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# Math 341: Probability Twenty-first Lecture (11/24/09)

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> Bronfman Science Center Williams College, November 24, 2009

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### Summary for the Day

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Summary	for the da	y				

### Benford's Law:

- Review.
- Inputs (equidistribution).
- Clicker question.
- Difference equations.
- Products.
- More Sum Than Difference Sets:
   > Definition.
   > Inputs (Chebyshev's Theorem).



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### Introduction

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Caveats!						

• Not all fraud can be detected by Benford's Law.



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Caveats!						

- Not all fraud can be detected by Benford's Law.
- A math test indicating fraud is *not* proof of fraud: unlikely events, alternate reasons.

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Caveats!						

- Not all fraud can be detected by Benford's Law.
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Notation						

- Logarithms:  $\log_B x = y$  means  $x = B^y$ . • Example:  $\log_{10} 100 = 2$  as  $100 = 10^2$ . •  $\log_B(uv) = \log_B u + \log_B v$ . •  $\log_{10}(100 \cdot 1000) = \log_{10}(100) + \log_{10}(1000)$ .
- Set Theory:
  - $\diamond \mathbb{Q} =$ rational numbers  $= \{p/q : p, q \text{ integers}\}.$
  - $\diamond x \in S$  means x belongs to S.
  - $\diamond [a,b] = \{x : a \le x \le b\}.$
- Modulo 1:
  - $\diamond$  Any *x* can be written as integer + fraction.
  - $\diamond x \mod 1$  means just the fractional part.
  - $\diamond$  Example:  $\pi$  mod 1 is about .14159.

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#### Benford's Law: Newcomb (1881), Benford (1938)

### Statement

For many data sets, probability of observing a first digit of *d* base *B* is  $\log_B(\frac{d+1}{d})$ ; base 10 about 30% are 1s.

- Not all data sets satisfy Benford's Law.
  - $\diamond$  Long street [1, *L*]: *L* = 199 versus *L* = 999.
  - $\diamond$  Oscillates between 1/9 and 5/9 with first digit 1.
  - Many streets of different sizes: close to Benford.

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Examples						

- recurrence relations
- special functions (such as n!)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the 3x + 1 map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

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Applicatio	ns					

# analyzing round-off errors

- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity



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# Clicker Question

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### **First 60 numbers of the form** $2^n$

digit	# Obs	# Pred	Obs Prob	Benf Prob
1	18	18.1	.300	.301
2	12	10.6	.200	.176
3	6	7.5	.100	.125
4	6	5.8	.100	.097
5	6	4.8	.100	.079
6	4	4.0	.067	.067
7	2	3.5	.033	.058
8	5	3.1	.083	.051
9	1	2.7	.017	.046

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As  $N \to \infty$ , is  $\{2^n\}_{n=0}^N$  Benford? (a) yes (b) no (c) open.

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As  $N \to \infty$ , is  $\{2^n\}_{n=0}^N$  Benford? (a) yes (b) no (c) open.

Are the 9s low in limit? (a) yes (b) no (c) open.

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### Benford's Law: Newcomb (1881), Benford (1938)

### Statement

For many data sets, probability of observing a first digit of *d* base *B* is  $\log_B(\frac{d+1}{d})$ .

### First 60 values of $2^n$ (only displaying 30)

1	1024	1048576	digit	#	Obs Prob	Benf Prob
2	2048	2097152	1	18	.300	.301
4	4096	4194304	2	12	.200	.176
8	8192	8388608	3	6	.100	.125
16	16384	16777216	4	6	.100	.097
32	32768	33554432	5	6	.100	.079
64	65536	67108864	6	4	.067	.067
128	131072	134217728	7	2	.033	.058
256	262144	268435456	8	5	.083	.051
512	524288	536870912	9	1	.017	.046

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Data Analysis							

χ<sup>2</sup>-Tests: Test if theory describes data

 ≿ Expected probability: p<sub>d</sub> = log<sub>10</sub> (d+1/d).
 ≿ Expect about Np<sub>d</sub> will have first digit d.
 ◊ Observe Obs(d) with first digit d.
 ◊ χ<sup>2</sup> = ∑<sup>9</sup><sub>d=1</sub> (Obs(d) - Np<sub>d</sub>)<sup>2</sup>/Np<sub>d</sub>.
 ◊ Smaller χ<sup>2</sup>, more likely correct model.

