\mathscr{P}} \pi_j (1-\pi_j-\pi_7)^{n-2} + \sum_{j\in\mathscr{P}} \pi_j \sum_{l=2}^{n-1} (1-\pi_j-\pi_7)^{l-2}\pi_j\, t(n-l) \tag{3}$$

for each $n \geq 3$. Indeed, for the event that the shooter sevens out in no fewer than n rolls to occur, consider the result of the initial come-out roll. If a natural or a craps number occurs, then, beginning with the next roll, the shooter must seven out in no fewer than $n-1$ rolls. If a point number occurs, then there are two possibilities. Either the point is still unresolved after $n-2$ additional rolls, or it is made at roll l for some $l \in \{2, 3, \ldots, n-1\}$ and the shooter subsequently sevens out in no fewer than $n-l$ rolls.

The second method, first suggested, to the best of our knowledge, by Peter A. Griffin in 1987 and rediscovered several times since, is based on a Markov chain. The state space is

$$S := \{\text{co}, \text{p4-10}, \text{p5-9}, \text{p6-8}, \text{7o}\},$$

whose five states represent the events that the shooter is coming out, has established the point 4 or 10, has established the point 5 or 9, has established the point 6 or 8, and has sevened out. The one-step transition matrix, which can be inferred from (1), is

$$\boldsymbol{P} := \frac{1}{36}\begin{pmatrix} 12 & 6 & 8 & 10 & 0 \\ 3 & 27 & 0 & 0 & 6 \\ 4 & 0 & 26 & 0 & 6 \\ 5 & 0 & 0 & 25 & 6 \\ 0 & 0 & 0 & 0 & 36 \end{pmatrix}.$$

The probability of sevening out in $n-1$ rolls or less is then just the probability that absorption in state 7o occurs by the $(n-1)$th step of the Markov chain, starting in state co. Thus, we have

$$t(n) = 1 - (\boldsymbol{P}^{n-1})_{1,5}, \tag{4}$$

where $(\boldsymbol{P}^{n-1})_{1,5}$ denotes the $(1,5)$ entry [or the $(\text{co}, \text{7o})$ entry] of the matrix $\boldsymbol{P}^{n-1}$.

Clearly, (3) is not a closed-form expression, and we do not regard (4) as being in closed form either. Is there a closed-form expression, simple enough to be used by a journalist the next time the record is broken?

A closed-form expression

We apply the spectral representation to (4). The eigenvalues of $\boldsymbol{P}$ include 1 and the four roots of the quartic equation

$$23328z^4 - 58320z^3 + 51534z^2 - 18321z + 1975 = 0.$$

We can use the quartic formula (or *Mathematica*) to find these eigenvalues. We notice that the complex number

$$\alpha := \frac{9829}{\zeta^{1/3}} + \zeta^{1/3},$$

where

$$\zeta := -710369 + 18i\sqrt{1373296647},$$

appears three times in each eigenvalue. Fortunately, α is positive, as we see by writing ζ in polar form, i.e., $\zeta = re^{i\theta}$. We obtain

$$\alpha = 2\sqrt{9829}\cos\left[\frac{1}{3}\cos^{-1}\left(-\frac{710369}{9829\sqrt{9829}}\right)\right],$$

The four nonunit eigenvalues can be expressed as

$$\begin{aligned} e_1 &:= e(1,1), \\ e_2 &:= e(1,-1), \\ e_3 &:= e(-1,1), \\ e_4 &:= e(-1,-1), \end{aligned}$$

where

$$e(u,v) := \frac{5}{8} + \frac{u}{72}\sqrt{\frac{349+\alpha}{3}} + \frac{v}{72}\sqrt{\frac{698-\alpha}{3} - u\,2136\sqrt{\frac{3}{349+\alpha}}}.$$

Next we need to find right eigenvectors corresponding to the five eigenvalues. Fortunately, these eigenvectors can be expressed in terms of the eigenvalues. Indeed, with $\boldsymbol{r}(x)$ defined to be the vector-valued function

$$\begin{pmatrix} -5+(1/5)x \\ -175+(581/15)x-(21/10)x^2+(1/30)x^3 \\ 275/2-(1199/40)x+(8/5)x^2-(1/40)x^3 \\ 1 \\ 0 \end{pmatrix}$$

we find that right eigenvectors corresponding to eigenvalues $1, e_1, e_2, e_3, e_4$ are

$$(1,1,1,1,1)^{\mathsf{T}},\ \boldsymbol{r}(36e_1),\ \boldsymbol{r}(36e_2),\ \boldsymbol{r}(36e_3),\ \boldsymbol{r}(36e_4),$$

respectively. Letting $\boldsymbol{R}$ denote the matrix whose columns are these right eigenvectors and putting $\boldsymbol{L} := \boldsymbol{R}^{-1}$, the rows of which are left eigenvectors, we know by (4) and the spectral representation that

$$t(n) = 1 - \{\boldsymbol{R}\,\mathrm{diag}(1, e_1^{n-1}, e_2^{n-1}, e_3^{n-1}, e_4^{n-1})\boldsymbol{L}\}_{1,5}.$$