• Will study 
$$\gamma^n$$
,  $e^n$ ,  $\pi^n$ .

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$\chi^2$ va	alues fo	or $\alpha^n$ , 1	$\leq n \leq n$	N (5% 1	5.5).
	N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$	
	100	0.72	0.30	46.65	
	200	0.24	0.30	8.58	
	400	0.14	0.10	10.55	
	500	0.08	0.07	2.69	
	700	0.19	0.04	0.05	
	800	0.04	0.03	6.19	
	900	0.09	0.09	1.71	
	1000	0.02	0.06	2.90	

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$$\log(\chi^2)$$
 vs *N* for  $\pi^n$  (red) and  $e^n$  (blue),  
 $n \in \{1, ..., N\}$ . Note  $\pi^{175} \approx 1.0028 \cdot 10^{87}$ , (5%,  
 $\log(\chi^2) \approx 2.74$ ).



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Logarithms and Benford's Law: Base 20								

$$\log(\chi^2)$$
 vs *N* for  $\pi^n$  (red) and  $e^n$  (blue),  
 $n \in \{1, ..., N\}$ . Note  $e^3 \approx 20.0855$ , (5%,  
 $\log(\chi^2) \approx 2.74$ ).



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# **General Theory**

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Mantissas						

Mantissa:  $x = M_{10}(x) \cdot 10^k$ , k integer.

 $M_{10}(x) = M_{10}(\tilde{x})$  if and only if x and  $\tilde{x}$  have the same leading digits.

Key observation:  $\log_{10}(x) = \log_{10}(\tilde{x}) \mod 1$  if and only if x and  $\tilde{x}$  have the same leading digits. Thus often study  $y = \log_{10} x$ .



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# Equidistribution

 $\{y_n\}_{n=1}^{\infty}$  is equidistributed modulo 1 if probability  $y_n \mod 1 \in [a, b]$  tends to b - a:

$$\frac{\#\{n \le N : y_n \bmod 1 \in [a, b]\}}{N} \rightarrow b - a.$$

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# Equidistribution

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$$\frac{\#\{n \le N : y_n \bmod 1 \in [a, b]\}}{N} \to b - a.$$

• Thm:  $\beta \notin \mathbb{Q}$ ,  $n\beta$  is equidistributed mod 1.

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• Thm:  $\beta \notin \mathbb{Q}$ ,  $n\beta$  is equidistributed mod 1.

• Examples: 
$$\log_{10} 2$$
,  $\log_{10} \left(\frac{1+\sqrt{5}}{2}\right) \notin \mathbb{Q}$ .

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- Examples:  $\log_{10} 2$ ,  $\log_{10} \left(\frac{1+\sqrt{5}}{2}\right) \notin \mathbb{Q}$ . *Proof:* if rational:  $2 = 10^{p/q}$ .

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# Equidistribution

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- Examples:  $\log_{10} 2$ ,  $\log_{10} \left(\frac{1+\sqrt{5}}{2}\right) \notin \mathbb{Q}$ . *Proof:* if rational:  $2 = 10^{p/q}$ . Thus  $2^q = 10^p$  or  $2^{q-p} = 5^p$ , impossible.

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### Dense

A sequence  $\{z_n\}_{n=1}^{\infty}$  of numbers in [0, 1] is dense if for any interval [a, b] there are infinitely many  $z_n$  in [a, b].

- Dirichlet's Box (or Pigeonhole) Principle:
   If n + 1 objects are placed in n boxes, at least one box has two objects.
- Denseness of  $n\alpha$ :

Thm: If  $\alpha \notin \mathbb{Q}$  then  $z_n = n\alpha \mod 1$  is dense.

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<b>Proof</b> $n\alpha$ mod 1 dense if $\alpha \notin \mathbb{Q}$								

- Enough to show in [0, *b*] infinitely often for any *b*.
- Choose any integer Q > 1/b.
- Q bins:  $\left[0, \frac{1}{Q}\right], \left[\frac{1}{Q}, \frac{2}{Q}\right], \dots, \left[\frac{Q-1}{Q}, Q\right].$
- Q + 1 objects: { $\alpha \mod 1, 2\alpha \mod 1, \dots, (Q + 1)\alpha \mod 1$ }.
- Two in same bin, say  $q_1 \alpha \mod 1$  and  $q_2 \alpha \mod 1$ .
- Exists integer *p* with 0 < q<sub>2</sub>α − q<sub>1</sub>α − p < <sup>1</sup>/<sub>Q</sub>.
  Get (q<sub>2</sub> − q<sub>1</sub>)α mod 1 ∈ [0, *b*].

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### **Fundamental Equivalence**

Data set  $\{x_i\}$  is Benford base *B* if  $\{y_i\}$  is equidistributed mod 1, where  $y_i = \log_B x_i$ .

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#### Logarithms and Benford's Law

# **Fundamental Equivalence**

Data set  $\{x_i\}$  is Benford base *B* if  $\{y_i\}$  is equidistributed mod 1, where  $y_i = \log_B x_i$ .

# **Proof:**

- $x = M_B(x) \cdot B^k$  for some  $k \in \mathbb{Z}$ .
- $FD_B(x) = d$  iff  $d \le M_B(x) < d + 1$ .
- $\log_B d \le y < \log_B(d+1)$ ,  $y = \log_B x \mod 1$ .
- If Y ~ Unif(0, 1) then above probability is log<sub>B</sub> (<sup>d+1</sup>/<sub>d</sub>).

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Examples						

# • $2^n$ is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$ .

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Examples						

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Examples						

# • Fibonacci numbers are Benford base 10. $a_{n+1} = a_n + a_{n-1}$ .

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Examples						

# • Fibonacci numbers are Benford base 10. $a_{n+1} = a_n + a_{n-1}$ . Guess $a_n = n^r$ : $r^{n+1} = r^n + r^{n-1}$ or $r^2 = r + 1$ .

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Examples						

$$a_{n+1} = a_n + a_{n-1}.$$
  
Guess  $a_n = n^r$ :  $r^{n+1} = r^n + r^{n-1}$  or  $r^2 = r + 1.$   
Roots  $r = (1 \pm \sqrt{5})/2.$ 

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Examples						

 $a_{n+1} = a_n + a_{n-1}.$ Guess  $a_n = n^r$ :  $r^{n+1} = r^n + r^{n-1}$  or  $r^2 = r + 1.$ Roots  $r = (1 \pm \sqrt{5})/2.$ General solution:  $a_n = c_1 r_1^n + c_2 r_2^n.$ 

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Examples						

 $\begin{array}{l} a_{n+1} = a_n + a_{n-1}.\\ \text{Guess } a_n = n^r \colon r^{n+1} = r^n + r^{n-1} \text{ or } r^2 = r+1.\\ \text{Roots } r = (1 \pm \sqrt{5})/2.\\ \text{General solution: } a_n = c_1 r_1^n + c_2 r_2^n.\\ \text{Binet: } a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n. \end{array}$ 

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Examples						

$$\begin{array}{l} a_{n+1} = a_n + a_{n-1}.\\ \text{Guess } a_n = n^r \colon r^{n+1} = r^n + r^{n-1} \text{ or } r^2 = r+1.\\ \text{Roots } r = (1 \pm \sqrt{5})/2.\\ \text{General solution: } a_n = c_1 r_1^n + c_2 r_2^n.\\ \text{Binet: } a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n. \end{array}$$

• Most linear recurrence relations Benford:

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Examples						

 $\begin{array}{l} a_{n+1} = a_n + a_{n-1}.\\ \text{Guess } a_n = n^r: r^{n+1} = r^n + r^{n-1} \text{ or } r^2 = r+1.\\ \text{Roots } r = (1 \pm \sqrt{5})/2.\\ \text{General solution: } a_n = c_1 r_1^n + c_2 r_2^n.\\ \text{Binet: } a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n. \end{array}$ 