After much algebra (and with some help from *Mathematica*), we obtain

$$t(n) = c_1 e_1^{n-1} + c_2 e_2^{n-1} + c_3 e_3^{n-1} + c_4 e_4^{n-1}, \qquad (5)$$

where the coefficients are defined in terms of the eigenvalues and the function

$$\begin{aligned} f(w,x,y,z) := (-25+36w)[4835 - 5580(x+y+z) \\ + 6480(xy+xz+yz) - 7776xyz] \\ /[38880(w-x)(w-y)(w-z)] \end{aligned}$$

as follows:

$$\begin{aligned} c_1 &:= f(e_1, e_2, e_3, e_4), \\ c_2 &:= f(e_2, e_3, e_4, e_1), \\ c_3 &:= f(e_3, e_4, e_1, e_2), \\ c_4 &:= f(e_4, e_1, e_2, e_3). \end{aligned}$$

Of course, (5) is our closed-form expression. We find it a surprisingly elegant solution to what might be considered a rather prosaic problem. Incidentally, the fact that $t(1) = t(2) = 1$ implies that

$$c_1 + c_2 + c_3 + c_4 = 1 \qquad (6)$$

and

$$c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 = 1.$$

As we will see, $1 > e_1 > e_2 > e_3 > e_4 > 0$ and $c_1 > 0$, $c_2 < 0$, $c_3 < 0$, and $c_4 < 0$. In particular, we have the inequality

$$t(n) < c_1 e_1^{n-1}, \qquad n \geq 1, \qquad (7)$$

as well as the asymptotic formula

$$t(n) \sim c_1 e_1^{n-1} \quad \text{as } n \to \infty. \qquad (8)$$

The latter may be adequate for large n; it can be shown to give three significant digits for $n \geq 24$, six for $n \geq 55$, nine for $n \geq 104$, and 12 for $n \geq 156$.

Numerical approximations

Rounding to 18 decimal places, the nonunit eigenvalues are

$$\begin{aligned} e_1 &\approx 0.862473751659322030, \\ e_2 &\approx 0.741708271459795977, \\ e_3 &\approx 0.709206775794379015, \\ e_4 &\approx 0.186611201086502979, \end{aligned}$$

and the constants in (5) are

$$\begin{aligned} c_1 &\approx 1.211844812464518572, \\ c_2 &\approx -0.006375542263784777, \\ c_3 &\approx -0.004042671248651503, \\ c_4 &\approx -0.201426598952082292. \end{aligned}$$

These approximations will give very accurate results over a wide range of values of n.

But we would like something still simpler, usable on a handheld calculator. We use the approximation

$$\bar{t}(n) := \bar{c}_1(\bar{e}_1)^{n-1}, \qquad (9)$$

where

$$\bar{c}_1 := 1.211844813 \quad \text{and} \quad \bar{e}_1 := 0.862473752,$$

which are the nine-decimal-place upper bounds. Then

$$1 < \bar{t}(n)/t(n) < 1 + 10^{-6}, \qquad 59 \leq n \leq 2531,$$

so, for most users, (9) should suffice.

Applications

We can use our formula (5) to obtain other properties of L. First, the distribution of L is

$$\mathrm{P}(L=n)=t(n)-t(n+1)=\sum_{j=1}^{4} c_j e_j^{n-1}(1-e_j),$$

which is a linear combination [*not* a convex combination: (6) holds but three of the coefficients are negative] of four geometric distributions. In particular, the probability generating function is the same linear combination of the geometric pgfs:

$$h(z):=\mathrm{E}[z^L]=\sum_{j=1}^{4} c_j \frac{(1-e_j)z}{1-e_j z}.$$

It also follows that

$$\mathrm{E}[L]=\sum_{j=1}^{4} c_j \frac{1}{1-e_j}\approx 8.525510204$$

and

$$\begin{aligned}\mathrm{Var}(L)&=\sum_{j=1}^{4} c_j \frac{1+e_j}{(1-e_j)^2}-\left(\sum_{j=1}^{4} c_j \frac{1}{1-e_j}\right)^2\\&\approx 46.040738234.\end{aligned}$$

Different expressions were obtained for these quantities in [2], namely

$$\mathrm{E}[L]=\frac{1671}{196} \quad \text{and} \quad \mathrm{Var}(L)=\frac{1768701}{38416}.$$

Of course, the results are consistent.

References

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