Most linear recurrence relations Benford:
 \$\$\$ a\_{n+1} = 2a\_n\$

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Examples						

 $\begin{array}{l} a_{n+1} = a_n + a_{n-1}.\\ \text{Guess } a_n = n^r: r^{n+1} = r^n + r^{n-1} \text{ or } r^2 = r+1.\\ \text{Roots } r = (1 \pm \sqrt{5})/2.\\ \text{General solution: } a_n = c_1 r_1^n + c_2 r_2^n.\\ \text{Binet: } a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n. \end{array}$ 

• Most linear recurrence relations Benford:  $\diamond a_{n+1} = 2a_n - a_{n-1}$ 

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 $a_{n+1} = a_n + a_{n-1}.$ Guess  $a_n = n^r$ :  $r^{n+1} = r^n + r^{n-1}$  or  $r^2 = r + 1$ . Roots  $r = (1 \pm \sqrt{5})/2$ . General solution:  $a_n = c_1 r_1^n + c_2 r_2^n$ . Binet:  $a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$ .

Most linear recurrence relations Benford:

 a<sub>n+1</sub> = 2a<sub>n</sub> - a<sub>n-1</sub>

 take a<sub>0</sub> = a<sub>1</sub> = 1 or a<sub>0</sub> = 0, a<sub>1</sub> = 1.

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# Applications

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Stock Mar	ket					

Milestone	Date	Effective Rate from last milestone
108.35	Jan 12, 1906	
500.24	Mar 12, 1956	3.0%
1003.16	Nov 14, 1972	4.2%
2002.25	Jan 8, 1987	4.9%
3004.46	Apr 17, 1991	9.5%
4003.33	Feb 23, 1995	7.4%
5023.55	Nov 21, 1995	30.6%
6010.00	Oct 14, 1996	20.0%
7022.44	Feb 13, 1997	46.6%
8038.88	Jul 16, 1997	32.3%
9033.23	Apr 6, 1998	16.1%
10006.78	Mar 29, 1999	10.5%
11209.84	Jul 16, 1999	38.0%
12011.73	Oct 19, 2006	1.0%
13089.89	Apr 25, 2007	16.7%
14000.41	Jul 19, 2007	28.9%

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#### **Applications for the IRS: Detecting Fraud**

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R a john rates, so	erer's bits	a sease and initial						263-40-28
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1800 CER	1000							Paperwork Reda
City seas or past of	dista. mm	t see 29 com. If a farage	add talk live	Mile 33	111-2-			Aut Notice,
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### Applications for the IRS: Detecting Fraud

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#### **Applications for the IRS: Detecting Fraud**

Date of Check	Amount
October 9, 1992	\$ 1.927.48
Ŧ	27.902.31
October 14, 1992	86,241.90
	72,117.46
ing the second second second second	81,321.75
↓	97,473.96
October 19, 1992	93,249.11
	89,658.17
	87,776.89
	92,105.83
MURANE RECEIPTION DURING THE	79,949.16
	87,602.93
	96,879.27
	91,806.47
	84,991.67
	90,831.83
	93,766.67
	88,338.72
NAME OF STREET, NO. 1997, M.	94,639.49
	83,709.28
	96,412.21
	88,432.86
*	71,552.16
TOTAL	\$ 1,878,687.58

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Applications for the IRS: Detecting Fraud (cont)

- Embezzler started small and then increased dollar amounts.
- Most amounts below \$100,000 (critical threshold for data requiring additional scrutiny).
- Over 90% had first digit of 7, 8 or 9.

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Detecting	Fraud					

#### **Bank Fraud**

- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.
- Write-off limit of \$5,000. Officer had friends applying for credit cards, ran up balances just under \$5,000 then he would write the debts off.

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Detecting	Fraud					

#### Enron

- Benford's Law detected manipulation of revenue numbers.
- Results showed a tendency towards round Earnings Per Share (0.10, 0.20, etc.).
   Consistent with a small but noticeable increase in earnings management in 2002.



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#### Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



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Election F	raud: Iran	2009				

Numerous protests and complaints over Iran's 2009 elections.

Lot of analysis done; data is moderately suspicious. Tests done include

- First and second leading digits;
- Last two digits (should almost be uniform);
- Last two digits differing by at least 2.

Warning: do enough tests, even if nothing is wrong will find a suspicious result, but when all tests are on the boundary....



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### The Modulo 1 Central Limit Theorem

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#### **Needed Input: Poisson Summation Formula**

Poisson Summation Formula  

$$f$$
 nice:  

$$\sum_{\ell=-\infty}^{\infty} f(\ell) = \sum_{\ell=-\infty}^{\infty} \widehat{f}(\ell),$$
Fourier transform  $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$ 

Daily Summary	Introduction	Clicker Questions	Theory	Applications	Mod 1 CLT	Products $\mathcal{F}$
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#### **Needed Input: Poisson Summation Formula**

Poisson Summation Formula  

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 nice:  

$$\sum_{\ell=-\infty}^{\infty} f(\ell) = \sum_{\ell=-\infty}^{\infty} \hat{f}(\ell),$$
Fourier transform  $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$ 

#### What is 'nice'?

- f Schwartz more than enough.
- *f* twice continuously differentiable & *f*, *f'*, *f''* decay like  $x^{-(1+\eta)}$  for an  $\eta > 0$  (*g* decays like  $x^{-a}$  if  $\exists x_0, C$  st  $|x| > x_0, |g(x)| \le C/|x|^a$ ).

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#### Modulo 1 Central Limit Theorem

### The Modulo 1 Central Limit Theorem for Independent

Let  $\{Y_m\}$  be independent continuous random variables on [0, 1), not necessarily identically distributed, with densities  $\{g_m\}$ . A necessary and sufficient condition for  $Y_1 + \cdots + Y_M$  modulo 1 to converge to the uniform distribution as  $M \to \infty$  (in  $L_1([0, 1])$  is that for each  $n \neq 0$  we have  $\lim_{M\to\infty} \widehat{g_1}(n) \cdots \widehat{g_M}(n) = 0$ .

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#### Modulo 1 Central Limit Theorem

#### The Modulo 1 Central Limit Theorem for Independent

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Application to Benford's law: If  $X = X_1 \cdots X_M$  then

$$\log_{10} X = \log_{10} X_1 + \dots + \log_{10} X_M := Y_1 + \dots + Y_M.$$

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### Products of Random Variables and the Fourier Transform

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Prelimina	ries					

• 
$$X_1 \cdots X_n \Leftrightarrow Y_1 + \cdots + Y_n \mod 1, Y_i = \log_B X_i$$

• Density  $Y_i$  is  $g_i$ , density  $Y_i + Y_j$  is

$$(g_i * g_j)(y) = \int_0^1 g_i(t)g_j(y-t)dt.$$

- $h_n = g_1 * \cdots * g_n$ ,  $\widehat{g}(\xi) = \widehat{g}_1(\xi) \cdots \widehat{g}_n(\xi)$ .
- Dirac delta functional:  $\int \delta_{\alpha}(y)g(y)dy = g(\alpha)$ .

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Fourier in	put					

• Fejér kernel:

$$F_N(x) = \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}.$$

• Fejér series:

$$T_N f(\mathbf{x}) = (f * F_N)(\mathbf{x}) = \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) \widehat{f}(n) e^{2\pi i n \mathbf{x}}.$$

• Lebesgue's Theorem:  $f \in L^1([0, 1])$ . As  $N \to \infty$ ,  $T_N f$  converges to f in  $L^1([0, 1])$ .

• 
$$T_N(f * g) = (T_N f) * g$$
: convolution assoc.

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#### Modulo 1 Central Limit Theorem

#### Theorem (M– and Nigrini 2007)

{Y<sub>m</sub>} independent continuous random variables on [0, 1) (not necc. i.i.d.), densities {g<sub>m</sub>}. Y<sub>1</sub> + · · · + Y<sub>M</sub> mod 1 converges to the uniform distribution as  $M \to \infty$  in  $L^1([0, 1])$  iff  $\forall n \neq 0$ ,  $\lim_{M\to\infty} \widehat{g_1}(n) \cdots \widehat{g_M}(n) = 0$ .

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Generaliza	ations					

- Levy proved for i.i.d.r.v. just one year after Benford's paper.
- Generalized to other compact groups, with estimates on the rate of convergence.
   Stromberg: *n*-fold convolution of a regular probability measure on a compact Hausdorff group *G* converges to normalized Haar measure in weak-star topology iff support of the distribution not contained in a coset of a proper normal closed subgroup of *G*.

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### Theorem (M– and Nigrini 2007)

 $\{Y_m\}$  indep. discrete random variables on [0, 1), not necc. identically distributed, densities

$$g_m(x) = \sum_{k=1}^{r_m} w_{k,m} \delta_{\alpha_{k,m}}(x), w_{k,m} > 0, \sum_{k=1}^{r_m} w_{k,m} = 1.$$

Assume that there is a finite set  $A \subset [0, 1)$  such that all  $\alpha_{k,m} \in A$ .  $Y_1 + \cdots + Y_M \mod 1$  converges weakly to the uniform distribution as  $M \to \infty$  iff  $\forall n \neq 0$ ,  $\lim_{M\to\infty} \widehat{g_1}(n) \cdots \widehat{g_M}(n) = 0$ .

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Distribution of digits (base 10) of 1000 products  $X_1 \cdots X_{1000}$ , where  $g_{10,m} = \phi_{11^m}$ .  $\phi_m(x) = m$  if  $|x - 1/8| \le 1/2m$  (0 otherwise).



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Proof of Modulo 1 CLT								

- Density of sum is  $h_{\ell} = g_1 * \cdots * g_{\ell}$ .
- Suffices show  $\forall \epsilon$ :  $\lim_{M \to \infty} \int_0^1 |h_M(x) 1| dx < \epsilon$ .
- Lebesgue's Theorem: N large,

$$||h_1 - T_N h_1||_1 = \int_0^1 |h_1(x) - T_N h_1(x)| dx < \frac{\epsilon}{2}.$$

• Claim: above holds for  $h_M$  for all M.

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Proof of claim								

$$T_N h_{M+1} = T_N (h_M * g_{M+1}) = (T_N h_M) * g_{M+1}$$

$$||h_{M+1} - T_N h_{M+1}||_1 = \int_0^1 |h_{M+1}(x) - T_N h_{M+1}(x)| dx$$
  
=  $\int_0^1 |(h_M * g_{M+1})(x) - (T_N h_M) * g_{M+1}(x)| dx$   
=  $\int_0^1 \left| \int_0^1 (h_M(y) - T_N h_M(y)) g_{M+1}(x - y) \right| dy dx$   
 $\leq \int_0^1 \int_0^1 |h_M(y) - T_N h_M(y)| g_{M+1}(x - y) dx dy$   
=  $\int_0^1 |h_M(y) - T_N h_M(y)| dy \cdot 1 < \frac{\epsilon}{2}.$
Daily Summary o	Introduction	Clicker Questions	Theory 0000000	Applications	Mod 1 CLT oo	Products <i>F</i> ○○○○○○○●○

## Proof of Modulo 1 CLT (continued)

Show 
$$\lim_{M\to\infty} ||h_M - 1||_1 = 0$$
.  
Triangle inequality:

$$||h_M - 1||_1 \le ||h_M - T_N h_M||_1 + ||T_N h_M - 1||_1.$$

Choices of N and  $\epsilon$ :

$$||h_M - T_N h_M||_1 < \epsilon/2.$$

Show  $||T_N h_M - 1||_1 < \epsilon/2$ .

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$$||T_N h_M - 1||_1 = \int_0^1 \left| \sum_{\substack{n=-N\\n\neq 0}}^N \left( 1 - \frac{|n|}{N} \right) \widehat{h_M}(n) e^{2\pi i n x} \right| dx$$
$$\leq \sum_{\substack{n=-N\\n\neq 0}}^N \left( 1 - \frac{|n|}{N} \right) |\widehat{h_M}(n)|$$

 $\widehat{h_M}(n) = \widehat{g_1}(n) \cdots \widehat{g_M}(n) \longrightarrow_{M \to \infty} 0.$ For fixed *N* and  $\epsilon$ , choose *M* large so that  $|\widehat{h_M}(n)| < \epsilon/4N$  whenever  $n \neq 0$  and  $|n| \leq N$ .